# C.3 FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS

**Solve first-order linear differential equations. Use first-order linear differential equations to model and solve real-life problems.**

## First-Order Linear Differential Equations

**Definition of a First-Order Linear Differential Equation**

A **first-order linear differential equation** is an equation of the form

$$y' + P(x)y = Q(x)$$

where $P$ and $Q$ are functions of $x$. An equation that is written in this form is said to be in **standard form.**

**STUDY TIP**

The term "first-order" refers to the fact that the highest-order derivative of $y$ in the equation is the first derivative.

To solve a linear differential equation, write it in standard form to identify the functions $P(x)$ and $Q(x)$. Then integrate $P(x)$ and form the expression

$$u(x) = e^{\int P(x)\,dx} \qquad \text{Integrating factor}$$

which is called an **integrating factor.** The general solution of the equation is

$$y = \frac{1}{u(x)}\int Q(x)u(x)\,dx. \qquad \text{General solution}$$

**EXAMPLE 1** **Solving a Linear Differential Equation**

Find the general solution of

$$y' + y = e^x.$$

**SOLUTION** For this equation, $P(x) = 1$ and $Q(x) = e^x$. So, the integrating factor is

$$u(x) = e^{\int dx} \qquad \text{Integrating factor}$$

$$= e^x.$$

This implies that the general solution is

$$y = \frac{1}{e^x}\int e^x(e^x)\,dx$$

$$= e^{-x}\left(\frac{1}{2}e^{2x} + C\right)$$

$$= \frac{1}{2}e^x + Ce^{-x}. \qquad \text{General solution}$$

In Example 1, the differential equation was given in standard form. For equations that are not written in standard form, you should first convert to standard form so that you can identify the functions $P(x)$ and $Q(x)$.

**EXAMPLE 2** **Solving a Linear Differential Equation**

Find the general solution of

$$xy' - 2y = x^2.$$

Assume $x > 0$.

**SOLUTION** Begin by writing the equation in standard form.

$$y' - \left(\frac{2}{x}\right)y = x \qquad \text{Standard form, } y' + P(x)y = Q(x)$$

In this form, you can see that $P(x) = -2/x$ and $Q(x) = x$. So,

$$\begin{aligned}\int P(x)\,dx &= -\int \frac{2}{x}\,dx \\ &= -2\ln x \\ &= -\ln x^2\end{aligned}$$

which implies that the integrating factor is

$$\begin{aligned}u(x) &= e^{\int P(x)\,dx} \\ &= e^{-\ln x^2} \\ &= \frac{1}{x^2}. \qquad \text{Integrating factor}\end{aligned}$$

This implies that the general solution is

$$\begin{aligned}y &= \frac{1}{u(x)}\int Q(x)u(x)\,dx && \text{Form of general solution} \\ &= \frac{1}{1/x^2}\int x\left(\frac{1}{x^2}\right)dx && \text{Substitute.} \\ &= x^2\int \frac{1}{x}\,dx && \text{Simplify.} \\ &= x^2(\ln x + C). && \text{General solution}\end{aligned}$$

**DISCOVERY**

Solve for $y'$ in the differential equation in Example 2. Use this equation for $y'$ to determine the slopes of $y$ at the points $(1, 0)$ and $(e^{-1/2}, -1/2e)$. Now graph the particular solution $y = x^2 \ln x$ and estimate the slopes at $x = 1$ and $x = e^{-1/2}$. What happens to the slope of $y$ as $x$ approaches zero?

**TECHNOLOGY**

From Example 2, you can see that it can be difficult to solve a linear differential equation. Fortunately, the task is greatly simplified by symbolic integration utilities. Use a symbolic integration utility to find the particular solution of the differential equation in Example 2, given the initial condition $y = 1$ when $x = 1$.

**Guidelines for Solving a Linear Differential Equation**

1. Write the equation in standard form
$$y' + P(x)y = Q(x).$$
2. Find the integrating factor
$$u(x) = e^{\int P(x)\,dx}.$$
3. Evaluate the integral below to find the general solution.
$$y = \frac{1}{u(x)}\int Q(x)u(x)\,dx$$

**PREREQUISITE REVIEW C.3**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–4, simplify the expression.

**1.** $e^{-x}(e^{2x} + e^{x})$

**2.** $\dfrac{1}{e^{-x}}(e^{-x} + e^{2x})$

**3.** $e^{-\ln x^3}$

**4.** $e^{2\ln x + x}$

In Exercises 5–10, find the indefinite integral.

**5.** $\displaystyle\int e^{x}(2 + e^{-2x})\,dx$

**6.** $\displaystyle\int e^{2x}(xe^{x} + 1)\,dx$

**7.** $\displaystyle\int \frac{1}{2x + 5}\,dx$

**8.** $\displaystyle\int \frac{x + 1}{x^2 + 2x + 3}\,dx$

**9.** $\displaystyle\int (4x - 3)^2\,dx$

**10.** $\displaystyle\int x(1 - x^2)^2\,dx$

**EXERCISES C.3**

…ercises 1–6, write the linear differential equation in standard …m.

…$^3 - 2x^2y' + 3y = 0$ **2.** $y' - 5(2x - y) = 0$

…$y' + y = xe^{x}$ **4.** $xy' + y = x^3y$

…$+ 1 = (x - 1)y'$ **6.** $x = x^2(y' + y)$

…ercises 7–18, solve the differential equation.

…$\frac{dy}{dx} + 3y = 6$ **8.** $\dfrac{dy}{dx} + 5y = 15$

…$\frac{dy}{dx} + y = e^{-x}$ **10.** $\dfrac{dy}{dx} + 3y = e^{-3x}$

…$\frac{dy}{dx} + \frac{y}{x} = 3x + 4$ **12.** $\dfrac{dy}{dx} + \dfrac{2y}{x} = 3x + 1$

…$y' + 5xy = x$ **14.** $y' + 5y = e^{5x}$

…$(x - 1)y' + y = x^2 - 1$ **16.** $xy' + y = x^2 + 1$

…$x^3y' + 2y = e^{1/x^2}$ **18.** $xy' + y = x^2 \ln x$

…ercises 19–22, solve for $y$ in two ways.

…$y' + y = 4$ **20.** $y' + 10y = 5$

…$y' - 2xy = 2x$ **22.** $y' + 4xy = x$

In Exercises 23–26, match the differential equation with its solution.

| Differential Equation | Solution |
|---|---|
| **23.** $y' - 2x = 0$ | (a) $y = Ce^{x^2}$ |
| **24.** $y' - 2y = 0$ | (b) $y = -\frac{1}{2} + Ce^{x^2}$ |
| **25.** $y' - 2xy = 0$ | (c) $y = x^2 + C$ |
| **26.** $y' - 2xy = x$ | (d) $y = Ce^{2x}$ |

In Exercises 27–34, find the particular solution that satisfies the initial condition.

| Differential Equation | Initial Condition |
|---|---|
| **27.** $y' + y = 6e^{x}$ | $y = 3$ when $x = 0$ |
| **28.** $y' + 2y = e^{-2x}$ | $y = 4$ when $x = 1$ |
| **29.** $xy' + y = 0$ | $y = 2$ when $x = 2$ |
| **30.** $y' + y = x$ | $y = 4$ when $x = 0$ |
| **31.** $y' + 3x^2y = 3x^2$ | $y = 6$ when $x = 0$ |
| **32.** $y' + (2x - 1)y = 0$ | $y = 2$ when $x = 1$ |
| **33.** $xy' - 2y = -x^2$ | $y = 5$ when $x = 1$ |
| **34.** $x^2y' - 4xy = 10$ | $y = 10$ when $x = 1$ |