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Light on the Infinite Group Relaxation

II. Sufficient Conditions for Extremality, Sequences, and Algorithms

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Abstract This is the second part of a survey on the infinite group problem, an infinite-dimensional relaxation of integer linear optimization problems introduced by Ralph Gomory and Ellis Johnson in their groundbreaking papers titled *Some continuous functions related to corner polyhedra I, II* [Math. Programming 3 (1972), 23–85, 359–389]. The survey presents the infinite group problem in the modern context of cut generating functions. It focuses on the recent developments, such as algorithms for testing extremality and breakthroughs for the k-row problem for general $k \geq 1$ that extend previous work on the single-row and two-row problems. The survey also includes some previously unpublished results; among other things, it unveils piecewise linear extreme functions with more than four different slopes. An interactive companion program, implemented in the open-source computer algebra package Sage, provides an updated compendium of known extreme functions.

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Contents

This is the second part of the survey by the authors on recent progress in the infinite group problem. This part presents some of the exciting new discoveries in the structure of valid functions for the infinite group problem and algorithms to test their fundamental properties. We refer to the definitions, concepts and results from Part I using labels consistent with Part I. For example, Theorem 2.13 refers to the corresponding theorem from section 2 in Part I. Moreover, the citations made in this part are to be found in References from Part I with the corresponding number. Finally, the numbering of the sections in this part is a continuation of the numbering used in Part I, so this part consists of sections 5, 6, 7 and 8.

As a prelude to the results presented in this second part, we give a quick proof of the classical result that the Gomory Mixed-Integer (GMI) function for the one-dimensional ($k = 1$) infinite group problem is extreme. The GMI function, defined for any $f \in (0, 1)$, is a continuous piecewise linear minimal valid function for $R_f(\mathbb{R}, \mathbb{Z})$. It has a single breakpoint in $(0, 1)$ at f; see Figure 9 and also `gmic` in Table 1 in Part I.[1] This extremality proof will not only serve as an application of the machinery developed in Part I, but also expose the basic skeleton of the arguments used in extremality proofs in this part – see also the discussion in subsection 2.3 after Theorem 2.12. Of course, the deeper theorems surveyed in the following sections require more sophisticated techniques building on this general framework.

Let π be a minimal function of the type `gmic`. To show that π is extreme, we analyze the additivity domain $E(\pi)$ which is illustrated in green in Figure 9.

1. Assume $\pi = \frac{1}{2}(\pi^1 + \pi^2)$, where π^1, π^2 are valid functions. We will proceed to show that $\pi^1 = \pi$ and thus $\pi^1 = \pi^2 = \pi$, showing that π is extreme.
2. By Lemma 2.11 (i), π^1, π^2 are minimal and by Lemma 2.11 (ii), we have inclusion of additivity domains, i.e., $E(\pi) \subseteq E(\pi^1) \cap E(\pi^2)$. Therefore, $F_1, F_2 \subseteq E(\pi^1)$ where $F_1, F_2 \subseteq E(\pi)$ are depicted in Figure 9.
3. Applying Theorem 4.3 [Convex additivity domain lemma, full-dimensional version] with $k = 1$, $f = g = h = \pi^1$ and $F = F_1, F_2$, we see that π^1 is affine over the open intervals $(0, f) = \operatorname{int}(p_1(F_1))$ and $(f, 1) = \operatorname{int}(p_1(F_2))$ (see section 4 [Part I] for notation).

[1] The function is available in the Electronic Compendium [70] as `gmic`.

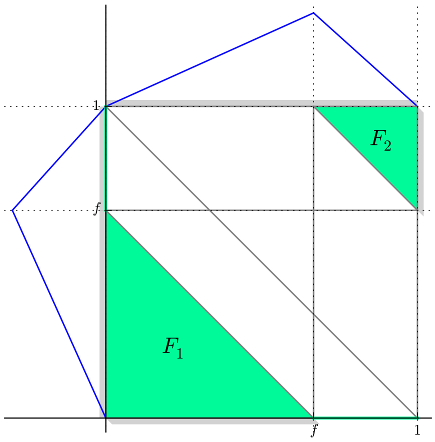

Fig. 9 A diagram of a function of the type `gmic` (*blue graphs on the top and the left*) and its polyhedral complex $\Delta\mathcal{P}$ (*gray solid lines*), as plotted by the command `plot_2d_diagram(gmic(f=2/3))`. There are three combinatorial types of these diagrams, depending on whether $f < \frac{1}{2}$, $f = \frac{1}{2}$, or $f > \frac{1}{2}$. No matter what f is, the additivity domain $E(\pi)$ is the union of the faces $F_1 = F([0, f], [0, f], [0, f])$ and $F_2 = F([f, 1], [f, 1], [1 + f, 2])$, *shaded in green*. At the borders of each diagram, the projections $p_i(F)$ of two-dimensional additive faces are shown as *gray shadows*: $p_1(F)$ at the top border, $p_2(F)$ at the left border, $p_3(F)$ at the bottom and the right borders.

4. By Lemma 2.11 (iv), π^1 is (Lipschitz) continuous. Moreover, by Theorem 2.6, $\pi^1(0) = 1$ and $\pi^1(f) = 1$. Combining this with the fact that π^1 is affine over $(0, f)$ and $(f, 1)$, we obtain that in fact $\pi^1 = \pi$.

5 Sufficient conditions for extremality in the k-row infinite group problem

5.1 The $(k + 1)$-Slope Theorem

The fact that `gmic` is extreme also follows from the classic Gomory–Johnson 2-Slope Theorem (Theorem 2.13), which states that for $k = 1$, if a continuous piecewise linear minimal valid function has only 2 slopes, then it is extreme. An analogous 3-Slope Theorem for $k = 2$ was proved by Cornuéjols and Molinaro [31]. We present here the $(k+1)$-Slope Theorem for the case of general k by Basu, Hildebrand, Köppe and Molinaro [21], along with the main ingredients of its proof.

Theorem 5.1 ([21, Theorem 1.7]) *Let $\pi\colon \mathbb{R}^k \to \mathbb{R}$ be a minimal valid function that is continuous piecewise linear and genuinely k-dimensional*[2] *with at most $k + 1$ slopes, i.e., at most $k + 1$ different values for the gradient of π where it exists. Then π is extreme and has exactly $k + 1$ slopes.*

[2] See Definition 3.7.

The proof will follow the basic roadmap of subsection 2.3 and use Lemma 2.11 just like the proof of extremality of `gmic` above. We give an outline here, before diving into the details. For the rest of this section, π is a continuous piecewise linear minimal function that is genuinely k-dimensional with at most $k+1$ slopes. Let $\mathcal{P}$ be the associated polyhedral complex.

1. Subadditivity and the property of being genuinely k-dimensional is used to first establish that π has exactly $k+1$ gradient values $\bar{\mathbf{g}}^1, \ldots, \bar{\mathbf{g}}^{k+1} \in \mathbb{R}^k$. This is a relatively easy step, and we refer to the reader to [21, Lemma 2.11] for the details.
2. Consider any minimal valid functions π^1, π^2 such that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$.
3. (Compatibility step) For each $i = 1, \ldots, k+1$, define $\mathcal{P}_i \subseteq \mathcal{P}$ to be the polyhedral complex formed by all the cells (and their faces) of $\mathcal{P}$ where the gradient of π is $\bar{\mathbf{g}}^i$. Show that there exist $\tilde{\mathbf{g}}^1, \ldots, \tilde{\mathbf{g}}^{k+1}$ such that π^1 is affine over every cell in $\mathcal{P}_i$ with gradient $\tilde{\mathbf{g}}^i$.
4. (Gradient matching step) We then highlight certain structures of genuinely k-dimensional functions with $k+1$ slopes that lead to a system of $k(k+1)$ equations that are satisfied by the coefficients of $\bar{\mathbf{g}}^1, \ldots, \bar{\mathbf{g}}^{k+1}$ and $\tilde{\mathbf{g}}^1, \ldots, \tilde{\mathbf{g}}^{k+1}$. Then, it is established that this system of equations has a unique solution, and thus, $\bar{\mathbf{g}}^i = \tilde{\mathbf{g}}^i$ for every $i = 1, \ldots, k+1$.
5. For every $\mathbf{r} \in \mathbb{R}^k$ there exist $\mu_1, \mu_2, \ldots, \mu_{k+1}$ such that μ_i is the fraction of the segment $[\mathbf{0}, \mathbf{r}]$ that lies in $\mathcal{P}_i$. Thus,
$$\pi(\mathbf{r}) = \pi(\mathbf{0}) + \sum_{i=1}^{k+1} \mu_i(\bar{\mathbf{g}}^i \cdot \mathbf{r}) = \pi^1(\mathbf{0}) + \sum_{i=1}^{k+1} \mu_i(\tilde{\mathbf{g}}^i \cdot \mathbf{r}) = \pi^1(\mathbf{r}).$$
This proves that $\pi = \pi^1$ and thus, $\pi = \pi^1 = \pi^2$, concluding the proof of Theorem 5.1.

Compatibility Step. The following observation is crucial:

Lemma 5.2 *Let $U, V \subseteq \mathbb{R}^k$ be full-dimensional convex sets such that $\mathbf{0} \in U$. Let $F = F(U, V, V)$*[3]*. Then $\mathbf{0} \in p_1(F)$, $V = p_2(F) = p_3(F)$ and $p_1(F)$ is full-dimensional. Furthermore, if $\pi \colon \mathbb{R}^k \to \mathbb{R}$ is such that $\pi(\mathbf{0}) = 0$ and is affine on U, V with the same slope, then $F \subseteq E(\pi)$.*

Proof By definition, $p_1(F) \subseteq U, p_2(F), p_3(F) \subseteq V$. Since $\mathbf{0} \in U$ and $\{\mathbf{0}\} + V = V$, we see that $p_2(F), p_3(F) = V$ and $\mathbf{0} \in p_1(F)$. Now, let $\mathbf{v} \in \operatorname{int}(V)$. Therefore there exists a ball $B(\mathbf{v}, \epsilon) \subseteq V$. Since U is full-dimensional, there exist k-linearly independent vectors $\mathbf{u}^1, \ldots, \mathbf{u}^k \in U$ with $\|\mathbf{u}^i\| \leq \epsilon$. But then $\mathbf{u}^i + \mathbf{v} \in V$. Therefore, $\mathbf{u}^i \in p_1(F)$. Finally, since F is convex and the projection of convex sets is convex, we have that $p_1(F)$ is full-dimensional.

For the second part of the lemma, observe that there exist $\mathbf{g} \in \mathbb{R}^k$ and $\delta \in \mathbb{R}$ be such that $\pi(\mathbf{u}) = \mathbf{g} \cdot \mathbf{u}$ (follows since $\mathbf{0} \in U$ and $\pi(\mathbf{0}) = 0$) for all $\mathbf{u} \in U$ and $\pi(\mathbf{v}) = \mathbf{g} \cdot \mathbf{v} + \delta$ for all $\mathbf{v} \in V$. Then for any $\mathbf{u} \in U, \mathbf{v} \in V$ with $\mathbf{u} + \mathbf{v} \in V$, we have $\pi(\mathbf{u}) + \pi(\mathbf{v}) - \pi(\mathbf{u} + \mathbf{v}) = (\mathbf{g} \cdot \mathbf{u}) + (\mathbf{g} \cdot \mathbf{v} + \delta) - (\mathbf{g} \cdot (\mathbf{u} + \mathbf{v}) + \delta) = 0$. □

[3] See the definition in (3.9).

The analysis of step (1) also shows that for every $i = 1, \ldots, k+1$, there exist $C_i \in \mathcal{P}_i$ such that $\mathbf{0} \in C_i$ (in other words, for every gradient value, there is a cell containing the origin with that gradient). Fix an arbitrary $i \in \{1, \ldots, k+1\}$ and consider any cell $P \in \mathcal{P}_i$. By Theorem 5.2 with $U = C_i$ and $V = P$, we obtain that $F = F(C_i, V, V) \subseteq E(\pi)$. By Lemma 2.11 (ii), $F \subseteq E(\pi^1)$ and by Lemma 2.11 (iii), π^1 is continuous. By Theorem 4.3 and continuity of π^1, we obtain that π^1 is affine on C_i and P with the same gradient. Since the choice of P was arbitrary, this establishes that for every cell $P \in P_i$, π^1 is affine with the same gradient; this is precisely the desired $\tilde{\mathbf{g}}^i$.

Gradient matching step. The system for step (4) has two sets of constraints, the first of which follows from the condition that $\pi(\mathbf{f} + \mathbf{w}) = 1$ for every $\mathbf{w} \in \mathbb{Z}^k$. The second set of constraints is more involved. Consider two adjacent cells $P, P' \in \mathcal{P}$ that contain a segment $[\mathbf{x}, \mathbf{y}] \subseteq \mathbb{R}^k$ in their intersection. Along the line segment $[\mathbf{x}, \mathbf{y}]$, the gradients of P and P' projected onto the line spanned by the vector $\mathbf{y} - \mathbf{x}$ must agree; the second set of constraints captures this observation. We will identify a set of vectors $\mathbf{r}^1, \ldots, \mathbf{r}^{k+1}$ such that every subset of k vectors is linearly independent and such that each vector $\mathbf{r}^i$ is contained in k cells of $\mathcal{P}$ with different gradients. We then use the segment $[\mathbf{0}, \mathbf{r}^i]$ to obtain linear equations involving the gradients of π and π'. The fact that every subset of k vectors is linearly independent will be crucial in ensuring the uniqueness of the system of equations.

Lemma 5.3 ([21, Lemma 3.10]) *There exist vectors $\mathbf{r}^1, \mathbf{r}^2, \ldots, \mathbf{r}^{k+1} \in \mathbb{R}^k$ with the following properties:*

(i) *For every $i, j, \ell \in \{1, \ldots, k+1\}$ with j, ℓ different from i, the equations $\mathbf{r}^i \cdot \bar{\mathbf{g}}^j = \mathbf{r}^i \cdot \bar{\mathbf{g}}^\ell$ and $\mathbf{r}^i \cdot \tilde{\mathbf{g}}^j = \mathbf{r}^i \cdot \tilde{\mathbf{g}}^\ell$ hold.*
(ii) *Every k-subset of $\{\mathbf{r}^1, \ldots, \mathbf{r}^{k+1}\}$ is linearly independent.*

The proof of Theorem 5.3 uses a nontrivial result known as the *Knaster–Kuratowski–Mazurkiewicz Lemma (KKM Lemma)* from fixed point theory, which exposes a nice structure in the gradient pattern of π. The KKM lemma states that if a d-dimensional simplex is covered by $d+1$ closed sets satisfying certain combinatorial conditions, then there is a point in the intersection of all $d+1$ sets. This lemma is applied to the facets of a certain simplex S containing the origin, where the closed sets form $S \cap \mathcal{P}_i$. The fixed points on the $k+1$ facets of this simplex give the vectors $\mathbf{r}^1, \ldots, \mathbf{r}^{k+1}$ from Theorem 5.3. The bulk of the technicality lies in showing that the hypothesis of the KKM lemma are satisfied by the gradient structure of π. A few more details are offered in Figure 10.

We finally present the system of linear equations that we consider.

Corollary 5.4 ([21, Corollary 3.13]) *Consider any $k+1$ affinely independent vectors $\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^{k+1} \in \mathbb{Z}^k + \mathbf{f}$. Also, let $\mathbf{r}^1, \mathbf{r}^2, \ldots, \mathbf{r}^{k+1}$ be the vectors given by Theorem 5.3. Then there exist $\mu_{ij} \in \mathbb{R}_+$, $i, j \in \{1, \ldots, k+1\}$*

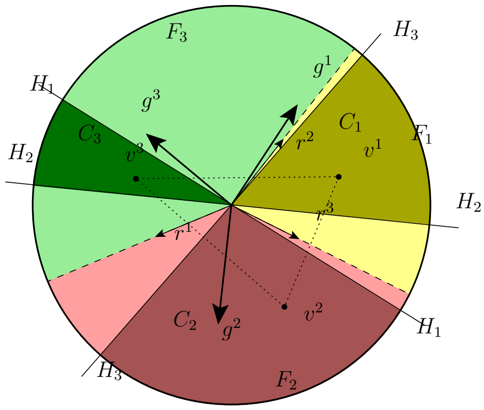

Fig. 10 The geometry of the proof of Theorem 5.3. Each cone C_i (shaded in dark colors) is the intersection of the halfspaces H_j (defined by the gradients $\bar{\mathbf{g}}^j$) for $j \neq i$. Near the origin (within the ball $B(\mathbf{0}, \varepsilon)$), each point of C_i lies in the set F_i of points where the function π has gradient $\bar{\mathbf{g}}^i$ (shaded in light colors). Picking points $\mathbf{v}_i$ near the origin in the interior of C_i, we construct a simplex Δ with $\mathbf{0}$ in its interior. By applying the KKM Lemma to each of its facets Δ_i, we show the existence of the vectors $\mathbf{r}^i$ with the desired properties.

with $\sum_{j=1}^{k+1} \mu_{ij} = 1$ *for all* $i \in \{1, \dots, k+1\}$ *such that both* $\tilde{\mathbf{g}}^1, \dots, \tilde{\mathbf{g}}^{k+1}$ *and* $\bar{\mathbf{g}}^1, \dots, \bar{\mathbf{g}}^{k+1}$ *are solutions to the linear system*

$$\begin{aligned} \textstyle\sum_{j=1}^{k+1} (\mu_{ij} \mathbf{a}^i) \cdot \mathbf{g}^j &= 1 && \textit{for all } i \in \{1, \dots, k+1\}, \\ \mathbf{r}^i \cdot \mathbf{g}^j - \mathbf{r}^i \cdot \mathbf{g}^\ell &= 0 && \textit{for all } i, j, \ell \in \{1, \dots, k+1\} \textit{ such that } i \neq j, \ell, \end{aligned} \tag{5.1}$$

with variables $\mathbf{g}^1, \dots, \mathbf{g}^{k+1} \in \mathbb{R}^k$.

We remark that we can always find vectors $\mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^{k+1} \in \mathbb{Z}^k + \mathbf{f}$ such that the set $\mathbf{a}^1, \dots, \mathbf{a}^{k+1}$ is affinely independent, so the system above indeed exists. Property (ii) in Theorem 5.3 and the fact that $\mathbf{a}^1, \dots, \mathbf{a}^{k+1}$ are affinely independent can be used to show that (5.1) has either no solutions or a unique solution. Since $\bar{\mathbf{g}}^1, \dots, \bar{\mathbf{g}}^{k+1}$ is a solution, the conclusion is that the system has a unique solution and so $\tilde{\mathbf{g}}^j = \bar{\mathbf{g}}^j$ for each $j = 1, \dots, k+1$.

Remark 5.5 Along almost identical lines, one can show that a $(k+1)$-slope function π is a facet – this is done in [21]. The only difference is that the continuity of π^1 in the proof above was obtained easily via Lemma 2.11 (iii). For the facetness proof, this continuity argument is slightly more involved.

5.2 Construction of extreme functions with the sequential-merge procedure

Dey and Richard [38] gave the first examples of facets in higher dimensions by combining facets from lower dimensions. We outline these concepts here. For a more detailed discussion, we refer the reader to [38] and also the survey [65].

Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be valid for $R_{f_{k+1}}(\mathbb{R}, \mathbb{Z})$ and $\pi\colon \mathbb{R}^k \to \mathbb{R}$ valid for $R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$. The *sequential merge* of φ and π is the function $\varphi \diamondsuit \pi\colon \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$ given by

$$(\varphi \diamondsuit \pi)(\mathbf{x}, x_{k+1}) = \frac{\pi(\mathbf{x}) \sum_{i=1}^{k} f_i + f_{k+1}\varphi\Big(\sum_{i=1}^{k+1} x_i - \pi(\mathbf{x}) \sum_{i=1}^{k} f_i\Big)}{\sum_{i=1}^{k+1} f_i}.$$

Here we assume, without loss of generality, that $f_{k+1} \in (0, 1)$, $\mathbf{f} \in [0, 1)^k \setminus \{\mathbf{0}\}$. The *lifting space representation* of a function $\pi\colon \mathbb{R}^k \to \mathbb{R}$ is given by $[\pi]_{\mathbf{f}}(\mathbf{x}) = \sum_{i=1}^{k} x_i - \pi(\mathbf{x}) \sum_{i=1}^{k} f_i$.[4]

Dey and Richard showed that the function $(\varphi \diamondsuit \pi)(\mathbf{x}, x_{k+1})$ is a facet for $R_{(\mathbf{f},f_{k+1})}(\mathbb{R}^{k+1}, \mathbb{Z}^{k+1})$ provided that φ and π are facets, their lifting representations are non-decreasing, and the perturbation spaces[5] $\bar{\Pi}^{E(\varphi)}(\mathbb{R}, \mathbb{Z})$ and $\bar{\Pi}^{E(\pi)}(\mathbb{R}^k, \mathbb{Z}^k)$ both contain only trivial solutions [38, Theorem 5]. They also show how to extend these sequential merge facets to facets of the mixed-integer problem [38, Proposition 15]. This produces a simple method to construct facets in higher dimensions from facets in lower dimensions.

Some sequential merge functions can be projected as well. Let $\xi\colon \mathbb{R} \to \mathbb{R}$ be the `gmic` function and let $\pi\colon \mathbb{R} \to \mathbb{R}$ be a valid function for $R_f(\mathbb{R}, \mathbb{Z})$. For any $n \in \mathbb{Z}_+$ such that $0 < f < 1/n$, we define the *projected sequential merge* function $\pi \diamondsuit_n^1 \xi\colon \mathbb{R} \to \mathbb{R}$ as $(\pi \diamondsuit_n^1 \xi)(x) = (\pi \diamondsuit \xi)(nx, x)$. Provided that π is a facet of $R_f(\mathbb{R}, \mathbb{Z})$ and $[\pi]_f$ is non-decreasing and $\bar{\Pi}^{E(\pi)}(\mathbb{R}, \mathbb{Z})$ has only the trivial solution, we have that $\pi \diamondsuit_n^1 \xi$ is a facet for $R_{nf}(\mathbb{R}, \mathbb{Z})$. See Table 5 for an example of a projected sequential merge inequality, `dr_projected_sequential_merge_3_slope`. Also `dg_2_step_mir` from Table 1 can be seen as the projected sequential merge function $\xi \diamondsuit_n^1 \xi$. We can state this idea in the following more general way. Consider $(\pi_1 \diamondsuit (\pi_2 \diamondsuit \ldots (\pi_{k-1} \diamondsuit \pi_k) \ldots))$, where π_i is a facet for $R_f(\mathbb{R}, \mathbb{Z})$ and $[\pi_i]_f$ is non-decreasing, and $\bar{\Pi}^{E(\pi^i)}(\mathbb{R}, \mathbb{Z})$ has only the trivial solution for $i = 1, \ldots, k$. Let $n \in \mathbb{Z}_+$ such that $0 < f_k < \frac{1}{n}$. Then $(\pi_1 \diamondsuit (\pi_2 \diamondsuit \ldots \diamondsuit (\pi_{k-1} \diamondsuit (\pi_k \diamondsuit_n^1 \xi)) \ldots))$ is a facet for $R_{\mathbf{f}'}(\mathbb{R}^k, \mathbb{Z}^k)$ where $\mathbf{f}' = (f_1, \ldots, f_{k-1}, nf_k)$ [38, Theorem 6].

6 Sequences of minimal valid and extreme functions

6.1 Minimality of limits of minimal valid functions

The most basic topology on the space $\mathbb{R}^G$ of real-valued functions on G is the product topology, or the topology of pointwise convergence. We first note that

[4] This is a superadditive pseudo-periodic function in the terminology of [66].

[5] See subsection 3.6.

the properties in the characterization of minimal valid functions (Theorem 2.6) are preserved under pointwise convergence.

Proposition 6.1 ([39, Proposition 4]) *Let $\pi_i \in \mathbb{R}^G$, $i \in \mathbb{N}$ be a sequence*[6] *of minimal valid functions that converge pointwise to $\pi \in \mathbb{R}^G$. Then π is a minimal valid function.*

Proof Since each π_i is nonnegative, π is nonnegative. We simply verify the conditions in Theorem 2.6 for π.

1. For any $\mathbf{w} \in S$, $\pi(\mathbf{w}) = \lim_{i\to\infty} \pi_i(\mathbf{w}) = \lim_{i\to\infty} 0 = 0$.
2. For any $\mathbf{x}, \mathbf{y} \in G$, $\pi(\mathbf{x}+\mathbf{y}) = \lim_{i\to\infty} \pi_i(\mathbf{x}+\mathbf{y}) \le \lim_{i\to\infty}(\pi_i(\mathbf{x})+\pi_i(\mathbf{y})) = \lim_{i\to\infty} \pi_i(\mathbf{x}) + \lim_{i\to\infty} \pi_i(\mathbf{y}) = \pi(\mathbf{x}) + \pi(\mathbf{y})$.
3. For any $\mathbf{x}$, $\pi(\mathbf{x})+\pi(\mathbf{f}-\mathbf{x}) = \lim_{i\to\infty} \pi_i(\mathbf{x})+\lim_{i\to\infty} \pi_i(\mathbf{f}-\mathbf{x}) = \lim_{i\to\infty}(\pi_i(\mathbf{x})+ \pi_i(\mathbf{f}-\mathbf{x})) = \lim_{i\to\infty} 1 = 1$.

□

This result can be used to prove Proposition 2.7 regarding the compactness of the set of minimal functions.

Proof (Proof of Proposition 2.7) Theorem 2.6 implies that all minimal valid functions π satisfy $0 \le \pi \le 1$. The set of functions in $\mathbb{R}^G$ bounded between 0 and 1 is compact by Tychonoff's theorem. Theorem 6.1 applies to nets of minimal functions also, which is a generalization of sequences; this shows that the set of minimal valid functions is a closed subset of the set of functions in $\mathbb{R}^G$ bounded between 0 and 1. As a closed subset of a compact set, the set of minimal functions is compact. □

6.2 Failure of extremality of limits of extreme functions

While minimality is preserved by limits, this is not true in general for extremality.

Dey and Wolsey [39, section 2.2, Example 2] give an example where a sequence of continuous piecewise linear extreme functions of type `gj_2_slope_repeat` converges pointwise to a discontinuous piecewise linear minimal valid function that is not extreme (Figure 11).[7]

This convergence, of course, is not uniform. One may then ask whether extremality is preserved by stronger notions of convergence. However, even uniform convergence (i.e., convergence in the sense of the space $C(\mathbb{R})$ of continuous functions) or convergence in the sense of the Sobolev space[8] $W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ do not suffice to ensure extremality of the limit function (Figure 12).

[6] The statement of Theorem 6.1 remains true for generalizations of sequential limits; for example, we may consider the convergence of nets of minimal functions.

[7] The sequence and its limit can be constructed using `drlm_gj_2_slope_extreme_limit_to_nonextreme`.

[8] See, for example, [57] for an introduction to Sobolev spaces.

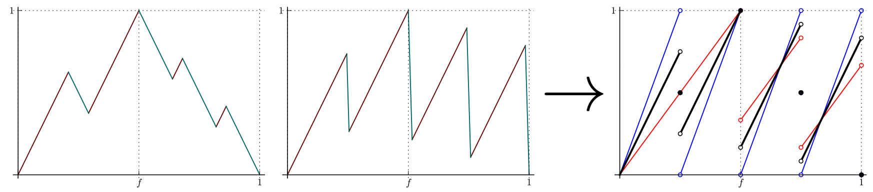

Fig. 11 A pointwise limit of extreme functions that is not extreme [39, section 2.2]. Consider the sequence of continuous extreme functions of type `gj_2_slope_repeat` set up for any $n \in \mathbb{Z}_+$ by `h = drlm_gj_2_slope_extreme_limit_to_nonextreme(n)`. For example, $n = 3$ *(left)* and $n = 50$ *(center)*. This sequence converges to a non-extreme discontinuous minimal valid function, set up with `h = drlm_gj_2_slope_extreme_limit_to_nonextreme()` *(right)*. The limit function π (*black*) is shown with two minimal functions π^1 (*blue*), π^2 (*red*) such that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$.

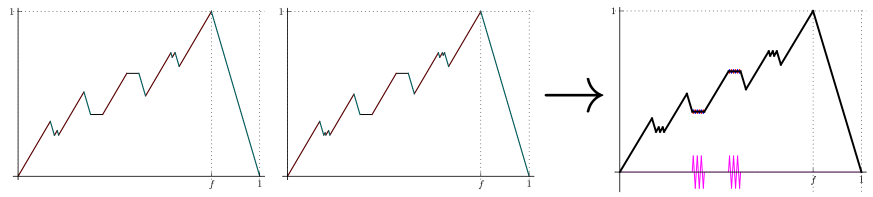

Fig. 12 A uniform limit of extreme functions that is not extreme. The sequence of extreme functions of type `bhk_irrational`, set up with `h = bhk_irrational_extreme_limit_to_rational_nonextreme(n)` where $n = 1$ *(left)*, $n = 2$ *(center)*, ... converges to a non-extreme function, set up with `h = bhk_irrational_extreme_limit_to_rational_nonextreme()` *(right)*. The limit function π (*black*) is shown with two minimal functions π^1 (*blue*), π^2 (*red*) such that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$ and a scaling of the perturbation function $\bar{\pi} = \pi^1 - \pi$ (*magenta*).

Proposition 6.2 (New result ♣) *There exists a sequence of continuous extreme functions of type* `bhk_irrational` *[18, section 5] that converges uniformly to a continuous non-extreme function of the same type. Further, even the sequence of generalized derivatives converges in the sense of* $L^1_{\mathrm{loc}}(\mathbb{R})$*; thus we have convergence in* $W^{1,1}_{\mathrm{loc}}(\mathbb{R})$.

The functions from Theorem 6.2 have the intriguing property that extremality depends, in addition to some inequalities in the parameters, on the $\mathbb{Q}$-linear independence of two real parameters [18, Theorems 5.3 and 5.4].[9] Thus it is easy to construct a sequence of parameters satisfying this condition whose limit is rational, making the limit function non-extreme.[10]

[9] These parameters are collected in the list `delta`, which is an argument to the function `bhk_irrational`. The parameters are $\mathbb{Q}$-linearly independent for example when one parameter is rational, e.g., `1/200`, the other irrational, e.g., `sqrt(2)/200`. When the irrational number is algebraic (for example, when it is constructed using square roots), the code will construct an appropriate real number field that is a field extension of the rationals. In this field, the computations are done in exact arithmetic.

[10] Such a sequence and the limit can be constructed using `bhk_irrational_extreme_limit_to_rational_nonextreme`.

6.3 Discontinuous extreme piecewise linear limit functions

Dey and Wolsey [39] give some general conditions under which the limit is indeed extreme. Recall that a function $\pi \in \mathbb{R}^{\mathbb{R}}$ is called *piecewise linear* (not necessarily continuous) if we can express $\mathbb{R}$ as the union of closed intervals with non-overlapping interiors such that any bounded subset of $\mathbb{R}$ intersects only finitely many intervals, and the function is affine linear over the *interior* of each interval.

Theorem 6.3 ([39, Theorem 7]) *Let $\pi_i \in \mathbb{R}^{\mathbb{R}}$, $i \in \mathbb{N}$ be a sequence of continuous piecewise linear, extreme valid functions for $R_f(\mathbb{R}, \mathbb{Z})$ and let ϕ be the pointwise limit of the sequence π_i, $i \in \mathbb{N}$ such that the following conditions hold:*

(i) ϕ is piecewise linear (not necessarily continuous).
(ii) ϕ has a finite right derivative at 0.[11]
(iii) There is a sequence of integers k_i, $i \in \mathbb{N}$ with $\lim_{i\to\infty} k_i = \infty$ such that for each $i \in \mathbb{N}$,
  (a) $\phi(u) = \pi_i(u)$ for all $u \in \frac{1}{k_i}\mathbb{Z}$ and
  (b) the set of nondifferentiable points of π_i is contained in $\frac{1}{k_i}\mathbb{Z}$.

Then ϕ is extreme.

The authors of [39] use the above theorem to construct families of discontinuous piecewise linear extreme functions for the single-row infinite group problem; see Table 3 for a list. The use of Theorem 6.3 does not seem to be essential, however; the extremality of all of these functions can also be established by following the algorithm of subsection 7.1.

6.4 Non–piecewise linear extreme limit functions

We now describe a construction based on limits of extreme functions that yields an extreme function that is not piecewise linear. The extremality of this limit function cannot be obtained by an application of Theorem 6.3 since the limit function is not piecewise linear.

This construction is motivated by a conjecture of Gomory and Johnson from 2003 that all facets are piecewise linear [49, section 6.1]. If true, this would justify focusing attention on piecewise linear minimal valid functions, for which we have developed many tools for analysis (see section 3). However, even for $k = 1$, this conjecture was disproved by Basu, Conforti, Cornuéjols and Zambelli [15]. We present their counterexample and a brief argument for its extremality.

[11] This can also be done with a finite left derivative. Note that not all extreme functions have a finite left or right derivative at the origin. That is, there exist extreme functions that are discontinuous on both sides of the origin. See Table 4 for examples.

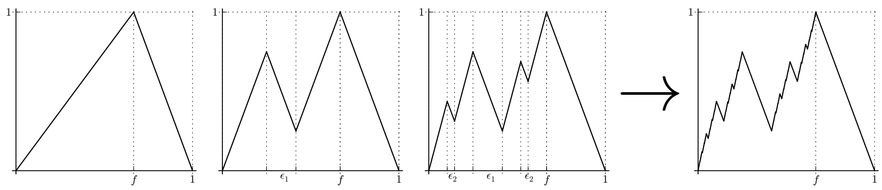

Fig. 13 First steps (ψ_0 = `gmic()`, ψ_1, ψ_2) in the construction of the continuous non–piecewise linear limit function ψ = `bccz_counterexample()`.

Remark 6.4 The arguments for its facetness are almost identical; the only difference is that some technical continuity arguments can be avoided in the proof of extremality because of Lemma 2.11 (iii).

We first define a sequence of valid functions $\psi_i \colon \mathbb{R} \to \mathbb{R}$ that are piecewise linear, and then consider the limit ψ of this sequence, which will be extreme but not piecewise linear.

Let $0 < f < 1$. Consider a geometric sequence of real numbers $\epsilon_1 > \epsilon_2 > \dots$ such that $\epsilon_1 \leq 1 - f$ and

$$\mu^- = (1 - f) + \sum_{i=1}^{+\infty} 2^{i-1}\epsilon_i \leq 1 \tag{6.2}$$

holds.[12] We distinguish two cases: $\mu^- < 1$ [15] and $\mu^- = 1$ [60]. An example for the first case is the sequence $\epsilon_i = (\frac{1}{4})^i f$ for $0 < f \leq \frac{4}{5}$; for the second case, $\epsilon_i = 2(\frac{1}{4})^i f$ for $0 < f \leq \frac{1}{2}$. Let ψ_0 be the `gmic` function with peak at f. We construct ψ_{i+1} from ψ_i by modifying each segment with positive slope in the graph of ψ_i in the manner of the `kf_n_step_mir` construction [59] as follows.[13] For every inclusion-maximal interval $[a, b]$ where ψ_i has constant positive slope we replace the line segment from $(a, \psi_i(a))$ to $(b, \psi_i(b))$ with the following three segments:

- a positive slope segment connecting $\bigl(a, \psi_i(a)\bigr)$ and $\bigl(\frac{(a+b)-\epsilon_{i+1}}{2}, \psi_i(\frac{a+b}{2}) + \frac{\epsilon_{i+1}}{2(1-f)}\bigr)$,
- a negative slope segment connecting $\bigl(\frac{(a+b)-\epsilon_{i+1}}{2}, \psi_i(\frac{a+b}{2}) + \frac{\epsilon_{i+1}}{2(1-f)}\bigr)$ and $\bigl(\frac{(a+b)+\epsilon_{i+1}}{2}, \psi_i(\frac{a+b}{2}) - \frac{\epsilon_{i+1}}{2(1-f)}\bigr)$,
- a positive slope segment connecting $\bigl(\frac{(a+b)+\epsilon_{i+1}}{2}, \psi_i(\frac{a+b}{2}) - \frac{\epsilon_{i+1}}{2(1-f)}\bigr)$ and $\bigl(b, \psi_i(b)\bigr)$.

Figure 13 shows the transformation of ψ_0 to ψ_1 and ψ_1 to ψ_2. Each ψ_i is nonnegative, subadditive and satisfies the symmetry condition [15, Lemma 4.5 and Fact 4.6], and thus is a minimal valid function. By construction, the new

[12] The first n terms of such a sequence of ϵ_i are generated by `e = generate_example_e_for_psi_n(n=`n`)`.

[13] The construction of ψ_n is furnished by `h = psi_n_in_bccz_counterexample_construction(e=e)`, where `e` is the list `[`$\epsilon_1, \dots, \epsilon_n$`]`.

negative slopes match the existing negative slopes, and the new positive slopes of each function have all the same slope. Thus ψ_i is a (continuous piecewise linear) 2-slope function and hence extreme. The functions ψ_i are therefore extreme functions by the Gomory–Johnson 2-Slope Theorem (Theorem 2.13).

The function ψ which we show to be extreme but not piecewise linear is defined as the pointwise limit of this sequence of functions, namely

$$\psi(x) = \lim_{i\to\infty} \psi_i(x). \tag{6.3}$$

This limit is well defined when (6.2) holds.[14] In fact, ψ_i converges uniformly to ψ. Since each ψ_i is continuous, this implies that ψ is also continuous.[15] The limit function has the following intriguing properties:

1. By Theorem 6.1, ψ is minimal.
2. For each integer $i \geq 0$, define X_i^- to be the subset of points of $[0,1]$ on which the function ψ_i is differentiable with a negative slope. From the construction of ψ_i, X_i^- is the union of 2^i open intervals [15, Fact 4.1]. Furthermore, $X_i^- \subseteq X_{i+1}^-$ for every $i \in \mathbb{N}$. The set $X^- \subseteq [0,1]$ defined by $X^- = \bigcup_{i=0}^{\infty} X_i^-$ is thus the set of points over which ψ has negative slope, and it is an open set since it is the union of open intervals. The set X^- is dense in $[0,1]$ [15, Fact 5.4]. Its Lebesgue measure is μ^-.
3. ψ is not piecewise linear. This is because each ψ_i is nonnegative, and therefore so is ψ. If ψ is piecewise linear, by definition of continuous piecewise linear functions from subsection 3.1 there exists $0 < \epsilon$ such that ψ is affine linear on $[0,\epsilon]$. Since X^- is dense, there exists a point from X^- in $(0,\epsilon)$ and so ψ has negative slope on this entire segment. But since $\psi(0) = 0$, this contradicts the fact that $\psi \geq 0$.
4. The complement $[0,1] \setminus X^-$ is a closed set, which does not contain any interval; hence $[0,1] \setminus X^-$ is a nowhere dense set. It has Lebesgue measure $\mu^+ = 1 - \mu^-$. Removing from $[0,1] \setminus X^-$ the countably many breakpoints of the negative-slope intervals, we obtain the set $X^+ = [0,1] \setminus \bigcup_{i=0}^{\infty} \operatorname{cl} X_i^-$, which is still a nowhere dense set of measure μ^+.
5. If $\mu^- < 1$, the set X^+ is of positive measure, and thus a *fat Cantor set*; in this case the derivative of ψ exists for all points in X^+ and equals the limit of the positive slopes of the functions ψ_i. Thus ψ is an absolutely continuous, measurable, non–piecewise linear "2-slope function."
6. On the other hand, if $\mu^- = 1$, the measure of X^+ is zero, and so the derivative of ψ equals the negative slopes of the functions ψ_i Lebesgue–almost everywhere. Thus ψ is a continuous (but not absolutely continuous), measurable, non–piecewise linear "1-slope function." This case is discussed in [60].

[14] The function can be created by `h = bccz_counterexample()`; however, `h(x)` can be exactly evaluated only on the set $\bigcup_{i=0}^{\infty} \operatorname{cl} N_i$ defined below; for other values, the function will return an approximation.

[15] In fact, if $\mu^- < 1$, then ψ is actually Lipschitz continuous and thus absolutely continuous and hence almost everywhere differentiable. The convergence then holds even in the sense of the space $W^{1,1}_{\mathrm{loc}}(\mathbb{R})$.

The proof of extremality of ψ proceeds along the roadmap of subsection 2.3 as follows.

1. Consider any minimal valid functions π^1, π^2 such that $\psi = \frac{\pi^1+\pi^2}{2}$. Since ψ is affine over the segments in $\operatorname{cl} X_i^-$, the additivity properties on these segments are inherited by π^1 using a one-dimensional version of Theorem 5.2 and Lemma 2.11 (ii).
2. One uses the Interval Lemma (Lemma 4.1) on π^1 to obtain that π^1 is affine over X^-, and moreover, since $\pi^1(0) = \psi(0) = 0$ and $\pi^1(f) = \psi(f) = 1$, one recursively establishes that $\pi^1(x) = \psi(x)$ for all $x \in X^-$.
3. Since X^- is dense in $[0,1]$, ψ is continuous and π^1 is continuous by Lemma 2.11 (iii) we obtain that $\pi^1 = \psi$. Therefore, $\pi^1 = \pi^2 = \psi$, establishing that ψ is extreme.

We end this section with a conjecture about limits of minimal functions, whose positive resolution would emphasize the importance of piecewise linear functions.

Conjecture 6.5 ([15, Conjecture 6.1]) Every extreme function (resp. facet) $\pi\colon \mathbb{R}^k \to \mathbb{R}$ is either piecewise linear or the limit of a sequence of piecewise linear extreme functions (resp. facets).

7 Algorithmic characterization of extreme functions

In this section we discuss recent algorithmic results for proving piecewise linear functions are either extreme or not extreme for the infinite group problem $R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$. In [18], the first algorithmic test for extremality was given for the single-row infinite group problem $R_f(\mathbb{R}, \mathbb{Z})$, followed by an extension to two-row infinite group problem $R_{\mathbf{f}}(\mathbb{R}^2, \mathbb{Z}^2)$ in [19]. We summarize these algorithmic ideas here in two lights. We will first discuss a general procedure to test for extremality and then in section 8 discuss specific classes of functions that have relations to finite group problems where extremality can be tested easily using linear algebra.

7.1 General procedure outline

We will outline here a general procedure for testing extremality of a continuous piecewise linear function $\pi\colon \mathbb{R}^k \to \mathbb{R}$ defined on a polyhedral complex $\mathcal{P}$. Similar techniques may apply to testing extremality and even facetness of discontinuous piecewise linear functions as well.

Let $E = E(\pi)$. Recall that π is not extreme if and only if there exists a nontrivial function $\bar\pi$ such that $\pi \pm \bar\pi$ is minimal. From Lemma 2.11 parts (i) and (ii) it follows that π is not extreme if and only if there exists a nontrivial continuous function $\bar\pi \in \bar\Pi^E(\mathbb{R}^k, \mathbb{Z}^k)$ such that $\pi \pm \bar\pi$ is minimal.

Let $\mathcal{T}$ be a triangulation of $\mathbb{R}^k$ that satisfies the hypotheses of Lemma 3.14, i.e., there exists $q \in \mathbb{N}$ such that $\operatorname{vert}(\mathcal{T}) = \frac{1}{q}\mathbb{Z}^k$ and $p_i(\operatorname{vert}(\Delta\mathcal{T})) \subseteq \frac{1}{q}\mathbb{Z}^k$ for

$i = 1, 2, 3$ and $\mathbf{f} \in \frac{1}{q}\mathbb{Z}^k$. The following algorithmic ideas are based on the decomposition in Lemma 3.14 part (2) of perturbations $\bar\pi \in \bar\Pi^E(\mathbb{R}^k, \mathbb{Z}^k)$ into $\bar\pi = \bar\pi_{\mathcal{T}} + \bar\pi_{\mathrm{zero}(\mathcal{T})}$ with $\bar\pi_{\mathcal{T}} \in \bar\Pi^E_{\mathcal{T}}(\mathbb{R}^k, \mathbb{Z}^k)$ and $\bar\pi_{\mathrm{zero}(\mathcal{T})} \in \bar\Pi^E_{\mathrm{zero}(\mathcal{T})}(\mathbb{R}^k, \mathbb{Z}^k)$. Since $\bar\pi$ and $\bar\pi_{\mathcal{T}}$ are continuous, $\bar\pi_{\mathrm{zero}(\mathcal{T})}$ is also continuous.

7.1.1 Finite-dimensional linear algebra for $\bar\Pi^E_{\mathcal{T}}(\mathbb{R}^k, \mathbb{Z}^k)$

We begin by looking for a perturbation function in $\bar\Pi^E_{\mathcal{T}}(\mathbb{R}^k, \mathbb{Z}^k)$. By Lemma 3.14, $\bar\pi \in \bar\Pi^E_{\mathcal{T}}(\mathbb{R}^k, \mathbb{Z}^k)$ if and only if $\bar\pi|_{\frac{1}{q}\mathbb{Z}^k} \in \bar\Pi^{E'}(\frac{1}{q}\mathbb{Z}^k, \mathbb{Z}^k)$ where $E' = E(\pi) \cap \frac{1}{q}\mathbb{Z}^k$. Thus we consider the linear system $\bar\Pi^{E'}(\frac{1}{q}\mathbb{Z}^k, \mathbb{Z}^k)$, which is finite-dimensional if we identify the variables $\bar\pi(\mathbf{x})$ and $\bar\pi(\mathbf{x} + \mathbf{t})$ for all $\mathbf{t} \in \mathbb{Z}^k$. Hence, this is a finite-dimensional linear system and any nontrivial solution can be computed by analyzing the null space of this system. If such a nontrivial solution exists, it can be interpolated to a piecewise linear function $\bar\pi \in \bar\Pi^E_{\mathcal{T}}(\mathbb{R}^k, \mathbb{Z}^k)$ because $\mathcal{T}$ is a triangulation of $\mathbb{R}^k$ that satisfies the hypotheses of Lemma 3.14, and so by Theorem 3.13 π is not extreme. This is demonstrated in Figure 3 for the case of $\mathcal{T} = \mathcal{P} = \mathcal{P}_{\frac{1}{q}\mathbb{Z}}$ where a perturbation is found on the complex $\mathcal{T}$.

Otherwise we have that $\bar\Pi^E_{\mathcal{T}}(\mathbb{R}^k, \mathbb{Z}^k) = \{0\}$. This scenario is depicted in Figure 15 where $\bar\Pi^{E'}_{\mathcal{P}_{\frac{1}{q}\mathbb{Z}}}(\frac{1}{q}\mathbb{Z}, \mathbb{Z}) = \{0\}$ with $E' = E(\pi) \cap \frac{1}{q}\mathbb{Z}^2$.

7.1.2 Projections and additivity for $\bar\Pi^E_{\mathrm{zero}(\mathcal{T})}(\mathbb{R}^k, \mathbb{Z}^k)$

From Lemma 3.14, any $\bar\pi \in \bar\Pi^E_{\mathrm{zero}(\mathcal{T})}(\mathbb{R}^k, \mathbb{Z}^k)$ satisfies $\bar\pi|_{\frac{1}{q}\mathbb{Z}^k} \equiv 0$.

We consider full-dimensional faces $F \in E(\pi, \mathcal{P})$. By Corollary 4.9, these full-dimensional faces imply that any $\bar\pi \in \bar\Pi^E_{\mathrm{zero}(\mathcal{T})}(\mathbb{R}^k, \mathbb{Z}^k)$ is affine on the projections $p_1(F), p_2(F)$, and $p_3(F)$. If a projection $p_i(F)$ contains $k+1$ affinely independent points in $\frac{1}{q}\mathbb{Z}^k$, then we conclude that $\bar\pi|_{p_i(F)} \equiv 0$ on this projection. This is because $\bar\pi|_{\frac{1}{q}\mathbb{Z}^k} \equiv 0$. Therefore, we learn certain polyhedral regions where $\bar\pi$ vanishes and we record these.

In the next step, we consider any faces F of $E(\pi, \mathcal{P})$ such that two of $p_1(F), p_2(F), p_3(F)$ are full-dimensional and one is zero-dimensional. In particular, if one of these full-dimensional projections intersects a region where $\bar\pi$ is zero, then that property is transferred to the other full-dimensional projection. For example, the relations $\pi(\mathbf{x}) + \pi(\mathbf{t}) = \pi(\mathbf{x} + \mathbf{t})$ for all $\mathbf{x} \in I$ corresponds to the face $F = F(I, \{\mathbf{t}\}, I + \{\mathbf{t}\})$ where $p_1(F) = I$, $p_2(F) = \{\mathbf{t}\}$, $p_3(F) = I + \{\mathbf{t}\}$. Hence, if I is full-dimensional in $\mathbb{R}^k$ then $I + \{\mathbf{t}\}$ is full-dimensional in $\mathbb{R}^k$. In this way the function values of $\bar\pi$ in $I + \{\mathbf{t}\}$ are dependent on the function values on I. For example, if we know that $\bar\pi$ is affine over I, then it is also affine over $I + \{\mathbf{t}\}$. This is the key step in this procedure. We continue transferring properties until no new affine properties are discovered.

If the procedure terminates, it may either show that $\bar\pi \equiv 0$, in which case π is extreme. Otherwise, we hope to find a perturbation function $\bar\pi$ that shows that π is not extreme. In fact, in certain cases, we can find a $\bar\pi$ that is piecewise linear on a refinement of $\mathcal{T}$. Showing termination of this procedure is non-trivial and it is an open question under what conditions this procedure is guaranteed to terminate. Subsections 7.2 and 7.3 discuss cases in which the procedure provably terminates.

The above procedure only considers certain faces of $E(\pi, \mathcal{P})$. Other faces of $E(\pi, \mathcal{P})$, as shown in Theorem 4.6, establish other affine properties about $\bar{\pi}$, but not necessarily full-dimensional affine properties. These properties can sometimes combine to create full-dimensional affine properties. This effect is investigated in the forthcoming paper [20] for the case of the two-row problem and general continuous piecewise linear functions over the complex $\mathcal{P}_q$.

7.2 One-row case with rational breakpoints

We will consider the one-dimensional polyhedral complex $\mathcal{P}_B$ for $B = \frac{1}{q}\mathbb{Z} \cap [0, 1)$ as defined in Example 3.2; we will call this complex $\mathcal{P}_{\frac{1}{q}\mathbb{Z}}$. Therefore, we consider piecewise linear functions (possibly discontinuous) with breakpoints in $\frac{1}{q}\mathbb{Z}$.

Theorem 7.1 ([18, Theorem 1.3]) *Consider the following problem.*

> *Given a minimal valid function π for $R_f(\mathbb{R}, \mathbb{Z})$ that is piecewise linear with a set of rational breakpoints with the least common denominator q, decide if π is extreme or not.*

There exists an algorithm for this problem whose running time is bounded by a polynomial in q.

Since the above algorithm is polynomial in the least common denominator q, it is only a pseudo-polynomial time algorithm.

Open question 7.2 *Does there exist a polynomial time algorithm to determine extremality of piecewise linear functions for $R_f(\mathbb{R}, \mathbb{Z})$?*

A more general version of the above algorithm is implemented in [56] for the case of piecewise linear functions, which are allowed to be continuous or discontinuous, and whose data may be algebraic irrational numbers.[16] The implementation will be described in more detail in a forthcoming article.

7.3 Two-row case using a standard triangulation of $\mathbb{R}^2$

For the case of the standard triangulations $\mathcal{P}_q$ of $\mathbb{R}^2$ (Example 3.3), [17, 19] describe an algorithm of the above scheme for a special class of piecewise linear functions over this complex, which are said to be *diagonally constrained*.

[16] If `h` is the function π, e.g., after typing `h = dg_2_step_mir()`, then the algorithm is invoked by typing `extremality_test(h, show_plots=True)`. In the irrational case no proof of finite convergence of the procedure is known.

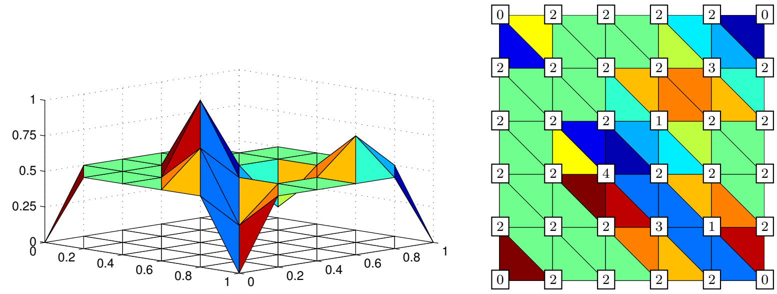

Fig. 14 A minimal valid, continuous, piecewise linear function over the polyhedral complex $\mathcal{P}_5$, which is diagonally constrained (subsection 7.3). *Left*, the three-dimensional plot of the function on $D = [0,1]^2$. *Right*, the complex $\mathcal{P}_5$, restricted to D and colored according to slopes to match the 3-dimensional plot, and decorated with values v at each vertex of $\mathcal{P}_5$ where the function takes value $\frac{v}{4}$.

Let

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}^T.$$

Then for every face $I \in \mathcal{P}_q$, there exists a vector $\mathbf{b} \in \frac{1}{q}\mathbb{Z}^6$ such that $I = \{\, \mathbf{x} \mid A\mathbf{x} \le \mathbf{b} \,\}$. Furthermore, for every vector $\mathbf{b} \in \frac{1}{q}\mathbb{Z}^6$, the set $\{\, \mathbf{x} \mid A\mathbf{x} \le \mathbf{b} \,\}$ is a union of faces of $\mathcal{P}_q$ (possibly empty), since each inequality corresponds to a hyperplane in the arrangement $\mathcal{H}_q$. The matrix A is totally unimodular and this fact plays a key role in proving the following lemma.

Lemma 7.3 *Let $F \in \Delta\mathcal{P}_q$. Then the projections $p_1(F)$, $p_2(F)$, and $p_3(F)$ are faces in the complex $\mathcal{P}_q$. In particular, let $(\mathbf{x}, \mathbf{y})$ be a vertex of $\Delta\mathcal{P}_q$. Then $\mathbf{x}, \mathbf{y}$ are vertices of the complex $\mathcal{P}_q$, i.e., $\mathbf{x}, \mathbf{y} \in \frac{1}{q}\mathbb{Z}^2$.*

Extremality is more easily studied if we restrict ourselves to a setting determined by the types of faces $F \in E_{\max}(\pi, \mathcal{P}_q)$. Recall that

$$E_{\max}(\pi, \mathcal{P}_q) = \{\, F \in E(\pi, \mathcal{P}_q) \mid F \text{ is a maximal face by set inclusion in } E(\pi, \mathcal{P}_q) \,\}.$$

Definition 7.4 A continuous piecewise linear function π on $\mathcal{P}_q$ is called *diagonally constrained* if for all $F \in E_{\max}(\pi, \mathcal{P}_q)$ and $i = 1, 2, 3$, the projection $p_i(F)$ is either a vertex, diagonal edge, or triangle from the complex $\mathcal{P}_q$.

The properties in Lemma 7.3 provide an easy method to compute $E(\pi, \mathcal{P}_q)$ and test if a function is *diagonally constrained* by using simple arithmetic and set membership operations on vertices of $\mathcal{P}_q$.

Example 7.5 Figure 14 shows the complex $\mathcal{P}_5$ with an example of a minimal valid continuous piecewise linear function on $\mathcal{P}_5$ with $\mathbf{f} = \binom{2/5}{2/5}$ that is periodic modulo $\mathbb{Z}^2$. Note that, due the periodicity of the function modulo $\mathbb{Z}^2$, the values of the function on the left and the right edge (and likewise on the bottom and the top edge) of $D = [0,1]^2$ match.

It can be checked that no relations appearing in the list of all maximal additive faces involve a vertical or horizontal edge; thus, the function is diagonally constrained. See [19, sections 4.1 and 4.2].

Theorem 7.6 ([19, Theorem 1.8]) *Consider the following problem.*

> *Given a minimal valid function π for $R_{\mathbf{f}}(\mathbb{R}^2, \mathbb{Z}^2)$ that is piecewise linear continuous on $\mathcal{P}_q$ and diagonally constrained with $\mathbf{f} \in \operatorname{vert}(\mathcal{P}_q)$, decide if π is extreme.*

There exists an algorithm for this problem whose running time is bounded by a polynomial in q.

As before, this algorithm is only a pseudo-polynomial time algorithm.

Open question 7.7 *For any fixed k, does there exist a polynomial time algorithm to determine extremality of piecewise linear functions for $R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$?*

Unlike in the one-row problem, even with all rational input, no algorithm is known for determining extremality of piecewise linear functions for $R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$ for $k \geq 3$ and, as mentioned in Theorem 7.6, only for certain cases is an algorithm known for $k = 2$.

8 Algorithm using restriction to finite group problems

In this section, we discuss connections between infinite group problems and finite group problems. We begin with a discussion of testing extremality for finite group problems. Later we show that in certain settings, a function is extreme for an infinite group problem if and only if its restriction to a finite group is extreme for the finite group problem. Hence, this connection provides an alternative algorithm from those described in section 7 for testing extremality and facetness.

8.1 Algorithm for finite group problem

When S has finite index in G, we call $R_{\mathbf{f}}(G, S)$ a *finite group problem.* As we noted in Remark 2.1, $R_{\mathbf{f}}(G, S)$ and $R_{\mathbf{f}}(G/S, 0)$ are closely related by aggregation of variables, and it is convenient to study the finite-dimensional problem $R_{\bar{\mathbf{f}}}(G/S, 0)$. The fundamental theorem of finitely generated abelian groups shows that $G/S \cong (\frac{1}{q_1}\mathbb{Z} \times \cdots \times \frac{1}{q_k}\mathbb{Z})/\mathbb{Z}^k$ for some $q_i \in \mathbb{N}$ for $i = 1, \dots, k$. Therefore, it suffices to consider $G = \frac{1}{q_1}\mathbb{Z} \times \cdots \times \frac{1}{q_k}\mathbb{Z}$ and $S = \mathbb{Z}^k$ where $q_i \in \mathbb{N}$. In the case of one row, $G/S = \frac{1}{q_1}\mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}/q_1\mathbb{Z}$ is a cyclic group. Cyclic group problems were originally studied by Gomory [45] and have been the subject of many later studies. See [65] for an excellent survey on these results.

The set of minimal valid functions $\pi\colon G/S \to \mathbb{R}$ is a (finite-dimensional) convex polytope [45]. Extreme functions are thus extreme points of this polytope. As we noted in subsubsection 2.2.4, standard polyhedral theory reveals that extreme functions are equivalent to weak facets and facets. Furthermore, extreme points of polytopes are characterized by points where the tight inequalities are of full rank. Therefore, testing extremality of a function for a finite group problem can be done with simple linear algebra.

Note that there is a bijection between the minimal valid functions of $R_{\mathbf{f}}(G,S)$ and minimal valid functions for $R_{\bar{\mathbf{f}}}(G/S,0)$. This is because minimal valid functions for $R_{\mathbf{f}}(G,S)$ are S-periodic functions by Theorem 2.6. Hence the extremality test translates into the following statement about $\bar{\Pi}^{E(\pi)}(G,\mathbb{Z}^k)$.

Theorem 8.1 *Let $G = \frac{1}{q_1}\mathbb{Z} \times \cdots \times \frac{1}{q_k}\mathbb{Z}$ and let $\mathbf{f} \in G$. Let $\pi\colon G \to \mathbb{R}$ be a minimal valid function for $R_{\mathbf{f}}(G,\mathbb{Z}^k)$. Then π is extreme if and only if $\bar{\Pi}^E(G,\mathbb{Z}^k) = \{0\}$ where $E = E(\pi)$.*

For any discrete group $G \supseteq \mathbb{Z}^k$ and subgroup G', the set $R_{\mathbf{f}}(G'/\mathbb{Z}^k,0)$ is a face of the polyhedron $R_{\mathbf{f}}(G/\mathbb{Z}^k,0)$. This observation implies the following theorem via the above bijection.

Theorem 8.2 *Let $G = \frac{1}{q_1}\mathbb{Z} \times \cdots \times \frac{1}{q_k}\mathbb{Z}$, let G' be any subgroup of G, and let $\mathbf{f} \in G'$. Let $\pi\colon G \to \mathbb{R}$.*

1. *If π is minimal for $R_{\mathbf{f}}(G,\mathbb{Z}^k)$, then π is minimal for $R_{\mathbf{f}}(G',\mathbb{Z}^k)$.*
2. *If π is extreme for $R_{\mathbf{f}}(G,\mathbb{Z}^k)$, then π is extreme for $R_{\mathbf{f}}(G',\mathbb{Z}^k)$.*

8.2 Restriction and interpolation in the one-row problem

Gomory and Johnson devised the infinite group problem as a way to study the finite group problem. They studied interpolations of valid functions of the finite group problems $R_f(\frac{1}{q}\mathbb{Z},\mathbb{Z})$ in order to connect the problems, but they never completed this program. Due to the ease of testing extremality in the finite group problems, having this connection is useful for algorithms. We encapsulate their results on this connection in the following theorem.

Theorem 8.3 ([47]) *Let π be a continuous piecewise linear function with breakpoints in $\frac{1}{q}\mathbb{Z}$ for some $q \in \mathbb{Z}_+$ and let $f \in \frac{1}{q}\mathbb{Z}$.[17] Then the following hold:*

1. *π is minimal for $R_f(\mathbb{R},\mathbb{Z})$ if and only if $\pi_{\frac{1}{q}\mathbb{Z}}$ is minimal for $R_f(\frac{1}{q}\mathbb{Z},\mathbb{Z})$.*
2. *If π is extreme for $R_f(\mathbb{R},\mathbb{Z})$, then $\pi|_{\frac{1}{q}\mathbb{Z}}$ is extreme for $R_f(\frac{1}{q}\mathbb{Z},\mathbb{Z})$.*

Part (1) shows that minimality can be tested on just points in $\frac{1}{q}\mathbb{Z}$, while part (2) yields a method of proving a function is not extreme. That is, if $\pi|_{\frac{1}{q}\mathbb{Z}}$ is not extreme for $R_f(\frac{1}{q}\mathbb{Z},\mathbb{Z})$, then π is not extreme for $R_f(\mathbb{R},\mathbb{Z})$. However, it is

[17] Under these hypotheses, π is the continuous interpolation of $\pi|_{\frac{1}{q}\mathbb{Z}}$.

not true in general that if $\pi|_{\frac{1}{q}\mathbb{Z}}$ is extreme for $R_f(\frac{1}{q}\mathbb{Z},\mathbb{Z})$, then π is extreme for $R_f(\mathbb{R},\mathbb{Z})$. See Figure 15 for an example. To obtain such a characterization, it turns out that we must restrict to a finer grid. The first result in this direction of relating the infinite and the finite group problems appeared in [39]; we state it in our notation.

Theorem 8.4 ([39, Theorem 6]) *Let π be a piecewise linear minimal valid function for $R_f(\mathbb{R},\mathbb{Z})$ with set B of rational breakpoints with the least common denominator q. Then π is extreme if and only if the restriction $\pi|_{\frac{1}{2^n q}\mathbb{Z}}$ is extreme for $R_f(\frac{1}{2^n q}\mathbb{Z},\mathbb{Z})$ for all $n \in \mathbb{N}$.*

The above condition cannot be checked in a finite number of steps and hence cannot be converted into a computational algorithm, because it potentially needs to test infinitely many finite group problems. In fact, this result holds even when just considering $n = 2$.

Theorem 8.5 ([18, Theorem 1.5]) *If the function π is continuous, then π is extreme for $R_f(\mathbb{R},\mathbb{Z})$ if and only if the restriction $\pi\big|_{\frac{1}{4q}\mathbb{Z}}$ is extreme for the finite group problem $R_f(\frac{1}{4q}\mathbb{Z},\mathbb{Z})$.*

This result demonstrates a tight connection between finite and infinite group problems, and in particular, yields an alternative algorithm to Theorem 7.1 for testing extremality. That is, to test extremality of π, simply test if $\pi|_{\frac{1}{4q}\mathbb{Z}}$ is extreme for $R_f(\frac{1}{4q}\mathbb{Z},\mathbb{Z})$ using linear algebra, as discussed in subsection 8.1. To prove Theorem 8.5, the authors construct certain perturbations functions that are piecewise linear with breakpoints in $1/4q$. In fact, this result can be improved by a different choice of perturbation function, to have the piecewise linear function have breakpoints in $1/3q$, or $1/mq$ for any fixed $m \in \mathbb{Z}_{\geq 3}$. This observation yields the following result for which we provide a proof.

Theorem 8.6 (New result ♣) *Let $m \in \mathbb{Z}_{\geq 3}$. Let π be a continuous piecewise linear minimal valid function for $R_f(\mathbb{R},\mathbb{Z})$ with breakpoints in $\frac{1}{q}\mathbb{Z}$ and suppose $f \in \frac{1}{q}\mathbb{Z}$. The following are equivalent:*

1. *π is a facet for $R_f(\mathbb{R},\mathbb{Z})$,*
2. *π is extreme for $R_f(\mathbb{R},\mathbb{Z})$,*
3. *$\pi|_{\frac{1}{mq}\mathbb{Z}}$ is extreme for $R_f(\frac{1}{mq}\mathbb{Z},\mathbb{Z})$.*

Proof As mentioned in subsection 2.2.4, facets are extreme functions [21, Lemma 1.3], and hence $1 \Rightarrow 2$. By Theorem 8.3, $2 \Rightarrow 3$. We now show $3 \Rightarrow 1$.

Set $E = E(\pi)$. Let $\pi|_{\frac{1}{mq}\mathbb{Z}}$ be extreme for $R_f(\frac{1}{mq}\mathbb{Z},\mathbb{Z})$ and suppose, for the sake of deriving a contradiction, that π is not a facet for $R_f(\mathbb{R},\mathbb{Z})$. Then, by the Facet Theorem (Theorem 2.12), $\bar{\Pi}^E(\mathbb{R},\mathbb{Z})$ contains a nontrivial element (see subsection 3.6). Since $\pi|_{\frac{1}{mq}\mathbb{Z}}$ is extreme for $R_f(\frac{1}{mq}\mathbb{Z},\mathbb{Z})$, by Theorem 8.1, $\bar{\Pi}^{E'}(\frac{1}{mq}\mathbb{Z},\mathbb{Z}) = \{0\}$ for $E' = E \cap \frac{1}{mq}\mathbb{Z}^2$. By Lemma 3.14 part 1 with $\mathcal{T} =$

$\mathcal{P}_{\frac{1}{mq}\mathbb{Z}}$, we have that $\bar{\Pi}_{\mathcal{T}}(\mathbb{R},\mathbb{Z}) = \{0\}$. Therefore $\bar{\Pi}^E(\mathbb{R},\mathbb{Z}) = \bar{\Pi}^E_{\mathrm{zero}(\mathcal{T})}(\mathbb{R},\mathbb{Z})$. Furthermore, Lemma 3.14 part 2 shows that

$$\bar{\Pi}^E(\mathbb{R},\mathbb{Z}) = \bar{\Pi}^E(\mathbb{R},\mathbb{Z}) \cap \{\bar{\pi} \mid \bar{\pi}|_{\frac{1}{mq}\mathbb{Z}} \equiv 0\}. \tag{8.4}$$

We divide $E(\pi)$ by the faces of $\Delta\mathcal{P}_{\frac{1}{q}\mathbb{Z}}$ using Lemma 3.12. For $i = 1, 2, 3$, define

$$E_i := \bigcup\{F \in E(\pi, \mathcal{P}_{\frac{1}{q}\mathbb{Z}}) \mid \dim(F) = i\}.$$

So $E = E_0 \cup E_1 \cup E_2$.

Step 1. Remove E_0: We claim that $\bar{\Pi}^E(\mathbb{R},\mathbb{Z}) = \bar{\Pi}^{E_1\cup E_2}(\mathbb{R},\mathbb{Z}) \cap \{\bar{\pi} \mid \bar{\pi}|_{\frac{1}{q}\mathbb{Z}} \equiv 0\}$.

First, for any $\bar{\pi} \in \bar{\Pi}^{E_1\cup E_2}(\mathbb{R},\mathbb{Z}) \cap \{\bar{\pi} \mid \bar{\pi}|_{\frac{1}{q}\mathbb{Z}} \equiv 0\}$, we have that $\frac{1}{q}\mathbb{Z}^2 \subseteq E(\bar{\pi})$. Furthermore, since $\mathrm{vert}(\Delta\mathcal{P}_{\frac{1}{q}}) = \frac{1}{q}\mathbb{Z}^2$, we have that $E_0 \subseteq \frac{1}{q}\mathbb{Z}^2$. Therefore, $E_0 \subseteq E(\bar{\pi})$. Hence $\bar{\pi} \in \bar{\Pi}^{E_0\cup E_1\cup E_2}(\mathbb{R},\mathbb{Z}) = \bar{\Pi}^E(\mathbb{R},\mathbb{Z})$.

On the other hand, for any $\bar{\pi} \in \bar{\Pi}^E(\mathbb{R},\mathbb{Z})$, trivially $\bar{\pi} \in \bar{\Pi}^{E_1\cup E_2}(\mathbb{R},\mathbb{Z})$. From (8.4), we see that $\bar{\pi} \in \{\bar{\pi} \mid \bar{\pi}|_{\frac{1}{q}\mathbb{Z}} \equiv 0\}$.

Step 2. Remove E_2: Define $X := \bigcup\{p_i(E_2) \colon i = 1, 2, 3\}$. The set X is called the "covered intervals" in [18]. We claim that $\bar{\Pi}^{E_1\cup E_2}(\mathbb{R},\mathbb{Z}) \cap \{\bar{\pi} \mid \bar{\pi}|_{\frac{1}{q}\mathbb{Z}} \equiv 0\} = \bar{\Pi}^{E_1}(\mathbb{R},\mathbb{Z}) \cap \{\bar{\pi} \mid \bar{\pi}|_{\frac{1}{q}\mathbb{Z}\cup X} \equiv 0\}$.

For any $\bar{\pi} \in \bar{\Pi}^{E_1}(\mathbb{R},\mathbb{Z}) \cap \{\bar{\pi} \mid \bar{\pi}|_{\frac{1}{q}\mathbb{Z}\cup X} \equiv 0\}$, we see that $E_2 \subseteq E(\bar{\pi})$ since $\bar{\pi}|_X \equiv 0$. Therefore $\bar{\pi} \in \bar{\Pi}^{E_1\cup E_2}(\mathbb{R},\mathbb{Z}) \cap \{\bar{\pi} \mid \bar{\pi}|_{\frac{1}{q}\mathbb{Z}} \equiv 0\}$.

On the other hand, let $\bar{\pi} \in \bar{\Pi}^{E_1\cup E_2}(\mathbb{R},\mathbb{Z}) \cap \{\bar{\pi} \mid \bar{\pi}|_{\frac{1}{q}\mathbb{Z}} \equiv 0\}$. By Step 1 and (8.4), $\bar{\pi}|_{\frac{1}{mq}\mathbb{Z}} \equiv 0$. For any $F \in E(\pi, \mathcal{P}_{\frac{1}{q}\mathbb{Z}})$ with $\dim(F) = 2$, by Theorem 4.8 the function $\bar{\pi}$ is affine on the projections $\mathrm{int}(p_i(F))$ for $i = 1, 2, 3$. The projections $p_i(F)$ are full intervals in the complex $\mathcal{P}_{\frac{1}{q}\mathbb{Z}}$ (see Figure 7). In particular, their endpoints lie in $\frac{1}{q}\mathbb{Z}$. Thus, $\mathrm{int}(p_i(F)) \cap \frac{1}{mq}\mathbb{Z}$ contains at least two points since $m \geq 3$. Since $\bar{\pi}|_{\frac{1}{mq}\mathbb{Z}} \equiv 0$ and $\bar{\pi}$ is affine on $\mathrm{int}(p_i(F))$, it follows that $\bar{\pi}|_{\mathrm{int}(p_i(F))} \equiv 0$. Furthermore, since the endpoints of $p_i(F)$ are in $\frac{1}{q}\mathbb{Z}$, we also have that $\bar{\pi}|_{p_i(F)} \equiv 0$. Finally, since E_2 is the union of all $F \in E(\pi, \mathcal{P}_{\frac{1}{q}\mathbb{Z}})$ with $\dim(F) = 2$, it follows that $\bar{\pi}|_X \equiv 0$, and hence $\bar{\pi} \in \bar{\Pi}^{E_1}(\mathbb{R},\mathbb{Z}) \cap \{\bar{\pi} \mid \bar{\pi}|_{\frac{1}{q}\mathbb{Z}\cup X} \equiv 0\}$.

Step 3. Write down E_1 relations: The additivity set E_1 corresponds to one-dimensional faces in $\Delta\mathcal{P}_{\frac{1}{q}\mathbb{Z}}$. These faces represent one the following two relations:

$$\begin{aligned} \pi(x) + \quad \pi(t) &= \pi(x+t) && \text{for all } x \in I,\\ \pi(x) + \pi(r-x) &= \pi(r) && \text{for all } x \in I, \end{aligned}$$

for some $I \in \mathcal{P}_q$ and $r, t \in \frac{1}{q}\mathbb{Z}$. Since $\bar{\pi}|_{\frac{1}{mq}} \equiv 0$, we have $\bar{\pi}(t) = 0$ and $\bar{\pi}(r) = 0$. Considering this, we can find sets $R_I, T_I \subseteq \frac{1}{q}\mathbb{Z}$ for every interval $I \in \mathcal{P}_{\frac{1}{q}\mathbb{Z},\mapsto}$

(see Example 3.2 for notation $\mathcal{P}_{\frac{1}{q}\mathbb{Z},\mapsto}$) such that

$$\bar{\Pi}^E(\mathbb{R},\mathbb{Z}) = \left\{ \bar{\pi}\colon \mathbb{R}\to\mathbb{R} \;\middle|\; \begin{array}{lr} \bar{\pi}(x) = 0 & \text{for all } x \in X \cup \frac{1}{q}\mathbb{Z} \\ \bar{\pi}(x) = \bar{\pi}(x+t) & \text{for all } x \in I, t \in T_I, I \in \mathcal{P}_{\frac{1}{q}\mathbb{Z},\mapsto} \\ \bar{\pi}(x) = -\bar{\pi}(r-x) & \text{for all } x \in I, r \in R_I, I \in \mathcal{P}_{\frac{1}{q}\mathbb{Z},\mapsto} \end{array} \right\}. \tag{8.5}$$

Note that taking $T_I \supseteq \mathbb{Z}$ for all $I \in \mathcal{P}_{\frac{1}{q}\mathbb{Z},\mapsto}$ covers the periodicity conditions.

Step 4. Derive contradiction: We define the orbit $\mathcal{O}(x) = (\{x\}\cup\{-x\}) + \frac{1}{q}\mathbb{Z}$. Thus, for any interval $I \in \mathcal{P}_{\frac{1}{q}\mathbb{Z},\mapsto}$ and $x \in I$, we have $x+t, r-x \in \mathcal{O}(x)$ for all $t \in T_I, r \in R_I$. Notice that $\mathcal{O}([0,\frac{1}{2q}]) := \bigcup_{x\in[0,\frac{1}{2q}]} \mathcal{O}(x) = \mathbb{R}$.

Let $\bar{\pi} \in \bar{\Pi}^E(\mathbb{R},\mathbb{Z})$ such that $\bar{\pi} \not\equiv 0$. By (8.4), $\bar{\pi}|_{\frac{1}{mq}\mathbb{Z}} \equiv 0$. Since $\bar{\pi} \not\equiv 0$ and $\mathcal{O}([0,\frac{1}{2q}]) = \mathbb{R}$, there exists an $x_0 \in [0,\frac{1}{2q}] \setminus \frac{1}{mq}\mathbb{Z}$ such that $\bar{\pi}|_{\mathcal{O}(x_0)} \not\equiv 0$. Define $\bar{\pi}_{x_0}\colon \mathbb{R}\to\mathbb{R}$ as

$$\bar{\pi}_{x_0}(x) = \begin{cases} \bar{\pi}(x) & \text{if } x \in \mathcal{O}(x_0), \\ 0 & \text{otherwise.} \end{cases}$$

The key idea here that we need to use is that in (8.5) the value of $\bar{\pi}$ at x is related only to the value at points in $\mathcal{O}(x)$. From that, it follows from (8.5) that $\bar{\pi}_{x_0} \in \bar{\Pi}^E(\mathbb{R},\mathbb{Z})$. We next will transform $\bar{\pi}_{x_0}$. By definition of $x \in \mathcal{O}(x_0)$ we have $x = x_0 + t$ for some $t \in \frac{1}{q}\mathbb{Z}$ or $x = -x_0 + r$ for some $r \in \frac{1}{q}\mathbb{Z}$. If $x_0 \in \frac{1}{2q}\mathbb{Z}$, both decompositions are possible, but otherwise, only one such decomposition is possible.

We now consider the orbit $\mathcal{O}(\frac{1}{mq}) = \{\frac{1}{mq}, -\frac{1}{mq}\} + \frac{1}{q}\mathbb{Z}$ and define $\varphi\colon \mathcal{O}(\frac{1}{mq}) \to \mathbb{R}$ as

$$\varphi(\tfrac{i}{mq} + t) = \begin{cases} \bar{\pi}_{x_0}(x_0 + t) & \text{if } i = 1, \\ \bar{\pi}_{x_0}(-x_0 + t) & \text{if } i = -1, \end{cases}$$

for all $t \in \frac{1}{q}\mathbb{Z}$. The description of φ transfers values of $\bar{\pi}$ in $\mathcal{O}(x_0)$ to values in $\mathcal{O}(\frac{1}{mq})$. Since $\bar{\pi}|_{\mathcal{O}(x_0)} \not\equiv 0$, we also have that $\varphi \not\equiv 0$. Finally, define $\bar{\varphi}\colon \mathbb{R}\to\mathbb{R}$ as

$$\bar{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x \in \mathcal{O}(\frac{1}{mq}), \\ 0 & \text{otherwise.} \end{cases}$$

Then, using the representation in (8.5), the fact that $\bar{\pi}_{x_0} \in \bar{\Pi}^E(\mathbb{R},\mathbb{Z})$ implies that $\bar{\varphi} \in \bar{\Pi}^E(\mathbb{R},\mathbb{Z})$. But notice that $\bar{\varphi}|_{\frac{1}{mq}\mathbb{Z}} \not\equiv 0$ since $\varphi \not\equiv 0$ and $\mathcal{O}(\frac{1}{mq}) \subseteq \frac{1}{mq}\mathbb{Z}$, which contradicts (8.4). Therefore, we conclude that $3 \Rightarrow 1$. $\square$

Figure 15 gives an example of a function π that is not extreme for $R_f(\mathbb{R},\mathbb{Z})$, but $\pi|_{\frac{1}{q}\mathbb{Z}}$ is extreme for $R_f(\frac{1}{q}\mathbb{Z},\mathbb{Z})$.

Using computer-based search, Köppe and Zhou [61] found a function that is not extreme for $R_f(\mathbb{R},\mathbb{Z})$, but whose restriction to $\frac{1}{2q}\mathbb{Z}$ is extreme for $R_f(\frac{1}{2q}\mathbb{Z},\mathbb{Z})$.[18] This proves the following result.

[18] The function is available in the electronic compendium [70] as `kzh_2q_example_1`.

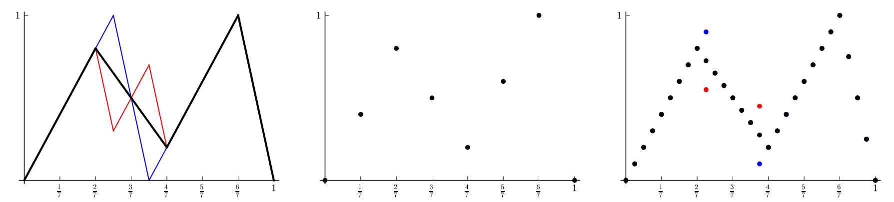

Fig. 15 This function (`h = drlm_not_extreme_1()`) is minimal, but not extreme (and hence also not a facet), as proved by `extremality_test(h, show_plots=True)` by demonstrating a perturbation. The red and blue perturbations describe the minimal functions π^1, π^2 that verify that π is not extreme. These minimal functions necessarily have more breakpoints than π. This is because $\pi|_{\frac{1}{q}\mathbb{Z}}$ with $q = 7$, as depicted in the middle figure, is extreme for the finite group problem $R_f(\frac{1}{q}\mathbb{Z}, \mathbb{Z})$. However, $\pi|_{\frac{1}{2q}\mathbb{Z}}$ is not extreme for $R_f(\frac{1}{2q}\mathbb{Z}, \mathbb{Z})$. The discrete perturbations, depicted on the right, are interpolated to obtain the continuous functions π^1, π^2.

Proposition 8.7 (Köppe and Zhou [61]) *The hypothesis $m \geq 3$ in Theorem 8.6 is best possible. The theorem does not hold for $m = 2$.*

8.3 Restriction and interpolation for $k \geq 2$

Some similar restriction results can be proved for the case of k rows, but this area is much more open. Restrictions seem to require the use of nice polyhedral complexes. The only results known are for the polyhedral complex $\mathcal{P}_q$ (Example 3.3) in $\mathbb{R}^2$.

Theorem 8.8 ([19, Theorem 4.5 and Theorem 5.16]) *Let $\pi \colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous piecewise linear function over $\mathcal{P}_q$ and suppose $\mathbf{f} \in \frac{1}{q}\mathbb{Z}^2$. Then the following hold:*

1. *π is minimal for $R_{\mathbf{f}}(\mathbb{R}^2, \mathbb{Z}^2)$ if and only if $\pi_{\frac{1}{q}\mathbb{Z}^2}$ is minimal for $R_{\mathbf{f}}(\frac{1}{q}\mathbb{Z}^2, \mathbb{Z}^2)$.*
2. *If π is extreme for $R_{\mathbf{f}}(\mathbb{R}^2, \mathbb{Z}^2)$, then $\pi|_{\frac{1}{q}\mathbb{Z}^2}$ is extreme for $R_{\mathbf{f}}(\frac{1}{q}\mathbb{Z}^2, \mathbb{Z}^2)$.*

For $k \geq 3$ rows, it is unclear when similar results are possible.

Open question 8.9 *Can Theorem 8.8 be generalized to other triangulations of $\mathbb{R}^k$ for $k \geq 2$?*

In the special case of diagonally constrained functions in $\mathbb{R}^2$, there is a similar result to Theorem 8.6.

Theorem 8.10 ([19, Theorem 1.9]) *Let π be a minimal continuous piecewise linear function over $\mathcal{P}_q$ that is diagonally constrained and $\mathbf{f} \in \operatorname{vert}(\mathcal{P}_q)$. Fix $m \in \mathbb{Z}_{\geq 3}$. Then π is extreme for $R_{\mathbf{f}}(\mathbb{R}^2, \mathbb{Z}^2)$ if and only if the restriction $\pi\big|_{\frac{1}{mq}\mathbb{Z}^2}$ is extreme for $R_{\mathbf{f}}(\frac{1}{mq}\mathbb{Z}^2, \mathbb{Z}^2)$.*

If we know that π is diagonally constrained, then this theorem produces an alternative algorithm to Theorem 7.6 to test extremality of π by simply restricting to $\frac{1}{3q}\mathbb{Z}^2$ and testing extremality in the finite dimensional setting. A generalization of this theorem that removes the condition of being diagonally constrained will appear in a forthcoming article [20].

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The bibliography appears in Part I of this survey.