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Light on the Infinite Group Relaxation

I. Foundations and Taxonomy

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Abstract This is a survey on the infinite group problem, an infinite-dimensional relaxation of integer linear optimization problems introduced by Ralph Gomory and Ellis Johnson in their groundbreaking papers titled *Some continuous functions related to corner polyhedra I, II* [Math. Programming 3 (1972), 23–85, 359–389]. The survey presents the infinite group problem in the modern context of cut generating functions. It focuses on the recent developments, such as algorithms for testing extremality and breakthroughs for the k-row problem for general $k \geq 1$ that extend previous work on the single-row and two-row problems. The survey also includes some previously unpublished results; among other things, it unveils piecewise linear extreme functions with more than four different slopes. An interactive companion program, implemented in the open-source computer algebra package Sage, provides an updated compendium of known extreme functions.

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1 Introduction

The importance and utility of mixed-integer optimization algorithms and software are widely acknowledged. The mathematical theory of cutting planes is one of the workhorses of state-of-the-art software for mixed-integer optimization. The historical development of cutting planes has seen surprising twists and turns.

Cutting planes for specially structured combinatorial optimization problems came first. In their landmark 1954 paper [34], Dantzig, Fulkerson, and S. Johnson spectacularly solved a (then) large-scale Traveling Salesman Problem using linear optimization with added cutting planes; see also [51].

Shortly after, Gomory began his seminal work [41–46] on general purpose cutting planes for integer programming. Gomory's integer rounding cuts, together with Chvátal's later contributions, became a cornerstone of the theory of cutting planes. Many hopes were set on this elegant theory, but eventually the computational results disappointed. The overwhelming opinion in the computationally minded community was that Gomory's general purpose cutting planes were a mathematical curiosity with not much practical value [29].

The 1970s saw important work on the polyhedral combinatorics for combinatorial optimization problems, which can be seen as a systematic continuation of the early Dantzig–Fulkerson–Johnson work. Important examples include the work on the Stable Set Problem [25, 63, 64, 68] and the Knapsack Problem [7, 69]. However, it took until the early 1980s for the method of polyhedral combinatorics, combined with branch-and-bound, to become the method of choice for solving hard combinatorial optimization problems. Again it were computational breakthroughs for the Traveling Salesman Problem that put the method in the spotlight [30, 32, 33, 52–55].

As a consequence, the prevailing paradigm in the 1980s and 1990s became that in order to solve hard combinatorial and mixed integer optimization problems, one needed to find classes of strong, preferably facet-defining, problem-specific cutting planes and use them in combination with branch and bound. Mixed integer solvers such as CPLEX allowed to deploy user-defined cutting-plane separators via callback functions, for example.

This picture changed in the mid 1990s when general purpose cutting planes, including Gomory's mixed integer cut [42], were demonstrated to be very effective with contemporary implementations of the simplex method, especially when combined with branch-and-bound [8–10]. The newly discovered numerical efficacy came as a surprise to many. This discovery led to a revolution that transformed mixed-integer optimization technology [29].

Subsequently, research activity on general purpose cutting planes increased and gained equal prominence with the study of strong cutting planes for problems with special structure. A key turning point in the 2000s was the emphasis on the so-called *multi-row cuts*, which hold the promise of making even further breakthroughs in algorithms for solving large-scale mixed-integer programs. This recent research is collectively referred to under the label of *cut-generating functions*, a term coined by Cornuéjols et al. [26]. Although the nature of this research is often very technical and theoretically inclined, the practical impact of this work is expected to be big given the successful role played by Gomory's initial research in solving real-world problems [22].

A central problem and a driving force behind this line of work has been the so-called *infinite group problem* (or *infinite relaxation*), introduced by Gomory and E. Johnson in two seminal papers in 1972 [47, 48]. In this sense, the infinite group problem was a visionary contribution that anticipated this modern trend in integer programming decades earlier. To make further progress in the elaborate research program of cut-generating functions, it is imperative to understand the infinite group problem even better. The bulk of Gomory and Johnson's contributions were in the *single-row infinite group problem*, and until recently the theory behind the *multi-row infinite group problem* was mostly in the dark. With the modern focus on multi-row cuts within cut-generating functions, it is very important to understand the multi-row infinite group problem. The last decade has seen some excellent progress in this question, and this survey attempts to present this story.

1.1 Cut-generating function pairs

We begin with a quick overview of the cut-generating function approach to unifying cutting plane theory. Let $d \in \mathbb{N}$ and I be a fixed subset of $\{1, \dots, d\}$. A mixed-integer optimization problem of the form

$$\max\{\, \mathbf{c} \cdot \mathbf{x} \mid A\mathbf{x} = \mathbf{b},\ \mathbf{x} \in \mathbb{R}^d_+,\ x_i \in \mathbb{Z} \quad \forall i \in I \,\} \tag{1.1}$$

is first solved by ignoring the integrality constraints and using the simplex algorithm. This leads to a simplex tableau reformulation:

$$\begin{aligned} A_B^{-1} A_N \mathbf{x}_N = A_B^{-1}\mathbf{b} - \mathbf{x}_B, \quad & \mathbf{x}_{B\cap I} \in \mathbb{Z}_+^{B\cap I},\ \mathbf{x}_{B\setminus I} \in \mathbb{R}_+^{B\setminus I}, \\ & \mathbf{x}_{N\cap I} \in \mathbb{Z}_+^{N\cap I},\ \mathbf{x}_{N\setminus I} \in \mathbb{R}_+^{N\setminus I} \end{aligned} \tag{1.2}$$

where the subscripts B and N denote the basic and non-basic parts of the solution $\mathbf{x}$ and matrix A, respectively. The following change of notation will be convenient: let $k = |B|$, $m = |N \setminus I|$, $\ell = |N \cap I|$, let R denote the submatrix of $A_B^{-1} A_N$ indexed by $N \setminus I$, P denote the submatrix of $A_B^{-1} A_N$ indexed by $N \cap I$, and set $\bar{S} = A_B^{-1}\mathbf{b} - (\mathbb{Z}_+^{B\cap I} \times \mathbb{R}_+^{B\setminus I})$. Then in the new notation, we describe system (1.2) as

$$X_{\bar{S}}(R,P) := \left\{ (\mathbf{s},\mathbf{y}) \in \mathbb{R}_+^m \times \mathbb{Z}_+^\ell \;\middle|\; R\mathbf{s} + P\mathbf{y} \in \bar{S} \right\}. \tag{1.3}$$

In the following, we will consider general systems of the form (1.3), where $m, \ell \in \mathbb{Z}_+$ and $k \in \mathbb{N}$, $R \in \mathbb{R}^{k\times m}$ and $P \in \mathbb{R}^{k\times \ell}$ are matrices, and $\bar{S}$ is a closed subset of $\mathbb{R}^k$ such that $\mathbf{0} \notin \bar{S}$. Instead of using the full simplex tableau (1.2), one can as well consider relaxations of (1.2), for example by taking a subset of the rows only. In the simplest case, one focuses on only one row, i.e., $k = 1$.

We denote the columns of matrices R and P by $\mathbf{r}^1, \dots, \mathbf{r}^m$ and $\mathbf{p}^1, \dots, \mathbf{p}^\ell$, respectively. Given $k \in \mathbb{N}$ and $\bar{S} \subseteq \mathbb{R}^k$, a *cut-generating function pair (or simply, cut-generating pair)* (ψ, π) for $\bar{S}$ is a pair of functions $\psi, \pi \colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\sum_{i=1}^m \psi(\mathbf{r}^i) s_i + \sum_{j=1}^\ell \pi(\mathbf{p}^j) y_j \geq 1 \tag{1.4}$$

is a valid inequality (also called a *cutting plane* or *cut*) for the set $X_{\bar{S}}(R,P)$ for every choice of $m, \ell \in \mathbb{Z}_+$ and for all matrices $R \in \mathbb{R}^{k\times m}$ and $P \in \mathbb{R}^{k\times\ell}$. We emphasize that cut-generating pairs depend on k and $\bar{S}$ and do *not* depend on m, ℓ, R and P. *A priori* it is not clear that such cut-generating function pairs can exist. However, it has been observed that for many special cases of model (1.3) the convex hull of points in $X_{\bar{S}}(R,P)$ can be completely described using cut-generating functions, i.e., not only do they exist, but they are sufficient for the purposes of optimization from a theoretical perspective.

Gomory and Johnson's joint work in the 1970s [47, 48], together with Johnson's independent results [58] in the same decade, shows that cut-generating pairs can be understood by studying infinite-dimensional convex sets parameterized by $k \in \mathbb{N}$ and $\bar{S} \subseteq \mathbb{R}^k$. For any index set I (not necessarily finite), $\mathbb{R}^I$ will denote the vector space of all real-valued functions with domain I, and $\mathbb{R}^{(I)}$ will denote the subspace of real-valued functions with domain I that have finite support, i.e., functions that take value zero except on a finite set.[1] For example, $\mathbb{R}^{(\mathbb{R}^k)}$ is the set of all functions $s \colon \mathbb{R}^k \to \mathbb{R}$ with finite support. The object of interest is

$$X_{\bar{S}} := \left\{ (s,y) \in \mathbb{R}^{(\mathbb{R}^k)} \times \mathbb{R}^{(\mathbb{R}^k)} \;\middle|\; \begin{array}{l} \displaystyle\sum_{\mathbf{r}\in\mathbb{R}^k} \mathbf{r} s(\mathbf{r}) + \sum_{\mathbf{p}\in\mathbb{R}^k} \mathbf{p} y(\mathbf{p}) \in \bar{S}, \\ s(\mathbf{r}) \in \mathbb{R}_+ \ \forall \mathbf{r} \in \mathbb{R}^k, \\ y(\mathbf{p}) \in \mathbb{Z}_+ \ \forall \mathbf{p} \in \mathbb{R}^k \end{array} \right\}. \tag{1.5}$$

The convex hull of points in $X_{\bar{S}}$ is an infinite-dimensional convex set in $\mathbb{R}^{(\mathbb{R}^k)} \times \mathbb{R}^{(\mathbb{R}^k)}$ that contains the convex hull of every $X_{\bar{S}}(R,P)$ (for every choice

[1] This notation for functions of finite support is used, for example, in [2].

of R and P) as a finite-dimensional face. Cut generating function pairs can then be interpreted as halfspaces in the vector space $\mathbb{R}^{(\mathbb{R}^k)} \times \mathbb{R}^{(\mathbb{R}^k)}$ that contain $X_{\bar{S}}$.

1.2 Approaches to understanding cut-generating function pairs

The setting of $\operatorname{conv}(X_{\bar{S}})$ where $\bar{S}$ is a translate of $\mathbb{Z}^k$ has received the most attention in the literature.[2] Fix a point $\mathbf{f} \in \mathbb{R}^k \setminus \mathbb{Z}^k$ and let $\bar{S} = \mathbf{f} + \mathbb{Z}^k$. Two distinct approaches have emerged within the study of the facial structure of $\operatorname{conv}(X_{\bar{S}})$, which we will compare below.

1. **The infinite group problem.** Gomory and Johnson, in their work in [47, 48], study the *infinite group problem*, which appears as the face of $\operatorname{conv}(X_{\bar{S}})$ given by $\operatorname{conv}(X_{\bar{S}}) \cap \{\, (s, y) \in \mathbb{R}^{(\mathbb{R}^k)} \times \mathbb{R}^{(\mathbb{R}^k)} \mid s = 0 \,\}$. This produces cut-generating functions $\pi\colon \mathbb{R}^k \to \mathbb{R}$ that are useful for the study of *pure* integer optimization problems. The structure of these functions π can be very complicated; it is the main topic of our survey.
   By Johnson's fundamental work [58], we know that these functions π can then be easily lifted to strong cut-generating pairs (ψ, π) for mixed-integer optimization problems using closed form formulas.
2. **Intersection cuts.** Another approach to cut-generating pairs has its roots in Balas' work on *intersection cuts* [6] and Balas and Jeroslow's work on *monoidal strengthening* [11]. More recent work by Andersen, Louveaux, Weismantel, and Wolsey [3] renewed the interest in this approach. Borozan and Cornuéjols [23] put it in the framework of cut generating functions, and Dey and Wolsey [40] interpreted monoidal strengthening in this setting. This line of research was developed further in many papers, including [12–14, 16, 28, 36].
   Consider again the case $\bar{S} = \mathbf{f} + \mathbb{Z}^k$. Then the face of $\operatorname{conv}(X_{\bar{S}})$ given by $\operatorname{conv}(X_{\bar{S}}) \cap \{\, (s, y) \in \mathbb{R}^{(\mathbb{R}^k)} \times \mathbb{R}^{(\mathbb{R}^k)} \mid y = 0 \,\}$ is studied first,[3] giving cut-generating functions $\psi\colon \mathbb{R}^k \to \mathbb{R}$. They are obtained as the gauge functions of maximal lattice-free convex bodies. The functions ψ are then lifted to cut-generating pairs (ψ, π) for $X_{\bar{S}}$.
   The advantage of the intersection cut approach, compared with Gomory–Johnson's infinite group problem, is that the gauge functions ψ can be evaluated using simpler formulas. Further, generalizations have been studied in which the set $\bar{S}$ is allowed to be more general than just a translated lattice – the most frequently studied $\bar{S}$ is of the form $C \cap (\mathbf{f} + \mathbb{Z}^k)$ where C is a convex subset of $\mathbb{R}^k$ and $\mathbf{f} \in \mathbb{R}^k \setminus \mathbb{Z}^k$ (for example, $C = \mathbb{R}^k_+$ would correspond to model (1.2)). In this case, the cut-generating functions are obtained from so-called maximal $\bar{S}$-free convex sets.

[2] This model is called the *mixed-integer infinite relaxation*, for example in the survey [27], or sometimes the *mixed-integer group problem*, but we shall not use either of these terms in the remainder of our survey.

[3] This model is called the *continuous infinite relaxation*, for example in the survey [27], or sometimes the *continuous group problem*.

The drawback of the intersection cut approach is that lifting a gauge function ψ to a strong cut-generating pair (ψ, π) can be rather difficult. This difficulty has been recently studied by [5, 12, 16, 28]. Moreover, this approach produces a much smaller subset of cut-generating pairs as compared with the infinite group approach when $\bar{S}$ is a translated lattice. In this case, there exist undominated cut-generating pairs (ψ, π) where ψ is not the gauge of a *maximal* lattice-free set – these can still be obtained in the context of the infinite group problem. In contrast the approach outlined above starts with a function ψ that is the gauge function of a maximal lattice-free set, and so the approach cannot derive such cut-generating functions.

Remark 1.1 The study of cut-generating functions for $k = 1$ is referred to as the *single-row* problem, and the general $k \geq 2$ case is referred to as the *multi-row* problem in the literature. Algorithms used in practice for solving mixed-integer problems have so far used only insights from the single-row problem. It is believed that the general multi-row analysis can lead to stronger cutting planes that can significantly boost the performance of state-of-the-art algorithms.

1.3 Outline of the survey

We will survey the recent progress made on the infinite group problem approach described in subsection 1.2. We view this as a follow-up to two excellent surveys, the first by Conforti, Cornuéjols, and Zambelli [27], which discusses the basic structure of the corner polyhedron and its relation with cut-generating functions, and the second by Richard and Dey [65], which focuses on the group-theoretic approach. Our survey focuses on the milestones that have been reached since [27, 65] were written. Although we do not intend [27, 65] to be prerequisites to this article, the reader who is familiar with the material from [27, 65] will certainly have a better context for the current article. The reader may use Table 6 in Appendix B as a reference to notation in these surveys and other literature.

Our survey is divided into two parts, of which this article comprises the first part. Part II will appear in a subsequent issue of the same journal. Part I consists of sections 1, 2, 3 and 4, which introduce the central concepts and develop the fundamental tools required for the study of the infinite group problem. Part II is divided into 4 sections (sections 5, 6, 7 and 8) devoted to algorithmic and structural results which build on the foundation laid in Part I.

Section 2 formally introduces the problem, the main objects of study such as *valid functions, minimal valid functions, extreme functions, and facets*, and their basic properties. We conclude the section with a discussion of families of valid functions and some open questions (subsection 2.4). The discussion references a compendium that summarizes known families from the literature (Appendix A), and contains some previously unknown families such as extreme functions with 5 or more slopes and some discontinuous extreme functions with left and right discontinuity at the origin. Section 3 introduces the notation and concepts from discrete geometry required for analyzing the problem, and collects foundational techniques for the general k-row problem. Section 4 surveys higher-dimensional variants of the celebrated *Interval Lemma*. Section 5 [Part II] introduces one of the most general sufficient conditions for the fundamental notion of *extremality*, illustrating how all the techniques introduced in the previous sections come together to analyze extremality. Section 6 [Part II] investigates some analytic properties of the problem and demonstrates the use of analytical ideas to construct extreme functions. Sections 7 and 8 [Part II] discuss important algorithmic and structural results known for the one-row and two-row problems. These results are based on recent breakthroughs in [17–19].

We highlight results that are new in this survey with the annotation "New result ♣". To the best of our knowledge, these do not appear elsewhere in the literature.

Due to constraints of space, we must limit the topics covered in this survey. We briefly mention some of the important highlights in the literature that are not discussed in this survey. A wealth of results on the *finite group problem* are closely related to the infinite group problem. We invite the reader to explore the survey by Richard and Dey [65] for more details about this direction. Furthermore, we focus on the structural results of the infinite group problem, as opposed to the implementation of these results to solve integer programming problems. This includes the so-called *shooting experiments* discussed in [50] to empirically judge quality of the cutting planes, and the discussion of relative strength in [49, section 6].

2 The Infinite Group Problem

As stated in subsection 1.2, Gomory and Johnson introduced the so-called *infinite group problem*. It has its roots in Gomory's *group problem* [45], which was introduced by him as an algebraic relaxation of pure integer linear optimization problems. We introduce this next as it will be useful for formulating many of our results in a unified language. One considers an abelian group G, written additively, and studies the set of functions $y\colon G \to \mathbb{R}$ satisfying the following constraints:

$$\begin{aligned} &\sum_{\mathbf{r}\in G} \mathbf{r}\, y(\mathbf{r}) \in \mathbf{f} + S \\ &y(\mathbf{r}) \in \mathbb{Z}_+ \quad \text{for all } \mathbf{r} \in G \\ &y \text{ has finite support,} \end{aligned} \tag{2.6}$$

where S is a subgroup of G and $\mathbf{f}$ is a given element in $G \setminus S$; so $\mathbf{f} + S$ is the coset containing the element $\mathbf{f}$. We are interested in studying the convex hull $R_{\mathbf{f}}(G, S)$ of the set of all functions $y\colon G \to \mathbb{R}$ satisfying the constraints in (2.6). $R_{\mathbf{f}}(G, S)$ is a convex subset of the vector space $\mathbb{R}^{(G)}$, which is infinite-dimensional when G is an infinite group, i.e., of infinite order. The nomenclature *k-row infinite group problem* is reserved for the situation when $G = \mathbb{R}^k$

is taken to be the group of real k-dimensional vectors under addition, and $S = \mathbb{Z}^k$ is the subgroup of the integer vectors. When $k = 1$, we refer to it as the *single-row infinite group problem.* Recall that the connection with the cut-generating function model (1.3) is made by setting $\bar{S} = \mathbf{f} + S$, whence we get $R_{\mathbf{f}}(G, S)$ as the projection of $\operatorname{conv}(X_{\bar{S}}) \cap \bigl\{(s, y) \in \mathbb{R}^{(\mathbb{R}^k)} \times \mathbb{R}^{(\mathbb{R}^k)} \mid s = 0\bigr\}$ onto the y space.

Remark 2.1 Note that there is a correspondence between the sets $R_{\mathbf{f}}(G, S)$ and $R_{\bar{\mathbf{f}}}(G/S, 0)$ where G/S is the quotient group with respect to the (normal) subgroup S and $\bar{\mathbf{f}}$ is the element corresponding to the coset $\mathbf{f} + S$, by standard aggregation of variables.[4]

In the earlier literature on the infinite group problem, the aggregated formulation $R_{\bar{\mathbf{f}}}(\mathbb{R}^k/\mathbb{Z}^k, 0)$ was used. The quotient $\mathbb{R}^k/\mathbb{Z}^k$ is the k-dimensional torus; it can be identified with the half-open unit cube $[0, 1)^k$, using coordinatewise arithmetic modulo 1. In this survey, however, we follow the trend in the recent literature [17–19, 27] to work with $R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$ instead. This removes the need for complicated notation for mapping between elements of $\mathbb{R}^k$ and elements of $\mathbb{R}^k/\mathbb{Z}^k$ (see Table 6 for an overview), and for complicated geometric notions, such as "wrap-around" line segments in Johnson's *cylindrical space* [49], in favor of the standard mathematical language of periodic, locally finite polyhedral complexes on $\mathbb{R}^k$ (subsection 3.1). We pay a small price for the simplicity and precision of this approach: We will often work with infinite objects where finite objects would suffice. However, it is very easy to go back to finite objects in the moments when we want to state algorithms.

The aggregated formulation is still of interest for the case where G/S is a finite group, as then $R_{\bar{\mathbf{f}}}(G/S, 0)$ is finite-dimensional and thus amenable to polyhedral techniques. This case is referred to as a *finite group problem*; it will appear in subsection 8.1. Due to the correspondence between the sets $R_{\mathbf{f}}(G, S)$ and $R_{\bar{\mathbf{f}}}(G/S, 0)$, we shall also refer to $R_{\mathbf{f}}(G, S)$ as a finite group problem whenever S has finite index in G, i.e., G/S is a finite group.

2.1 Valid inequalities and valid functions

Following Gomory and Johnson, we are interested in the description of $R_{\mathbf{f}}(G, S)$ as the intersection of halfspaces in $\mathbb{R}^{(G)}$. We first describe the general form that these halfspaces take and then a standard normalization that leads to the idea of cut-generating functions.

[4] Indeed, $y \in R_{\mathbf{f}}(G, S)$ gives an element $\bar{y} \in \mathbb{R}_{\bar{\mathbf{f}}}(G/S, 0)$ by setting $\bar{y}(C) = \sum_{\mathbf{r} \in C} y(\mathbf{r})$ for every coset $C \in G/S$. In the other direction, given $\bar{y} \in \mathbb{R}_{\bar{\mathbf{f}}}(G/S, 0)$ we get a solution $y \in R_{\mathbf{f}}(G, S)$ by simply picking a canonical representative $\mathbf{r}_C$ for each coset $C \in G/S$ and setting $y(\mathbf{r}_C) = \bar{y}(C)$. From aggregation of variables it follows that the strongest valid inequalities for the convex hull of $R_{\mathbf{f}}(G, S)$ will have identical coefficients on any coset; see Theorem 2.6.

2.1.1 Valid inequalities

Any halfspace in $\mathbb{R}^{(G)}$ is given by a pair (π, α), where $\pi \in \mathbb{R}^G$ and $\alpha \in \mathbb{R}$, and the halfspace is the set of all $y \in \mathbb{R}^{(G)}$ that satisfy $\sum_{\mathbf{r}\in G} \pi(\mathbf{r})y(\mathbf{r}) \geq \alpha$. The left-hand side of the inequality is a finite sum because y has finite support. Such an inequality is called a *valid inequality* for $R_{\mathbf{f}}(G,S)$ if $\sum_{\mathbf{r}\in G} \pi(\mathbf{r})y(\mathbf{r}) \geq \alpha$ for all $y \in R_{\mathbf{f}}(G,S)$, i.e., $R_{\mathbf{f}}(G,S)$ is contained in the halfspace defined by (π, α). Note that the set of all valid inequalities (π, α) is a cone in the space $\mathbb{R}^G \times \mathbb{R}$.

2.1.2 Sign of the coefficients of valid inequalities

If S has finite index in G, then it can be shown that if (π, α) gives a valid inequality, then $\pi \geq 0$. An even stronger statement is easily seen to be true: if $\mathbf{r} \in G$ is such that there exists $n \in \mathbb{N}$ satisfying $n\mathbf{r} \in S$, then $\pi(\mathbf{r}) \geq 0$ [27, section 5]. However, when this is not the case, there may exist valid inequalities (π, α) where π takes negative values. We give an explicit example below for the one-row infinite group problem ($G = \mathbb{R}$ and $S = \mathbb{Z}$).

It is well-known that there exist functions $h\colon \mathbb{R} \to \mathbb{R}$ such that they satisfy $h(a+b) = h(a) + h(b)$ for all $a, b \in \mathbb{R}$ and whose graph is dense in $\mathbb{R}^2$. These are the non-regular solutions to the so-called *Cauchy functional equation* [1, chapter 2, Theorem 3]. This functional equation is discussed further in section 4.

Proposition 2.2 (New result ♣) *Let f be any rational number. Let $h\colon \mathbb{R} \to \mathbb{R}$ be any function such that $h(a+b) = h(a) + h(b)$ for all $a, b \in \mathbb{R}$ and the graph of h is dense in $\mathbb{R}^2$. Define $\pi^*\colon \mathbb{R} \to \mathbb{R}$ as $\pi^*(a) = h(a) - h(1)a$ for all $a \in \mathbb{R}$. Then $(\pi^*, 0)$ defines an* implicit equality *of $\mathbb{R}_{\mathbf{f}}(G,S)$, i.e., the equation*

$$\sum_{\mathbf{r}\in G} \pi^*(\mathbf{r})y(\mathbf{r}) = 0 \quad \textit{holds for } y \in \mathbb{R}_{\mathbf{f}}(G,S).$$

Thus both $(\pi^*, 0)$ and $(-\pi^*, 0)$ define valid inequalities for $\mathbb{R}_f(G,S)$. Moreover π^* has a dense graph in $\mathbb{R}^2$.

Proof Using additivity, $h(a) = h(1)a$ for all rational a and therefore we have $\pi^*(f + w) = 0$ for any $w \in \mathbb{Z}$. Moreover, since $h(a+b) = h(a) + h(b)$ for all $a, b \in \mathbb{R}$, we also have $\pi^*(a+b) = \pi^*(a) + \pi^*(b)$ for all $a, b \in \mathbb{R}$. Consider any $y \in \mathbb{R}^{(\mathbb{R})}$ such that $\sum_{r\in\mathbb{R}} r\,y(r) = f + w$ for some $w \in \mathbb{Z}$, and $y(r) \in \mathbb{Z}_+$ for all $r \in \mathbb{R}$. Then $0 = \pi^*(f+w) = \pi^*(\sum_{r\in\mathbb{R}} r\,y(r)) = \sum_{r\in\mathbb{R}} \pi^*(r)\,y(r)$. This establishes that $(\pi^*, 0)$ defines an implicit equality for $R_f(G,S)$. The graph of π^* is dense in $\mathbb{R}^2$ because the graph of h is dense in $\mathbb{R}^2$. □

In fact, for the infinite group problem $R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$ with rational $\mathbf{f}$ we show that the set of implicit equalities (equivalently, the lineality space of the cone of valid inequalities) consists of the (π, α) such that π is additive and $\alpha = 0$. The discussion above says that for any valid inequality given by the pair (π, α) we have $\pi(\mathbf{r}) \geq 0$ for every $\mathbf{r} \in \mathbb{Q}^k$.

Proposition 2.3 (New result ♣) *Let* $\mathbf{f}$ *be a rational vector. A pair* $(\pi, \alpha) \in \mathbb{R}^{\mathbb{R}^k} \times \mathbb{R}$ *satisfies* $\sum_{\mathbf{r}\in\mathbb{R}^k} \pi(\mathbf{r})y(\mathbf{r}) = \alpha$ *for all* $y \in R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$ *if and only if* π *is additive, i.e.,* $\pi(\mathbf{r}^1) + \pi(\mathbf{r}^2) = \pi(\mathbf{r}^1 + \mathbf{r}^2)$ *for all* $\mathbf{r}^1, \mathbf{r}^2 \in \mathbb{R}^k$*, and* $\alpha = 0$.

Proof The "if" direction can be proved using the same calculations as in the proof of Proposition 2.2. We prove the "only if" direction.

For $\mathbf{r} \in \mathbb{R}^k$, let $e_{\mathbf{r}}$ denote the finite support function which takes value 1 at $\mathbf{r}$ and 0 everywhere else. Then $y^1 = e_{\mathbf{r}^1+\mathbf{r}^2} + e_{\mathbf{f}-\mathbf{r}^1-\mathbf{r}^2} \in R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$. Therefore, $\alpha = \sum_{\mathbf{r}\in\mathbb{R}^k} \pi(\mathbf{r})y^1(\mathbf{r}) = \pi(\mathbf{r}^1 + \mathbf{r}^2) + \pi(\mathbf{f} - \mathbf{r}^1 - \mathbf{r}^2)$. Similarly, $e_{\mathbf{r}^1}+e_{\mathbf{r}^2}+e_{\mathbf{f}-\mathbf{r}^1-\mathbf{r}^2} \in R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$ and therefore $\alpha = \pi(\mathbf{r}^1)+\pi(\mathbf{r}^2)+\pi(\mathbf{f}-\mathbf{r}^1-\mathbf{r}^2)$. Therefore, $\pi(\mathbf{r}^1) + \pi(\mathbf{r}^2) = \pi(\mathbf{r}^1 + \mathbf{r}^2)$.

Additive functions take value 0 at the origin: $\pi(\mathbf{0}) + \pi(\mathbf{0}) = \pi(\mathbf{0})$ which implies $\pi(\mathbf{0}) = 0$. Since, $\pi(\mathbf{r}) \geq 0$ for every $\mathbf{r} \in \mathbb{Q}^k$ and for any rational $\mathbf{r}$, $\pi(\mathbf{r}) + \pi(-\mathbf{r}) = \pi(\mathbf{0}) = 0$ we must have $\pi(\mathbf{r}) = 0$ for every rational $\mathbf{r}$. Thus, using the fact that $e_{\mathbf{f}} \in R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$, we have $\alpha = \sum_{\mathbf{r}\in\mathbb{R}^k} \pi(\mathbf{r})e_{\mathbf{f}}(\mathbf{r}) = \pi(\mathbf{f}) = 0$ since $\mathbf{f}$ is rational. □

We next show that the intersection of all halfspaces of the form $\sum_{\mathbf{r}\in G} \pi(\mathbf{r})y(\mathbf{r}) \geq \alpha$ with $\pi \geq 0$ is a much larger superset of $R_{\mathbf{f}}(G, S)$. Our example is for $R_f(\mathbb{R}, \mathbb{Z})$.

Proposition 2.4 (New result ♣) *Let* f *be any rational number. Let* $h\colon \mathbb{R} \to \mathbb{R}$ *be any function such that* $h(a + b) = h(a) + h(b)$ *for all* $a, b \in \mathbb{R}$ *and the graph of* h *is dense in* $\mathbb{R}^2$*. Define* $\pi^*\colon \mathbb{R} \to \mathbb{R}$ *as* $\pi^*(a) = h(a) - h(1)a$ *for all* $a \in \mathbb{R}$*. Let* $r^1, \ldots, r^k$ *be a finite set of real numbers such that* $\pi^*(r^i) < 0$ *for all* $i = 1, \ldots, k$*. Define* $y^* \in \mathbb{R}^{(\mathbb{R})}$ *as* $y^*(r) = 1$ *if* $r \in \{r^1, \ldots, r^k\} \cup \{f\}$*, and* $y^*(r) = 0$ *otherwise. Then*

1. y^* *violates the implicit equality* $\sum_{r\in G} \pi^*(r)\, y(r) = 0$ *and thus, does not lie in* $R_f(\mathbb{R}, \mathbb{Z})$*,*
2. y^* *satisfies all valid inequalities* $\sum_{r\in\mathbb{R}} \pi(r)y(r) \geq \alpha$ *where* $\pi \geq 0$*.*

Proof Observe that $\sum_{r\in G} \pi^*(r)\, y^*(r) = \sum_{i=1}^k \pi^*(r^i) + \pi^*(f) = \sum_{i=1}^k \pi^*(r^i) < 0$. By Proposition 2.2, $(\pi^*, 0)$ is an implicit equality for $R_f(\mathbb{R}, \mathbb{Z})$ and therefore, $y^* \notin R_f(\mathbb{R}, \mathbb{Z})$.

On the other hand, for any valid inequality given by (π, α) such that $\pi \geq 0$, we have $\pi(f) \geq \alpha$ (since $e_f \in R_f(\mathbb{R}, \mathbb{Z})$). So, for any such valid function π, we have $\sum_{r\in\mathbb{R}} \pi(r)y^*(r) = \sum_{i=1}^k \pi(r^i) + \pi(f) \geq \pi(f) \geq \alpha$ (since $\pi \geq 0$). □

The above example takes points that do not satisfy the implicit equalities, i.e., we consider points outside the affine hull of the feasible region. If we restrict ourselves to satisfy the implicit equalities, are the nonnegative valid inequalities sufficient? This is an open question.

Open question 2.5 *Is every valid inequality* (π, α) *the sum of a nonnegative valid inequality* (π^+, α) *and an implicit equality* $(\pi^=, 0)$*?*

2.1.3 Valid functions

Since data in finite-dimensional integer programs is usually rational, and this is our main motivation for studying the infinite group problem, it is customary to concentrate on valid inequalities with $\pi \geq 0$; then we can choose, after a scaling, $\alpha = 1$ (otherwise, the inequality is implied by the nonnegativity of y). Thus, we only focus on valid inequalities of the form $\sum_{\mathbf{r}\in G} \pi(\mathbf{r})y(\mathbf{r}) \geq 1$ with $\pi \geq 0$. Such functions $\pi \in \mathbb{R}^G$ are called *valid functions* for $R_{\mathbf{f}}(G, S)$. We remind the reader that this choice comes at a price because of Proposition 2.4; however, it can be shown that for rational *corner polyhedra*, which form an important family of relaxations for integer programs, all valid inequalities are restrictions of nonnegative valid functions for the infinite group problem. See [27] for a discussion.

2.2 Minimal functions, extreme functions and facets

Gomory and Johnson [47, 48] defined a hierarchy on the set of valid functions, capturing the strength of the corresponding valid inequalities, which we summarize now.

2.2.1 Minimal functions

A valid function π for $R_{\mathbf{f}}(G, S)$ is said to be *minimal* for $R_{\mathbf{f}}(G, S)$ if there is no valid function $\pi' \neq \pi$ such that $\pi'(\mathbf{r}) \leq \pi(\mathbf{r})$ for all $\mathbf{r} \in G$. For every valid function π for $R_{\mathbf{f}}(G, S)$, there exists a minimal valid function π' such that $\pi' \leq \pi$ [21, Theorem 1.1], and thus non-minimal valid functions are redundant in the description of $R_{\mathbf{f}}(G, S)$. Note that π' is not uniquely determined (Figure 1).

Minimal functions for $R_{\mathbf{f}}(G, S)$ were characterized by Gomory for the case where S has finite index in G in [45], and later for $R_{\mathbf{f}}(\mathbb{R}, \mathbb{Z})$ by Gomory and Johnson [47]. We state these results in a unified notation in the following theorem.

A function $\pi\colon G \to \mathbb{R}$ is *subadditive* if $\pi(\mathbf{x} + \mathbf{y}) \leq \pi(\mathbf{x}) + \pi(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in G$. We say that π is *symmetric* (or *satisfies the symmetry condition*) if $\pi(\mathbf{x}) + \pi(\mathbf{f} - \mathbf{x}) = 1$ for all $\mathbf{x} \in G$.

Theorem 2.6 (Gomory and Johnson [47]) *Let G be an abelian group, S be a subgroup of G and $\mathbf{f} \in G \setminus S$. Let $\pi\colon G \to \mathbb{R}$ be a nonnegative function. Then π is a minimal valid function for $R_{\mathbf{f}}(G, S)$ if and only if $\pi(\mathbf{z}) = 0$ for all $\mathbf{z} \in S$, π is subadditive, and π satisfies the symmetry condition. (The first two conditions imply that π is periodic modulo S, that is, $\pi(\mathbf{x}) = \pi(\mathbf{x} + \mathbf{z})$ for all $\mathbf{z} \in S$, and the symmetry condition implies that the values of minimal functions are bounded between* 0 *and* 1*.)*

See [27, Theorem 5.4] for a proof.

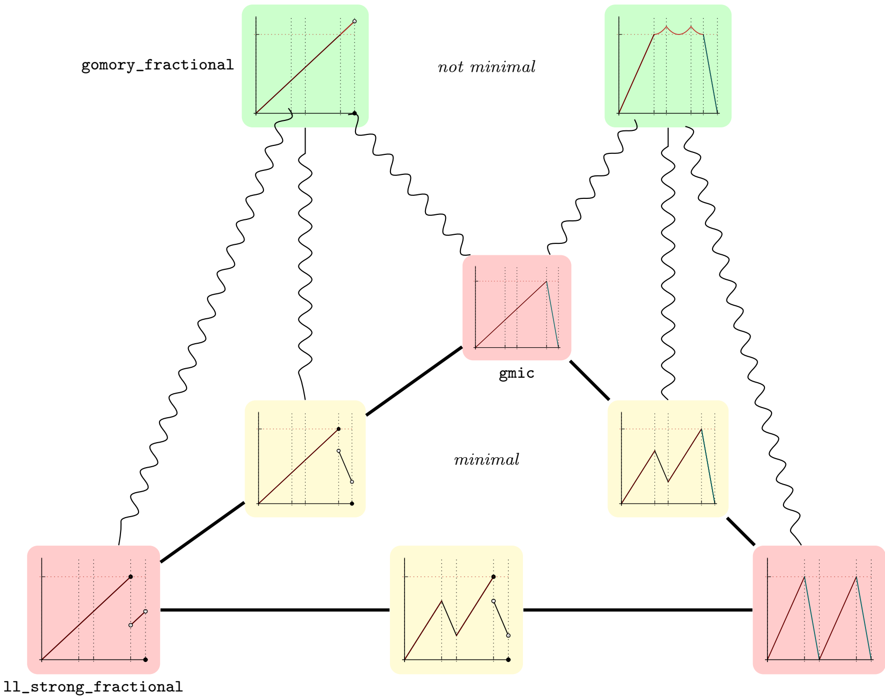


Fig. 1 The hierarchy of valid, minimal, and extreme functions by example for the case $R_f(\mathbb{R}, \mathbb{Z})$. Pairwise convex combinations (*solid lines forming the bottom triangle*) of three extreme functions (*graphs on red background at the corners*) give non-extreme, minimal functions (*graphs on yellow background on the edges*). These functions dominate (*wavy lines*) various non-minimal, valid functions (*graphs on green background, top*). Even without checking the dominance, it is easy to see that the functions shown on the top cannot be minimal: they have some function values larger than 1 (*international orange*), but minimal valid functions are upper bounded by 1 by Theorem 2.6. Since minimal valid functions for $R_f(\mathbb{R}, \mathbb{Z})$ are periodic with respect to $\mathbb{Z}$, we only show the interval $[0, 1]$.

2.2.2 Extreme functions

In polyhedral combinatorics, one is interested in classifying the facet-defining inequalities of a polytope, which are the strongest inequalities and provide a finite minimal description. In the infinite group problem literature, three notions analogous to that of a facet-defining inequality have been proposed, which are not known to be equivalent. We start with the notion of an *extreme function*.

A valid function π is *extreme* for $R_{\mathbf{f}}(G, S)$ if it cannot be written as a convex combination of two other valid functions for $R_{\mathbf{f}}(G, S)$, i.e., $\pi = \frac{1}{2}(\pi^1 + \pi^2)$ implies $\pi = \pi^1 = \pi^2$ (see Figure 1 and Figure 3). Extreme functions are easily seen to be minimal. In fact we may view this definition from a convex geometry perspective. By Theorem 2.6, the set of minimal valid functions is a convex subset of the infinite-dimensional space $\mathbb{R}^G$ of real-valued functions

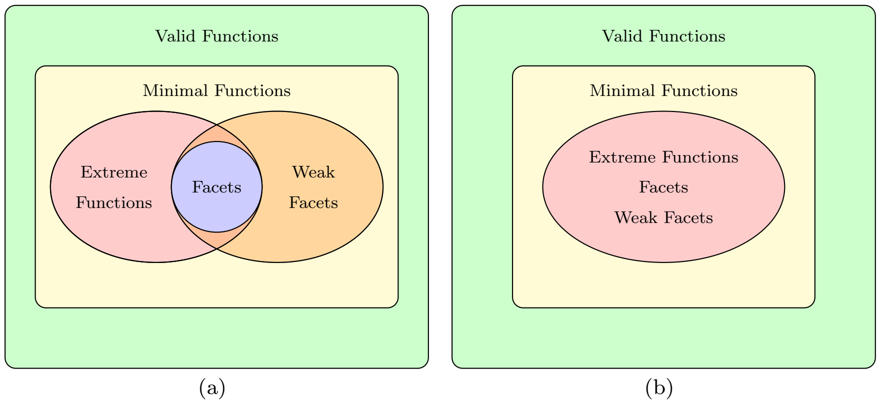


Fig. 2 The hierarchy of valid, minimal, and extreme functions and facets and weak facets. (a) General case. (b) Situation in the finite-dimensional case and in the case of continuous piecewise linear functions with rational breakpoints.

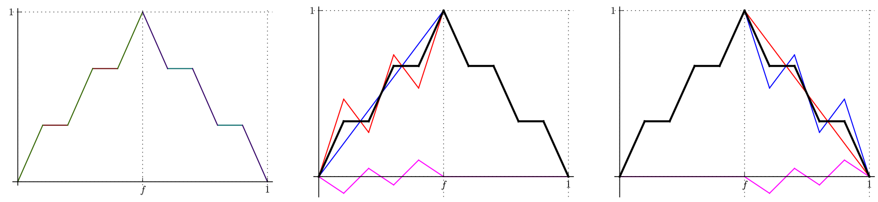


Fig. 3 This function (`h = not_extreme_1()`) is minimal, but not extreme (and hence also not a facet), as proved by `extremality_test(h, show_plots=True)`. The procedure first shows that for any distinct minimal $\pi^1 = \pi + \bar\pi$ (*blue*), $\pi^2 = \pi - \bar\pi$ (*red*) such that $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$, the functions π^1 and π^2 are continuous piecewise linear with the same breakpoints as π (in the terminology of [18], π is *affine imposing* on all intervals between breakpoints). A finite-dimensional extremality test then finds two linearly independent perturbations $\bar\pi$ (*magenta*), as shown.

on G; this follows from the observation that all the properties in Theorem 2.6 are preserved under taking convex combinations of functions.

Proposition 2.7 (New result ♣) *The set of minimal valid functions is a compact convex set under the product topology on the space* $\mathbb{R}^G$ *of real-valued functions on* G.

The proof appears in subsection 6.1. In the light of Proposition 2.7, it is natural to study the extreme points of this compact convex set of minimal valid functions. These are precisely the extreme functions. By an application of the Krein–Milman theorem, all minimal valid functions are either convex combinations of extreme functions or pointwise limits of such convex combinations (i.e., limits in the product topology).

2.2.3 Facets and weak facets

A related notion is that of a *facet*. Let $P(\pi)$ denote the set of all feasible solutions $y \in \mathbb{R}^{(G)}$ satisfying (2.6) such that $\sum_{r\in G} \pi(r)y(r) = 1$. A valid function π is called a *facet* if for every valid function π' such that $P(\pi) \subseteq P(\pi')$ we have that $\pi' = \pi$, as defined in [49]. Equivalently, a valid function π is a facet if this condition holds for all such *minimal* valid functions π' (cf. [21]).

A similar facet definition, which we call a *weak facet*, is given in [37] and in fact was used in an erroneous proof of the so-called *Facet Theorem* in [49, Theorem 3] (see Theorem 2.12)[5]. In particular, a valid function π is called a weak facet if for every valid function π' such that $P(\pi) \subseteq P(\pi')$ we have that $P(\pi) = P(\pi')$.

2.2.4 Relation between the three notions

Facets are extreme functions (cf. [21, Lemma 1.3]), but it is unknown if all extreme functions are facets. A facet is also a weak facet, but it is unknown if all weak facets are facets. Thus, facets are a subset of the intersection of extreme functions and weak facets, but nothing further is known in general; see Figure 2 (a). When G is a finite abelian group, the set of minimal functions is a finite-dimensional polyhedron (given by constraints coming from Theorem 2.6); see subsection 8.1. In this setting, it is well known that the three notions of weak facets, facets and extreme inequalities are equivalent, and form the extreme points of this polyhedron; see Figure 2 (b). In the one-row infinite group problem, we can also establish some equivalence as stated below, which is a consequence of Theorem 8.6. The result is new and has not been published before.

Proposition 2.8 (New result ♣) *Suppose $\pi\colon \mathbb{R} \to \mathbb{R}$ is a continuous piecewise linear function[6] with rational breakpoints in $\frac{1}{q}\mathbb{Z}$ for some $q \in \mathbb{N}$. Then π is extreme if and only if π is a facet.*

Open question 2.9 *Are the definitions of facets, weak facets, and extreme functions equivalent?*

2.3 A roadmap for proving extremality and facetness

An understanding of the set of points for which the subadditivity relations of a minimal function hold at equality is crucial to the study of both extreme functions and facets. This motivates the following definition.

[5] In a proof by contradiction, they say that if π is not a facet, then there exists a valid function π^* and a $y^* \in R_{\mathbf{f}}(G, S)$ such that $y^* \in P(\pi^*) \setminus P(\pi)$. This works when π is not a weak facet, but does not work if we assume that π is not a facet.

[6] See subsection 3.1 for the definition that we use.

Definition 2.10 Define the *subadditivity slack* of π as

$$\Delta\pi(\mathbf{x},\mathbf{y}) := \pi(\mathbf{x}) + \pi(\mathbf{y}) - \pi(\mathbf{x}+\mathbf{y}) \tag{2.7}$$

and the *additivity domain* of π as

$$E(\pi) := \{\, (\mathbf{x},\mathbf{y}) \mid \Delta\pi(\mathbf{x},\mathbf{y}) = 0 \,\}. \tag{2.8}$$

Additivity domains are used by Gomory and Johnson to define the notion of *merit index* in [49]. The merit index is the volume of $E(\pi)$ (modulo $\mathbb{Z}^n$) and can be taken as a quantitative measure of strength of minimal valid functions. Work on the merit index also appears in [39]. We will not discuss the merit index in this survey; however, the set $E(\pi)$ will be crucial in what follows.

The main technique used to show a function π is extreme is to assume that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$ where π^1, π^2 are valid functions, and then show that $\pi = \pi^1 = \pi^2$. One then employs the following lemma to infer important properties of π^1, π^2. These following facts can be found in the literature for the one-row problem; the extension to general k is straightforward.

Lemma 2.11 *Let $\pi : \mathbb{R}^k \to \mathbb{R}_+$ be minimal, $\pi = \frac{1}{2}(\pi^1 + \pi^2)$, and π^1, π^2 valid functions. Then the following hold:*

(i) *π^1, π^2 are minimal [47, Lemma 1.4].*
(ii) *All subadditivity relations $\pi(\mathbf{x}+\mathbf{y}) \leq \pi(\mathbf{x}) + \pi(\mathbf{y})$ that are tight for π are also tight for π^1, π^2, i.e., $E(\pi) \subseteq E(\pi^1) \cap E(\pi^2)$ [47, proof of Theorem 3.3].*[7]
(iii) *Suppose there exists a real number M such that $\limsup_{h\to 0} \left|\frac{\pi(h\mathbf{r})}{h}\right| \leq M$ for all $\mathbf{r} \in \mathbb{R}^k$ such that $\|\mathbf{r}\| = 1$. Then π is Lipschitz continuous. Furthermore, this condition holds for π^1 and π^2 and π^1, π^2 are Lipschitz continuous [18, Theorem 2.9].*
(iv) *If π is continuous piecewise linear*[8]*, then π, π^1, π^2 are all Lipschitz continuous [19, Lemma 1.4].*
(v) *Suppose $k = 1$, i.e., $\pi \colon \mathbb{R} \to \mathbb{R}_+$ and π is piecewise linear*[9] *and continuous from the right at 0 or continuous from the left at 0.*[10] *Then π^1 and π^2 are continuous at all points at which π is continuous [39, Theorem 2].*

To prove that a valid inequality is a facet, the main tool is the so-called *Facet Theorem*, originally proved by Gomory and Johnson [49] for the one-row

[7] When π is a discontinuous piecewise linear function, subadditivity gives certain relations on the limit values of the function. We omit this more subtle discussion in this survey; see [18] for more details.

[8] See subsection 3.1 for the definition that we use.

[9] See subsection 3.1 for the definition that we use, which includes certain discontinuous functions.

[10] This condition is also not always true for piecewise linear functions. See Table 4 for examples of extreme functions that are discontinuous on both sides of the origin. The condition of one-sided continuity at the origin cannot be removed from the hypothesis of Lemma 2.11 (v) (New result ♣). This is illustrated by example `zhou_two_sided_discontinuous_cannot_assume_any_continuity`, constructed by Zhou (2014, unpublished).

case; it extends verbatim to the k-row case.[11] We present a stronger version of the theorem, which first appeared in [21].[12]

Theorem 2.12 (Facet Theorem [49], [21, Theorem 3.1]) *Let π be a minimal valid function. Suppose for every minimal valid function π', $E(\pi) \subseteq E(\pi')$ implies $\pi' = \pi$. Then π is a facet.*

In the light of Lemma 2.11 and Theorem 2.12, if one can establish that for a minimal valid function π, $E(\pi) \subseteq E(\pi')$ implies $\pi' = \pi$ for every minimal valid function π', then π is extreme, as well as a facet. Indeed, if $\pi = \frac{1}{2}(\pi^1 + \pi^2)$ where π^1, π^2 are valid functions, by Lemma 2.11 (i), π^1 and π^2 are minimal and by Lemma 2.11 (ii), $E(\pi) \subseteq E(\pi^1) \cap E(\pi^2)$, and so $\pi = \pi^1 = \pi^2$. The facetness follows directly from Theorem 2.12, and gives an alternate proof of extremality since all facets are extreme.

The condition that $E(\pi) \subseteq E(\pi')$ implies $\pi' = \pi$ for every minimal valid function π' is established along the following lines. First, structural properties of π can be used to obtain a structured description of $E(\pi)$. For example, the fact that π is piecewise linear often shows that $E(\pi)$ is the union of many full-dimensional convex sets. $E(\pi')$ shares this structure with $E(\pi)$ because of the assumption that $E(\pi) \subseteq E(\pi')$. Then, results such as the *Interval Lemma*, discussed in section 4, are used to show that π' must be affine on the set of points contributing to $E(\pi')$. Finally, the conditions that all minimal valid functions are 0 at the origin and 1 at $\mathbf{f} + \mathbb{Z}^k$ puts further restrictions on the values that π' can take, and ultimately force $\pi' = \pi$.[13]

2.4 Classification and taxonomy of facets and extreme functions

The main goal in the study of the infinite group problem is to obtain a classification of facets and extreme valid functions. We do not believe that a simple classification exists like Theorem 2.6 for minimal valid functions. In spite of this, several beautiful theorems have been obtained regarding the structure of facets and extreme valid functions, and there is a lot more to be discovered. This survey attempts to highlight the most important known results in this research area and outline some of the challenging open problems.

Inspired by the survey by Richard and Dey [65, p. 786], we provide an updated compendium, or "taxonomy," of known extreme functions at the end of this survey (Appendix A). The focus lies on the case of the one-row ($k = 1$) infinite group problem, $R_f(\mathbb{R}, \mathbb{Z})$, for which many types of extreme functions have been discovered and analyzed (Table 1, 2, 3, 4). Also a number of "procedures"

[11] Gomory and Johnson's original proof actually holds only for weak facets, and not for facets as claimed in [49].

[12] In contrast to Gomory–Johnson's Facet Theorem, the condition that $E(\pi) \subseteq E(\pi')$ implies $\pi' = \pi$ only needs to be tested on minimal valid functions, not all valid functions.

[13] Sometimes certain continuity arguments need to be made, where results like Lemma 2.11 (iii), (iv) and (v) are helpful. In such situations, the proof of extremality is usually slightly simpler than a proof for facetness, owing to Lemma 2.11 (iii); see Theorem 5.5 and Theorem 6.4.

(operations) have been studied in the literature that preserve extremality under some conditions; we present these in Table 5.

We do not provide explicit constructions or descriptions of these functions here. Instead, we invite the interested reader to investigate the functions in an interactive companion program [56], including the electronic compendium of extreme functions [70]. The program and the electronic compendium are implemented in the free (open-source) computer algebra package Sage [67].[14]

Most facets and extreme functions described in the literature are piecewise linear functions.[15] The number of slopes (i.e., the different values of the derivative) of a function is a statistic that has received much attention in the literature. In fact, one of the classic results in the study of extreme functions for the single-row problem is the following:[16]

Theorem 2.13 (Gomory–Johnson 2-Slope Theorem [47]) *If a continuous piecewise linear minimal function of $R_f(\mathbb{R}, \mathbb{Z})$ has only 2 values for the derivative wherever it exists (2 slopes), then the function is extreme.*

Among the types of extreme functions that are piecewise linear functions, there are discontinuous and continuous ones. In the single-row case ($k = 1$), continuous piecewise linear extreme functions with 2, 3 and 4 different slopes were previously known, and discontinuous piecewise linear extreme functions with 1 and 2 slopes were previously known. Moreover, all previously known examples of extreme discontinuous functions were continuous on one side of the origin. Hildebrand (2013, unpublished) found continuous piecewise linear extreme functions with 5 slopes using computer-based search, as well as various discontinuous piecewise linear extreme functions. Köppe and Zhou [61] later found continuous piecewise linear extreme functions with up to 28 slopes.

Proposition 2.14 (New result ♣) *There exist continuous piecewise linear extreme functions with 5, 6, 7, and 28 slopes. There exist discontinuous piecewise linear extreme functions with 3 slopes and discontinuous piecewise linear extreme functions that are discontinuous on both sides at the origin. See Table 4.*

This prompts the following question.

Open question 2.15 *For the single-row problem $R_f(\mathbb{R}, \mathbb{Z})$, do there exist continuous and discontinuous extreme functions with s slopes for every $s \geq 2$?*

The additivity domain $E(\pi)$ for any minimal function π (see (2.8)) can be decomposed as the union of its maximal convex subsets. The first 5-slope func-

[14] The program [56] can be run on a local installation of Sage, or online via SageMathCloud. The help system provides a discussion of parameters of the extreme functions, bibliographic information, etc. It is accessed by typing the function name as shown in the table, followed by a question mark. Example: `gmic?`

[15] See subsection 3.1 for the definition that we use, which includes certain discontinuous functions.

[16] See Theorem 5.1 for a general k-row result.

tions found by Hildebrand (2013) have an additivity domain which contains lower-dimensional maximal convex components.[17] This begs the question:

Open question 2.16 *For the single-row problem $R_f(\mathbb{R},\mathbb{Z})$, do there exist continuous piecewise linear extreme functions of $R_f(\mathbb{R},\mathbb{Z})$ with s slopes such that $E(\pi)$ is the union of full-dimensional convex sets for every $s \geq 2$?*

Not all facets and extreme functions are piecewise linear though. Basu, Conforti, Cornuéjols, and Zambelli [15] constructed a family of facets that are not piecewise linear, yet the derivatives (where they exist) only take 2 values; see subsection 6.4. A function $\tilde{\pi}$ from this family is absolutely continuous and therefore it is differentiable almost everywhere (a.e.). The derivative $\tilde{\pi}'$ happens to take only two different values a.e., so $\tilde{\pi}$ is a "generalized 2-slope function." This suggests the following refined version of Gomory and Johnson's original piecewise linear conjecture for extreme functions.

Conjecture 2.17 For every absolutely continuous extreme function $\pi\colon \mathbb{R} \to \mathbb{R}$, the derivative π' is a simple function. Thus, there exists a finite partition of $\mathbb{R}$ into measureable subsets $M_0, \dots, M_t$ such that M_0 is of measure zero and π' is constant over each of $M_1, \dots, M_t$.

The fact that the derivative of the counterexample from [15] happens to take only two different values a.e. also gives rise to the following *generalized 2-slope conjecture*. This conjecture would generalize Theorem 2.13.

Conjecture 2.18 Let $\pi\colon \mathbb{R} \to \mathbb{R}$ be a minimal function that is absolutely continuous and whose derivative π' only takes two values outside of a set of measure zero. Then π is extreme.

The key difficulty in answering the above questions is that the tools of functional equations (such as the *Interval Lemma* as discussed in section 4) no longer directly apply and new tools will most likely need to be employed for the resolution. Thus, there are still substantial questions left to be explored, even for the single-row ($k = 1$) problem.

Much less is known about the k-row problem $R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$ for general k. Dey and Richard [38] pioneered the construction of extreme functions for the k-row problem. Their *sequential-merge* procedure constructs extreme functions and facets for $k \geq 2$ dimensions by combining extreme functions and facets for smaller k; see subsection 5.2. As mentioned earlier, a breakthrough was made when Theorem 5.1 was proved in [21, Theorem 1.7], generalizing Gomory and Johnson's single-row result (Theorem 2.13) to the general k-row problem, giving a very general sufficient condition for extremality and facetness.

[17] The functions are available in the electronic compendium [70] as `hildebrand_5_slope...`

3 The k-dimensional theory of piecewise linear minimal valid functions

3.1 Polyhedral complexes and piecewise linear functions

We introduce the notion of polyhedral complexes, which serves two purposes. First, it provides a framework to define piecewise linear functions, generalizing the familiar situation of functions of a single real variable. Second it is a tool for studying subadditivity and additivity relations of these functions. This exposition follows [19].

Definition 3.1 A (locally finite) *polyhedral complex* is a collection $\mathcal{P}$ of polyhedra in $\mathbb{R}^k$ such that:

(i) $\emptyset \in \mathcal{P}$,
(ii) if $I \in \mathcal{P}$, then all faces of I are in $\mathcal{P}$,
(iii) the intersection $I \cap J$ of two polyhedra $I, J \in \mathcal{P}$ is a face of both I and J,
(iv) any compact subset of $\mathbb{R}^k$ intersects only finitely many faces in $\mathcal{P}$.

A polyhedron I from $\mathcal{P}$ is called a *face* of the complex. A polyhedral complex $\mathcal{P}$ is said to be *pure* if all its maximal faces (with respect to set inclusion) have the same dimension. In this case, we call the maximal faces of $\mathcal{P}$ the *cells* of $\mathcal{P}$. The zero-dimensional faces of $\mathcal{P}$ are called *vertices* and the set of vertices of $\mathcal{P}$ will be denoted by $\operatorname{vert}(\mathcal{P})$. A polyhedral complex $\mathcal{P}$ is said to be *complete* if the union of all faces of the complex is $\mathbb{R}^k$. A pure and complete polyhedral complex $\mathcal{P}$ is called a *triangulation* of $\mathbb{R}^k$ if every maximal cell is a simplex.

Example 3.2 (Breakpoint intervals in $\mathbb{R}^1$ [18]) Let $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ be a list of "breakpoints" in $[0, 1]$. We extend it periodically as $B = \{\, x_0 + t, x_1 + t, \ldots, x_{n-1} + t \mid t \in \mathbb{Z} \,\}$. Define the set of 0-dimensional faces to be the collection of singletons, $\mathcal{P}_{B,\bullet} = \big\{ \{x\} \mid x \in B \big\}$, and the set of one-dimensional faces to be the collection of closed intervals, $\mathcal{P}_{B,\vdash\!\dashv} = \big\{ [x_i + t, x_{i+1} + t] \mid i = 0, \ldots, n-1 \text{ and } t \in \mathbb{Z} \big\}$. Then $\mathcal{P}_B = \{\emptyset\} \cup \mathcal{P}_{B,\bullet} \cup \mathcal{P}_{B,\vdash\!\dashv}$ is a locally finite polyhedral complex.

Example 3.3 (Standard triangulations of $\mathbb{R}^2$ [19]) Let q be a positive integer. Consider the arrangement $\mathcal{H}_q$ of all hyperplanes (lines) of $\mathbb{R}^2$ of the form $\binom{0}{1} \cdot \mathbf{x} = b$, $\binom{1}{0} \cdot \mathbf{x} = b$, and $\binom{1}{1} \cdot \mathbf{x} = b$, where $b \in \frac{1}{q}\mathbb{Z}$. The complement of the arrangement $\mathcal{H}_q$ consists of two-dimensional cells, whose closures are the triangles

$$_{\mathbf{0}}\blacktriangle = \tfrac{1}{q}\operatorname{conv}(\{\tbinom{0}{0}, \tbinom{1}{0}, \tbinom{0}{1}\}) \qquad \text{and} \qquad _{\mathbf{0}}\blacktriangledown = \tfrac{1}{q}\operatorname{conv}(\{\tbinom{1}{0}, \tbinom{0}{1}, \tbinom{1}{1}\})$$

and their translates by elements of the lattice $\frac{1}{q}\mathbb{Z}^2$. We denote by $\mathcal{P}_q$ the collection of these triangles and the vertices and edges that arise as intersections of the triangles, and the empty set. Thus $\mathcal{P}_q$ is a locally finite polyhedral complex. Since all nonempty faces of $\mathcal{P}_q$ are simplices, it is a triangulation of the space $\mathbb{R}^2$.

We give a precise definition of affine linear functions over a domain, suitable for the general k-dimensional case.

Definition 3.4 Let $U \subseteq \mathbb{R}^k$. We say $\pi\colon U \to \mathbb{R}$ is *affine (or affine linear)* over U if there exists a *gradient* $\mathbf{c} \in \mathbb{R}^k$ such that for any $\mathbf{u}_1, \mathbf{u}_2 \in U$ we have

$$\pi(\mathbf{u}_2) - \pi(\mathbf{u}_1) = \mathbf{c} \cdot (\mathbf{u}_2 - \mathbf{u}_1).$$

Given a pure and complete polyhedral complex $\mathcal{P}$, we call a function $\pi\colon \mathbb{R}^k \to \mathbb{R}$ *piecewise linear* over $\mathcal{P}$ if it is affine linear over the *relative interior* of each face of the complex. Under this definition, piecewise linear functions can be discontinuous. We say the function π is *continuous piecewise linear over* $\mathcal{P}$ if it is affine over each of the cells of $\mathcal{P}$ (thus automatically imposing continuity). Most of the results presented in this survey will be about continuous piecewise linear functions.

Motivated by Gomory–Johnson's characterization of minimal valid functions (Theorem 2.6), we are interested in functions $\pi\colon \mathbb{R}^k \to \mathbb{R}$ that are periodic modulo $\mathbb{Z}^k$, i.e., for all $\mathbf{x} \in \mathbb{R}^k$ and all vectors $\mathbf{t} \in \mathbb{Z}^k$, we have $\pi(\mathbf{x} + \mathbf{t}) = \pi(\mathbf{x})$. If π is periodic modulo $\mathbb{Z}^k$ and continuous piecewise linear over a pure and complete complex $\mathcal{P}$, then we can assume without loss of generality that $\mathcal{P}$ is also *periodic modulo* $\mathbb{Z}^k$, i.e., for all $I \in \mathcal{P}$ and all vectors $\mathbf{t} \in \mathbb{Z}^k$, the translated polyhedron $I + \mathbf{t}$ also is a face of $\mathcal{P}$. This is the case in Examples 3.2 and 3.3.

Remark 3.5 If all the cells of the polyhedral complex are bounded, the value of a continuous piecewise linear function at any point $\mathbf{x}$ can be obtained by interpolating the values of the function at the vertices of the minimal face containing $\mathbf{x}$. This is utilized in subsection 8.2. The assumption of boundedness of the cells can be made without loss of generality; see subsection 3.3. Moreover, for a periodic continuous piecewise linear function over a periodic complex, we can give a finite description for π by further restricting to the values in $\operatorname{vert}(\mathcal{P}) \cap D$ where $D = [0,1]^k$ or any set such that $D + \mathbb{Z}^k = \mathbb{R}^k$. The finiteness of the set $\operatorname{vert}(\mathcal{P}) \cap D$ is guaranteed by the assumption of local finiteness in Definition 3.1 (iv).

3.2 The extended complex $\Delta\mathcal{P}$

For any $I, J, K \subseteq \mathbb{R}^k$, we define the set

$$F(I,J,K) = \left\{\, (\mathbf{x},\mathbf{y}) \in \mathbb{R}^k \times \mathbb{R}^k \;\middle|\; \mathbf{x} \in I,\ \mathbf{y} \in J,\ \mathbf{x}+\mathbf{y} \in K \,\right\}. \tag{3.9}$$

When I, J, K are polyhedra, $F(I,J,K)$ is also a polyhedron. Let $\mathcal{P}$ be a pure, complete polyhedral complex of $\mathbb{R}^k$ and let π be a continuous piecewise linear function over $\mathcal{P}$. In order to study the additivity domain $E(\pi)$, we define the family of polyhedra in $\mathbb{R}^k \times \mathbb{R}^k$,

$$\Delta\mathcal{P} = \{\, F(I,J,K) \mid I,J,K \in \mathcal{P} \,\},$$

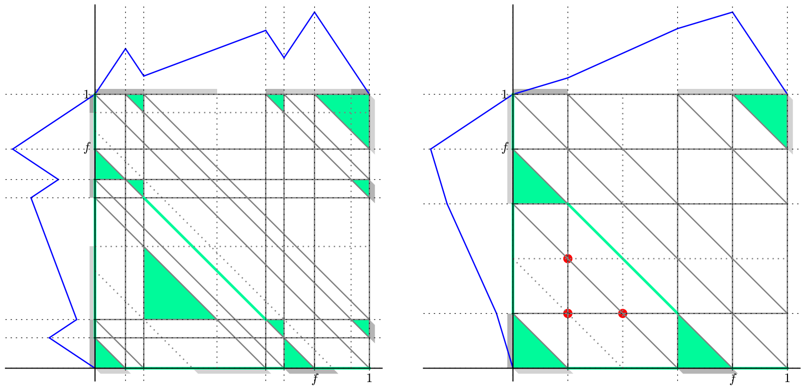


Fig. 4 Two diagrams of a function (*blue graphs on the top and the left*) and its polyhedral complex $\Delta\mathcal{P}$ (*gray solid lines*), as plotted by the command `plot_2d_diagram(h)`. *Left*, `h = gj_forward_3_slope()` *(left)*. *Right*, `h = not_minimal_2()`. The set $E(\pi)$ in both cases is the union of the faces shaded in green. The *heavy diagonal green line* $x+y=f$ corresponds to the symmetry condition. Vertices of $\Delta\mathcal{P}$ do not necessarily project (*dotted gray lines*) to breakpoints; compare with Figure 7. Vertices of the complex on which $\Delta\pi < 0$, i.e., subadditivity is violated, are shown as *red dots*; see Theorem 3.11. At the borders of each diagram, the projections $p_i(F)$ of two-dimensional additive faces are shown as *gray shadows*: $p_1(F)$ at the top border, $p_2(F)$ at the left border, $p_3(F)$ at the bottom and the right borders.

which is also polyhedral complex [19, Lemma 3.6]; see Figure 4.

Define the projections $p_1, p_2, p_3 \colon \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ as

$$p_1(\mathbf{x}, \mathbf{y}) = \mathbf{x}, \quad p_2(\mathbf{x}, \mathbf{y}) = \mathbf{y}, \quad p_3(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y}; \tag{3.10}$$

see Figure 5. Now let $I, J, K \subseteq \mathbb{R}^k$ and let $F = F(I, J, K)$. Simple formulas for the projections of F are available [19, Proposition 3.3]:

$$I' := p_1(F(I, J, K)) = (K + (-J)) \cap I \subseteq I, \tag{3.11a}$$
$$J' := p_2(F(I, J, K)) = (K + (-I)) \cap J \subseteq J, \tag{3.11b}$$
$$K' := p_3(F(I, J, K)) = (I + J) \cap K \quad\quad \subseteq K. \tag{3.11c}$$

The inclusions $I' \subseteq I$, $J' \subseteq J$, $K' \subseteq K$ may be strict. This possibility is illustrated by the largest shaded triangle in Figure 4 (left). We see that the projections I', J', K' give us a canonical, minimal way of representing F as $F(I', J', K')$ [19, Lemma 3.5]. Note that I', J', K' are not faces of $\mathcal{P}$ in general, even if I, J, K were faces; see again Figure 4 (left).

We will study the function $\Delta\pi \colon \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$, as defined in (2.7), which measures the slack in the subadditivity constraints. When π is continuous piecewise linear over $\mathcal{P}$, we have that $\Delta\pi$ is continuous piecewise linear over $\Delta\mathcal{P}$ (Lemma 3.7 in [19]).

Remark 3.6 If π and $\mathcal{P}$ are periodic modulo $\mathbb{Z}^k$, then $\Delta\pi$ and $\Delta\mathcal{P}$ are periodic modulo $\mathbb{Z}^k \times \mathbb{Z}^k$. Echoing Remark 3.5, one can make the description of $\Delta\pi$ finite by recording the values of $\Delta\pi$ on a smaller set; for example, the set

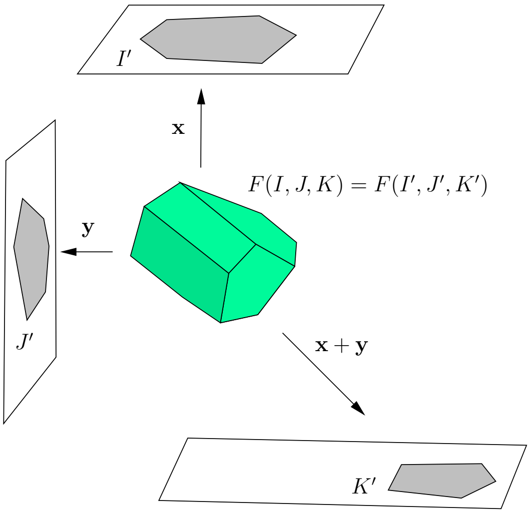


Fig. 5 A face $F = F(I, J, K)$ and its projections $I' = p_1(F)$, $J' = p_2(F)$, $K' = p_3(F)$. This is an abstract picture; note that if $I', J', K' \subseteq \mathbb{R}^2$ are full-dimensional, then $F(I, J, K)$ is actually a full-dimensional set of $\mathbb{R}^4$.

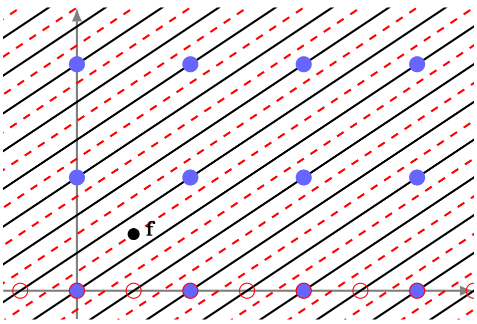


Fig. 6 A piecewise linear function $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ defined by interpolation between values of 0 in on the black solid lines and 1 on the dashed red lines. The blue dots depict the lattice $\mathbb{Z}^2$. In particular, $\pi(x, 0) = \min(4x, 2 - 4x)$ for $x \in [0, \frac{1}{2}]$. This is extended to the x-axis by $\pi(x, 0) = \pi(x \bmod \frac{1}{2}, 0)$. The points $(x, 0)$ for $x \in \frac{1}{2}\mathbb{Z}$ are shown as red circles. Finally, we can write $\pi(x, y) = \pi(\frac{1}{2}(2x - 3y), 0)$ for all $(x, y) \in \mathbb{R}^2$. This function is not genuinely two-dimensional, which is demonstrated by a function $\phi\colon \mathbb{R} \to \mathbb{R}$ and a linear map $T\colon \mathbb{R}^2 \to \mathbb{R}$ such that $\pi = \phi \circ T$. Many choices for this pair ϕ, T are possible. For $\phi(t) = \pi(\frac{1}{2}t, 0)$ and $T(x, y) = 2x - 3y$, we have $T\mathbb{Z}^2 = \mathbb{Z}$, which satisfies the conditions in Proposition 3.8.

$\operatorname{vert}(\Delta\mathcal{P}) \cap ([0,1]^k \times [0,1]^k)$. One may also replace $[0,1]^k \times [0,1]^k$ by $D \times D$ for any D satisfying $D + \mathbb{Z}^k = \mathbb{R}^k$.

3.3 Genuinely k-dimensional functions

In this subsection we show that when analyzing minimal functions it suffices to consider "full-dimensional" minimal functions. We formalize this in the following definition and proposition.

Definition 3.7 A function $\pi\colon \mathbb{R}^k \to \mathbb{R}$ is *genuinely k-dimensional* if there does not exist a function $\phi\colon \mathbb{R}^{k-1} \to \mathbb{R}$ and a linear map $T\colon \mathbb{R}^k \to \mathbb{R}^{k-1}$ such that $\pi = \phi \circ T$.

An example of a function that is not genuinely k-dimensional is described in Figure 6.

Proposition 3.8 (Dimension reduction; [19, Proposition B.9]) *Let $\mathcal{P}$ be a pure and complete polyhedral complex in $\mathbb{R}^k$ that is periodic modulo $\mathbb{Z}^k$. Let $\pi\colon \mathbb{R}^k \to \mathbb{R}$ be a continuous piecewise linear function over $\mathcal{P}$, such that π is nonnegative, subadditive, periodic modulo $\mathbb{Z}^k$ and $\pi(\mathbf{0}) = 0$. If π is not genuinely k-dimensional, then there exists a natural number $0 \leq \ell < k$, a pure and complete polyhedral complex $\mathcal{X}$ in $\mathbb{R}^\ell$ that is periodic modulo $\mathbb{Z}^\ell$, a nonnegative and subadditive function $\phi\colon \mathbb{R}^\ell \to \mathbb{R}$ that is continuous piecewise linear over $\mathcal{X}$, and a point $\mathbf{f}' \in \mathbb{R}^\ell \setminus \mathbb{Z}^\ell$ with the following properties:*

1. *π is minimal for $R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$ if and only if ϕ is minimal for $R_{\mathbf{f}'}(\mathbb{R}^\ell, \mathbb{Z}^\ell)$.*
2. *π is extreme for $R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$ if and only if ϕ is extreme for $R_{\mathbf{f}'}(\mathbb{R}^\ell, \mathbb{Z}^\ell)$.*

The above idea first appears in [37, Construction 6.3], where the authors give a construction to obtain two-dimensional minimal functions from one-dimensional minimal functions, and show that all minimal functions for $k = 2$ with 2 slopes can be obtained using such a construction [37, Theorem 6.4]. The construction is exactly via the use of a linear map as described in Definition 3.7. In fact, their result is a special case of Proposition 3.8 and the simple observation that subadditive, genuinely k-dimensional functions have at least $k + 1$ slopes or gradient values (see also the conclusion of Theorem 5.1).

Remark 3.9 (Dimension reduction; [19, Remark B.10]) Using Proposition 3.8, the extremality/minimality question for π that is not genuinely k-dimensional can be reduced to the same question for a lower-dimensional genuinely ℓ-dimensional function with $\ell < k$. When $\mathcal{P}$ is a rational polyhedral complex, this reduction can be done algorithmically.

Next, we show that genuinely k-dimensional functions that are continuous piecewise linear enjoy several regularity properties which can often simplify the investigation of minimal valid functions that are continuous piecewise linear functions.

Theorem 3.10 ([19, Theorem B.11]) *Let $\mathcal{P}$ be a pure and complete polyhedral complex in $\mathbb{R}^k$ that is periodic modulo $\mathbb{Z}^k$. Let $\theta\colon \mathbb{R}^k \to \mathbb{R}$ be a minimal valid function for $R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$ that is continuous piecewise linear over $\mathcal{P}$, and is genuinely k-dimensional. Then,*

(i) $\mathbf{f} \in \operatorname{vert}(\mathcal{P})$.
(ii) The cells of $\mathcal{P}$ and $\Delta\mathcal{P}$ are full-dimensional polytopes.

3.4 Finite test for minimality

One of the main advantages of working with minimal valid functions that are piecewise linear is their combinatorial structure, which avoids many analytical complexities. Moreover, it is possible to give a finite description of π. For example, it suffices to know the values of π on the unit hypercube $D = [0,1]^k$, which can in turn be broken into a finite number of polytopes over which π is simply an affine function. Of course, any choice of D such that $D + \mathbb{Z}^k = \mathbb{R}^k$ suffices to obtain such a finite description, and $D = [0,1]^k$ is just one such choice. In certain situations, other choices of D may be more natural, and provide a shorter description.

By Theorem 2.6, we can test whether a periodic function is minimal by testing subadditivity, the symmetry condition, and the value at the origin. These properties are easy to test when the function is continuous piecewise linear. The first of such tests came from Gomory and Johnson [49, Theorem 7] for the case $k = 1$.[18] Richard, Li, and Miller [66, Theorem 22] extended it to the case of discontinuous piecewise linear functions.[19] A test for subadditivity of continuous piecewise linear functions for the two-row problem was given in [37, Proposition 10] that reduces to testing subadditivity at vertices, edges, and the so-called *supplemental vertices.* We present a minimality test for continuous piecewise linear functions for general k. To simplify notation, we restrict ourselves to the continuous case.[20] The test is stated in terms of the set of vertices $\operatorname{vert}(\Delta\mathcal{P})$ of the complex $\Delta\mathcal{P}$; see again Figure 4 for an illustration. This uses the observation made in Remark 3.5 that the function values for a continuous piecewise linear function can be obtained by interpolating the values at $\operatorname{vert}(\mathcal{P})$.[21]

Theorem 3.11 (Minimality test [19, Theorem 3.10, Remark 3.11]) *Let $\mathcal{P}$ be a pure, complete, polyhedral complex in $\mathbb{R}^k$ that is periodic modulo $\mathbb{Z}^k$ and every cell of $\mathcal{P}$ is bounded.*[22] *Let $\Delta D = [0,1]^k \times [0,1]^k$.*[23] *Let $\pi\colon \mathbb{R}^k \to \mathbb{R}$*

[18] Note that in [49], the word "minimal" needs to be replaced by "satisfies the symmetry condition" throughout the statement of their theorem and its proof.

[19] They present it in a setting of pseudo-periodic superadditive functions, rather than periodic subadditive functions.

[20] A discontinuous version of Theorem 3.11 appears in [18, Theorem 2.5], where it is stated for the case $k = 1$; it extends verbatim to general k. All relevant limits of the function at discontinuities are taken care of by testing

$$\Delta\pi_F(u,v) = \lim_{\substack{(x,y)\to(u,v)\\ (x,y)\in \operatorname{rel\,int}(F)}} \Delta\pi(x,y) \tag{3.12}$$

for all faces $F \in \Delta\mathcal{P}$ that contain the vertex (u,v). For $k = 1$, by analyzing the possible faces F, one recovers the explicit limit relations stated in [66, Theorem 22].

[21] A different approach is taken in [37, Proposition 10] where the subadditivity test uses so-called *supplemental vertices* which are introduced to get around the problem of unbounded cells.

[22] This is not restrictive due to Theorem 3.10(ii) and Proposition 3.8(1).

[23] Instead of $\Delta D = [0,1]^k \times [0,1]^k$, one can choose $\Delta D = D \times D$ for any D such that $D + \mathbb{Z}^k = \mathbb{R}^k$; see the discussion in [19].

be a nonnegative continuous piecewise linear function over $\mathcal{P}$ that is periodic modulo $\mathbb{Z}^k$. Let $\mathbf{f} \in \operatorname{vert}(\mathcal{P})$.[24] *Then π is minimal for $R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$ if and only if the following conditions hold:*

1. $\pi(\mathbf{0}) = 0$,
2. *Subadditivity test:* $\Delta\pi(\mathbf{u}, \mathbf{v}) \geq 0$ *for all* $(\mathbf{u}, \mathbf{v}) \in \Delta D \cap \operatorname{vert}(\Delta\mathcal{P})$.
3. *Symmetry test:* $\pi(\mathbf{f}) = 1$ *and*

$$\Delta\pi(\mathbf{u}, \mathbf{v}) = 0 \quad \text{for all} \quad (\mathbf{u}, \mathbf{v}) \in \Delta D \cap \operatorname{vert}(\Delta\mathcal{P}) \quad \text{with} \quad \mathbf{u}+\mathbf{v} \equiv \mathbf{f} \pmod{\mathbb{Z}^k}. \tag{3.13}$$

Here $\pmod{\mathbb{Z}^k}$ *denotes componentwise equivalence modulo* 1.

3.5 Combinatorializing the additivity domain

Let $\pi \colon \mathbb{R}^k \to \mathbb{R}$ be a continuous piecewise linear function over a pure, complete polyhedral complex $\mathcal{P}$. Recall the definition of the *additivity domain* of π,

$$E(\pi) = \{\, (\mathbf{x}, \mathbf{y}) \mid \Delta\pi(\mathbf{x}, \mathbf{y}) = 0 \,\}.$$

We now give a combinatorial representation of this set using the faces of $\mathcal{P}$. Let

$$E(\pi, \mathcal{P}) = \{\, F \in \Delta\mathcal{P} \mid \Delta\pi|_F \equiv 0 \,\}.$$

We consider $E(\pi, \mathcal{P})$ to include $F = \emptyset$, on which $\Delta\pi|_F \equiv 0$ holds trivially. Then $E(\pi, \mathcal{P})$ is another polyhedral complex, a subcomplex of $\Delta\mathcal{P}$. As mentioned, if π is continuous, then $\Delta\pi$ is continuous. Under this continuity assumption, we can consider only the set of maximal faces in $E(\pi, \mathcal{P})$. We define

$$E_{\max}(\pi, \mathcal{P}) = \bigl\{\, F \in E(\pi, \mathcal{P}) \bigm| F \text{ is a maximal face by set inclusion in } E(\pi, \mathcal{P}) \,\bigr\}.$$

Lemma 3.12 ([19, Lemma 3.12])

$$E(\pi) = \bigcup \{F \in E(\pi, \mathcal{P})\} = \bigcup \{F \in E_{\max}(\pi, \mathcal{P})\}.$$

This combinatorial representation can then be made finite by choosing representatives as in Remark 3.6.

[24] For $k = 1$, necessarily $\mathbf{f} \in \operatorname{vert}(\mathcal{P})$ [18, Lemma 2.4]. The same is true for genuinely k-dimensional functions (Theorem 3.10). If, however, $\mathbf{f} \notin \operatorname{vert}(\mathcal{P})$, then the condition (3.13) in the symmetry test must be replaced by a slightly more complicated condition (as stated in [19, Theorem 3.10, Remark 3.11]). Let $S = \{\, (\mathbf{u}, \mathbf{v}) \mid \mathbf{u} + \mathbf{v} \equiv \mathbf{f} \pmod{\mathbf{1}} \,\}$. Then $\Delta\mathcal{P} \cap S := \{\, F \cap S : F \in \Delta\mathcal{P} \,\}$ is again a polyhedral complex. The condition (3.13) is then replaced by:

$$\Delta\pi(\mathbf{u}, \mathbf{v}) = 0 \quad \text{for all} \quad (\mathbf{u}, \mathbf{v}) \in \Delta D \cap \operatorname{vert}(\Delta\mathcal{P} \cap S).$$

3.6 Perturbation functions

We now discuss how to prove that a given minimal function is not a facet or not extreme. We consider the *space of perturbation functions* with prescribed additivities $E \subseteq G \times G$

$$\bar{\Pi}^E(G,S) = \left\{ \bar\pi\colon G \to \mathbb{R} \;\middle|\; \begin{array}{rl} \bar\pi(\mathbf{0}) = 0 & \\ \bar\pi(\mathbf{f}) = 0 & \\ \bar\pi(\mathbf{x}) + \bar\pi(\mathbf{y}) = \bar\pi(\mathbf{x}+\mathbf{y}) & \text{for all } (\mathbf{x},\mathbf{y}) \in E \\ \bar\pi(\mathbf{x}) = \bar\pi(\mathbf{x}+\mathbf{t}) & \text{for all } \mathbf{x} \in G,\, \mathbf{t} \in S \end{array} \right\}. \tag{3.14}$$

Later we will use this notation even if G is not a group and only require that $\mathbf{0}, \mathbf{f} \in G$, and $S \subseteq G$. Clearly $\bar{\Pi}^E(G,S)$ is a linear space.

The third condition implies that $E \subseteq E(\bar\pi)$ for all $\bar\pi \in \bar{\Pi}^E(G,S)$. From Lemma 2.11 it follows that π is not extreme if and only if there exists a $\bar\pi \in \Pi^{E(\pi)}(G,S) \setminus \{0\}$ such that $\pi^1 = \pi + \bar\pi$ and $\pi^2 = \pi - \bar\pi$ are minimal valid functions. In a similar vein, if π is not a facet of $R_{\mathbf{f}}(G,S)$, then by the Facet Theorem, Theorem 2.12, there exists a nontrivial $\bar\pi \in \bar{\Pi}^{E(\pi)}(G,S)$ such that $\pi' = \pi + \bar\pi$ is a minimal valid function. Note that this last statement is not an if and only if statement.

Suppose π is piecewise linear on a polyhedral complex $\mathcal{P}$. We will often consider a refinement $\mathcal{P}'$ of $\mathcal{P}$ on which we can find a continuous piecewise linear perturbation $\bar\pi$ such that π is not extreme.

The basic idea is that if one can find a non-zero function $\bar\pi$ in the linear subspace of functions $\bar{\Pi}^{E(\pi)}(\mathbb{R}^k, \mathbb{Z}^k)$ then the finite, combinatorial description of $\Delta\pi$ (since π and therefore $\Delta\pi$ is piecewise linear) allows small perturbations from π in the direction of $\bar\pi$ while maintaining minimality.

Theorem 3.13 (Perturbation [19, Theorem 3.13]) *Let $\mathcal{P}$ be a pure, complete, polyhedral complex in $\mathbb{R}^k$ that is periodic modulo $\mathbb{Z}^k$ and every cell of $\mathcal{P}$ is bounded. Suppose π is minimal and continuous piecewise linear over $\mathcal{P}$. Suppose $\bar\pi \not\equiv 0$ is continuous piecewise linear over a refinement $\mathcal{P}'$ of $\mathcal{P}$, is periodic modulo $\mathbb{Z}^k$ and satisfies $\bar\pi \in \bar{\Pi}^E(\mathbb{R}^k, \mathbb{Z}^k)$ where $E = E(\pi)$. Then π is not extreme. Furthermore, given $\bar\pi$, there exists an $\epsilon > 0$ such that $\pi^1 = \pi + \epsilon\bar\pi$ and $\pi^2 = \pi - \epsilon\bar\pi$ are distinct minimal functions that are continuous piecewise linear over $\mathcal{P}$ such that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$.*

When π is a continuous piecewise linear function over a polyhedral complex $\mathcal{P}$, for certain refinements $\mathcal{T}$ of $\mathcal{P}$ we can decompose perturbation functions $\bar\pi$ into piecewise linear perturbations over $\mathcal{T}$ and other perturbations that vanish on the vertices of $\mathcal{T}$. For a triangulation $\mathcal{T}$ define the vector spaces

$$\bar{\Pi}^E_{\mathcal{T}}(\mathbb{R}^k, \mathbb{Z}^k) := \{\, \bar\pi \in \bar{\Pi}^E(\mathbb{R}^k, \mathbb{Z}^k) \mid \bar\pi \text{ is continuous piecewise linear on } \mathcal{T} \,\}$$

and

$$\bar{\Pi}^E_{\operatorname{zero}(\mathcal{T})}(\mathbb{R}^k, \mathbb{Z}^k) := \{\, \bar\pi \in \bar{\Pi}^E(\mathbb{R}^k, \mathbb{Z}^k) \mid \bar\pi|_{\operatorname{vert}(\mathcal{T})} \equiv 0 \,\}.$$

Lemma 3.14 (New result ♣) *Suppose $\pi\colon \mathbb{R}^k \to \mathbb{R}$ is a minimal valid function that is piecewise linear over $\mathcal{P}$. Suppose $\mathcal{T}$ is a triangulation of $\mathbb{R}^k$ such that there exists $q \in \mathbb{N}$ such that $\operatorname{vert}(\mathcal{T}) = \frac{1}{q}\mathbb{Z}^k$ and $p_i(\operatorname{vert}(\Delta\mathcal{T})) \subseteq \frac{1}{q}\mathbb{Z}^k$ for $i = 1, 2, 3$ and $\mathbf{f} \in \frac{1}{q}\mathbb{Z}^k$. Let $E = E(\pi)$ and $E' = E(\pi) \cap \operatorname{vert}(\Delta\mathcal{T})$, and suppose $\mathcal{T}$ is a refinement of $\mathcal{P}$.*

1. *$\bar\pi \in \bar\Pi^E_{\mathcal{T}}(\mathbb{R}^k, \mathbb{Z}^k)$ if and only if $\bar\pi|_{\frac{1}{q}\mathbb{Z}^k} \in \bar\Pi^{E'}(\frac{1}{q}\mathbb{Z}^k, \mathbb{Z}^k)$,*
2. *For every $\bar\pi \in \bar\Pi^E(\mathbb{R}^k, \mathbb{Z}^k)$, there exist unique $\bar\pi_{\mathcal{T}} \in \Pi^E_{\mathcal{T}}(\mathbb{R}^k, \mathbb{Z}^k)$ and $\bar\pi_{\operatorname{zero}(\mathcal{T})} \in \bar\Pi^E_{\operatorname{zero}(\mathcal{T})}(\mathbb{R}^k, \mathbb{Z}^k)$ such that*

$$\bar\pi = \bar\pi_{\mathcal{T}} + \bar\pi_{\operatorname{zero}(\mathcal{T})}.$$

Proof Let $\bar\pi_{\mathcal{T}}$ be a continuous piecewise linear function over $\mathcal{T}$. Since $\mathcal{T}$ is a refinement of $\mathcal{P}$, we have that π is continuous piecewise linear over $\mathcal{T}$ as well. By Lemma 3.12, for any φ that is continuous piecewise linear on $\mathcal{T}$ we have that $E(\varphi) = \bigcup\{\, F \in \Delta\mathcal{T} \mid \Delta\varphi|_F \equiv 0 \,\}$. Since $\Delta\varphi$ is affine on F, we have that $\varphi|_F \equiv 0$ if and only if $\varphi|_{\operatorname{vert}(F)} \equiv 0$. Therefore, it follows that $E(\pi) \subseteq E(\bar\pi)$ if and only if $\Delta\pi|_{\operatorname{vert}(F)} \equiv 0$ implies that $\Delta\bar\pi|_{\operatorname{vert}(F)} \equiv 0$ for all $F \in \Delta\mathcal{T}$. Since $\operatorname{vert}(\mathcal{T}) = \frac{1}{q}\mathbb{Z}^k$, this establishes part (1).

Next, let $\bar\pi \in \bar\Pi^E(\mathbb{R}^k, \mathbb{Z}^k)$. Let $\bar\pi_{\mathcal{T}}$ be the unique extension of $\bar\pi|_{\frac{1}{q}\mathbb{Z}^k}$ to $\mathbb{R}^k$ via the triangulation $\mathcal{T}$. Note that $\bar\pi_{\mathcal{T}}$ is the unique piecewise linear function over $\mathcal{T}$ such that $(\bar\pi - \bar\pi_{\mathcal{T}})|_{\frac{1}{q}\mathbb{Z}^k} \equiv 0$. Define $\bar\pi_{\operatorname{zero}(\mathcal{T})} = \bar\pi - \bar\pi_{\mathcal{T}}$. It is left to show that $\bar\pi_{\mathcal{T}}, \bar\pi_{\operatorname{zero}(\mathcal{T})} \in \bar\Pi^E(\mathbb{R}^k, \mathbb{Z}^k)$.

Since $\bar\pi \in \bar\Pi^E(\mathbb{R}^k, \mathbb{Z}^k)$, it follows that $\bar\pi_{\mathcal{T}}|_{\frac{1}{q}\mathbb{Z}^k} = \bar\pi|_{\frac{1}{q}\mathbb{Z}^k} \in \bar\Pi^{E'}(\frac{1}{q}\mathbb{Z}^k, \mathbb{Z}^k)$. Therefore, by part (1), $\bar\pi_{\mathcal{T}} \in \bar\Pi^E(\mathbb{R}^k, \mathbb{Z}^k)$. Since $\bar\Pi^E(\mathbb{R}^k, \mathbb{Z}^k)$ is a vector space containing $\bar\pi$ and $\bar\pi_{\mathcal{T}}$, we have that $\bar\pi_{\operatorname{zero}(\mathcal{T})} = \bar\pi - \bar\pi_{\mathcal{T}} \in \bar\Pi^E(\mathbb{R}^k, \mathbb{Z}^k)$ which establishes part (2). □

Due to the decomposition in part (2) of Lemma 3.14, we can determine if a non-trivial perturbation function $\bar\pi \in \bar\Pi^{E(\pi)}(\mathbb{R}^k, \mathbb{Z}^k)$ exists by considering separately the spaces $\bar\Pi^{E(\pi)}_{\mathcal{T}}(\mathbb{R}^k, \mathbb{Z}^k)$ and $\bar\Pi^{E(\pi)}_{\operatorname{zero}(\mathcal{T})}(\mathbb{R}^k, \mathbb{Z}^k)$. This is used in a procedure to test extremality described in subsection 7.1.

Remark 3.15 The polyhedral complexes $\mathcal{P}_B$ for $B = \frac{1}{q}\mathbb{Z} \cap [0, 1)$ from Example 3.2 and $\mathcal{P}_q$ from Example 3.3 are triangulations of $\mathbb{R}^1$ and $\mathbb{R}^2$, respectively, and satisfy the hypotheses of Lemma 3.14. This fact can be seen in Figure 7 for the case of $\mathcal{P}_B$. The polyhedral complex $\mathcal{P}_q$ will be discussed more in section 7.

4 The Interval Lemma and its k-dimensional generalizations

In order to prove that a given minimal valid function π is a facet (or an extreme function), we make use of the additivity domain $E(\pi)$ of a subadditive function $\pi\colon \mathbb{R}^k \to \mathbb{R}$. As discussed in the roadmap (subsection 2.3), we would like to establish that $E(\pi) \subseteq E(\pi')$ implies $\pi = \pi'$ for every minimal valid

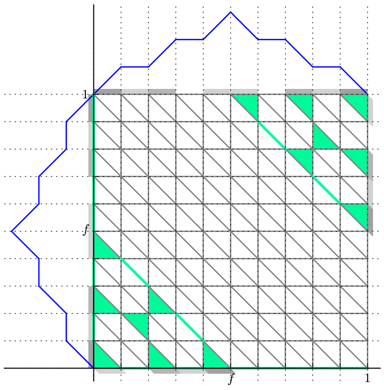


Fig. 7 Diagram of a function (*blue graphs on the top and the left*) on the evenly spaced complex $\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$ and the corresponding complex $\Delta\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$ (*gray solid lines*), as plotted by the command `plot_2d_diagram(h)`, where `h = not_extreme_1()`. Faces of the complex on which $\Delta\pi = 0$, i.e., additivity holds, are *shaded green*. The *heavy diagonal green lines* $x + y = f$ and $x + y = 1 + f$ correspond to the symmetry condition. At the borders, the projections $p_i(F)$ of two-dimensional additive faces are shown as *gray shadows*: $p_1(F)$ at the top border, $p_2(F)$ at the left border, $p_3(F)$ at the bottom and the right borders. Since the breakpoints of $\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$ are equally spaced, also $\Delta\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$ is very uniform, consisting only of points, lines, and triangles, and the projections are either a breakpoint in $\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$ or an interval in $\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$; compare with Figure 4.

function π'. An important ingredient in this step is to infer that π' is an affine function when restricted to projections of $E(\pi)$. For this purpose, it is convenient to separate the additivity domain into convex sets, which we then study independently. In the important case of continuous piecewise linear functions, we already know from subsection 3.5 that it suffices to study the maximal additive faces of the complex $\Delta\mathcal{P}$.

The primary object of investigation is the *functional equation* known as the *(additive) Cauchy functional equation*, which in its most general form is the study of real-valued functions θ satisfying

$$\theta(\mathbf{u}) + \theta(\mathbf{v}) = \theta(\mathbf{u} + \mathbf{v}), \quad (\mathbf{u}, \mathbf{v}) \in F \tag{4.15}$$

where F is some subset of $\mathbb{R}^k \times \mathbb{R}^k$. We focus on convex sets F that can be used as building blocks to cover $E(\pi)$ or other non-convex domains. The simplest convex sets F of $\mathbb{R}^k \times \mathbb{R}^k$ are direct (Cartesian) products $U \times V$, where U and V are convex sets of $\mathbb{R}^k$. For $k = 1$, this means we consider intervals $U \subseteq \mathbb{R}$ and $V \subseteq \mathbb{R}$ and set $F = U \times V$, i.e., we consider the functional equation $\theta(u) + \theta(v) = \theta(u + v)$ for all $u \in U$ and $v \in V$.

4.1 The classical case: Cauchy's functional equation

Classically (see, e.g., [1, Chapter 2]), (4.15) is studied for the case $F = \mathbb{R} \times \mathbb{R}$. In addition to the obvious *regular solutions* to (4.15), which are the (homogeneous) linear functions $\theta(x) = cx$, there exist certain pathological solutions, which are highly discontinuous [1, Chapter 2, Theorem 3]; these were used in Propositions 2.2 and 2.4. In order to rule out these solutions, one imposes a regularity hypothesis. Various such regularity hypotheses have been proposed in the literature; for example, it is sufficient to assume that the function θ is bounded on bounded intervals [1, Chapter 2, Theorem 8].

4.2 The bounded case: Gomory–Johnson's Interval Lemma in $\mathbb{R}^1$

The so-called Interval Lemma was introduced by Gomory and Johnson in [49] (the result appears implicitly in the proof of [48, Theorem 3.3]). This result concerns the Cauchy functional equation (4.15) on a bounded domain, i.e., the arguments u, v, and $u + v$ come from bounded intervals U, V, and their sum $U + V$, rather than the entire real line, i.e., additivity is on the set $F = U \times V$. In this case, we find that regular solutions are affine on these intervals; we lose homogeneity of the solutions. In fact, instead of equation (4.15), one can consider the more general equation $f(u) + g(v) = h(u + v)$, with three functions f, g, and h instead of one function θ.

Lemma 4.1 (Interval Lemma, [19, Lemma 2.2]) *Given real numbers $u_1 < u_2$ and $v_1 < v_2$, let $U = [u_1, u_2]$, $V = [v_1, v_2]$, and $U + V = [u_1 + v_1, u_2 + v_2]$. Let $f: U \to \mathbb{R}$, $g: V \to \mathbb{R}$, $h: U + V \to \mathbb{R}$ be bounded functions. If $f(u) + g(v) = h(u + v)$ for every $(u, v) \in U \times V$, then there exists $c \in \mathbb{R}$ such that $f(u) = f(u_1) + c(u - u_1)$ for every $u \in U$, $g(v) = g(v_1) + c(v - v_1)$ for every $v \in V$, $h(w) = h(u_1 + v_1) + c(w - u_1 - v_1)$ for every $w \in U + V$. In other words, f, g and h are affine with gradient c over U, V, and $U + V$ respectively.*

We provide a brief justification of this result under the assumption that f, g and h are in $C^2(\mathbb{R})$ (continuous first and second derivatives). We differentiate the relation $f(u) + g(v) = h(u + v)$ with respect to u (holding v fixed in the interval V) to obtain $f'(u) = h'(u + v)$ for all $u \in \operatorname{int}(U)$. Since the choice of v was arbitrary, this actually means $f'(u) = h'(u + v)$ for all $u \in \operatorname{int}(U)$ and $v \in \operatorname{int}(V)$. But then differentiating this relation with respect to v we obtain $0 = h''(u + v)$. This implies that h is affine over $U + V$, and f is affine with the same slope over U. Similarly, fixing u in U and differentiating with respect to v we obtain $g'(v) = h'(u + v)$ for all $v \in \operatorname{int}(V)$, implying that g is affine with the same slope over V. The result under the weaker assumption of boundedness of the functions is obtained by making a discrete version of these derivative arguments; the details are complicated and we refer the reader to [19, Lemma 2.2] for a full proof.

4.3 The full-dimensional Cartesian case: Higher-dimensional Interval Lemma

We now discuss generalization of the Interval Lemma (Lemma 4.1) presented in the previous section to the k-dimensional setting. The first higher dimensional versions of Lemma 4.1 in the literature appear in [31, 37] for the case of $k = 2$ and in [21] for general k, all of which apply when either U or V contains the origin. The result in [37] applies allows also for so-called *star-shaped* sets that contain the origin. We will follow the results of [19], which all for more general types of convex sets. Similar proofs of these results allow for star-shaped sets as well, but this is not presented here.

Theorem 4.2 (Higher-dimensional Interval Lemma, full-dimensional version [19, Theorem 1.6]) *Let $f, g, h\colon \mathbb{R}^k \to \mathbb{R}$ be bounded functions. Let U and V be convex subsets of $\mathbb{R}^k$ such that $f(\mathbf{u}) + g(\mathbf{v}) = h(\mathbf{u} + \mathbf{v})$ for all $(\mathbf{u}, \mathbf{v}) \in U \times V$. Assume that $\operatorname{aff}(U) = \operatorname{aff}(V) = \mathbb{R}^k$. Then there exists a vector $\mathbf{c} \in \mathbb{R}^k$ such that f, g and h are affine over U, V and $W = U + V$, respectively, with the same gradient $\mathbf{c}$.*

4.4 The full-dimensional convex case: Cauchy's functional equation on convex additivity domains in $\mathbb{R}^k$

The most direct generalization applies to full dimensional convex sets F.The general idea of the proof is to consider a point $(\mathbf{x}, \mathbf{y})$ in such a convex additivity domain F, and consider a finite set of smaller subsets $F_1, \dots, F_k \subseteq F$ that are Cartesian products, such that $\mathbf{x} \in F_1$, $\mathbf{y} \in F_k$ and $\operatorname{int}(F_i) \cap \operatorname{int}(F_{i+1}) \neq \emptyset$ for each $i = 1, \dots, k = 1$. Applying Theorem 4.2 on each F_i, we can deduce that the functions are affine over all of F. This idea of "patching" together simple additivity domains to obtain affine properties over a more complicated domain was first introduced in [37, Proposition 23], and then used again in [31, Lemma 10] and [21, Lemmas 3.5, 3.6].

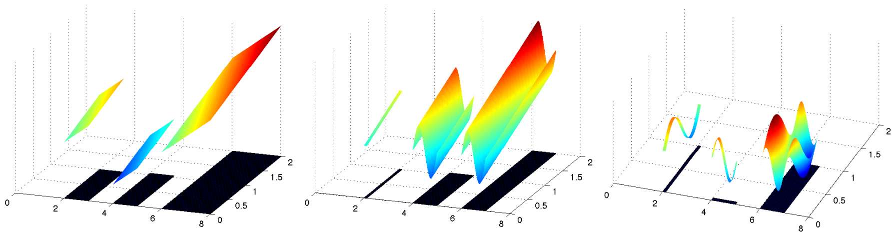
Theorem 4.3 (Convex additivity domain lemma, full-dimensional version [19, Theorem 1.7]) *Let $f, g, h\colon \mathbb{R}^k \to \mathbb{R}$ be bounded functions. Let $F \subseteq \mathbb{R}^k \times \mathbb{R}^k$ be a full-dimensional convex set such that $f(\mathbf{u}) + g(\mathbf{v}) = h(\mathbf{u} + \mathbf{v})$ for all $(\mathbf{u}, \mathbf{v}) \in F$. Then there exists a vector $\mathbf{c} \in \mathbb{R}^k$ such that f, g and h are affine with the same gradient $\mathbf{c}$ over $\operatorname{int}(p_1(F))$, $\operatorname{int}(p_2(F))$ and $\operatorname{int}(p_3(F))$, respectively.*

This theorem is obtained by applying the "patching" idea to subsets F_i that are Cartesian products. Theorem 4.2 is applied to the individual subsets F_i to deduce affine properties.

It is notable that we can only deduce affine linearity over the *interiors* of the projections in Theorem 4.3, as opposed to the conclusion of Theorem 4.2. This is best possible, as is illustrated in [19, Remark 2.12]. If continuity is assumed for the functions, then one easily extends the affine-ness property to the boundary (subsection 4.6).

4.5 The lower-dimensional case: Affine properties with respect to subspaces L

Theorems 4.2 and 4.3 can be established in a significantly more general setting, which takes care of situations in which the set F is not full-dimensional



(a) (b) (c)

Fig. 8 Cauchy's functional equation on bounded domains. In each part (a), (b), and (c), we depict 3 domains in the plane, $U, V, U+V$, left to right, and an function that is additive over these domains. (a) Full-dimensional situation. (b) Sum of a one-dimensional and a two-dimensional set; not a direct sum. (c) Direct sum of (non-parallel) one-dimensional sets.

(Theorems 4.6 and 4.8). Affine properties are deduced with respect to certain subspaces, which is important for the classification of extreme functions in two or more dimensions.

We start with a result obtained in [19], in which the additivity domain is $U \times V$ for convex sets $U \subseteq \mathbb{R}^k$ and $V \subseteq \mathbb{R}^k$, which are not necessarily of the same dimension. In this general setting we cannot expect to deduce that the solutions are affine over U, V, and $U+V$.

Remark 4.4 Indeed, if $U+V$ is a direct sum, i.e., for every $\mathbf{w} \in U+V$ there is a unique pair $\mathbf{u} \in U$, $\mathbf{v} \in V$ with $\mathbf{w} = \mathbf{u}+\mathbf{v}$, then $f(\mathbf{u}) + g(\mathbf{v}) = h(\mathbf{u}+\mathbf{v})$ merely expresses a form of separability of h with respect to certain subspaces, and f and g can be arbitrary functions; see Figure 8 (c).

Definition 4.5 Let $U \subseteq \mathbb{R}^k$. Given a linear subspace $L \subseteq \mathbb{R}^k$, we say $\pi\colon U \to \mathbb{R}$ is *affine with respect to L over U* if there exists $\mathbf{c} \in \mathbb{R}^k$ such that $\pi(\mathbf{u}^2) - \pi(\mathbf{u}^1) = \mathbf{c} \cdot (\mathbf{u}^2 - \mathbf{u}^1)$ for any $\mathbf{u}^1, \mathbf{u}^2 \in U$ such that $\mathbf{u}^2 - \mathbf{u}^1 \in L$.

Theorem 4.6 (Higher-dimensional Interval Lemma; [19, Theorem 2.5]) *Let $f, g, h\colon \mathbb{R}^k \to \mathbb{R}$ be bounded functions. Let U and V be convex subsets of $\mathbb{R}^k$ such that $f(\mathbf{u}) + g(\mathbf{v}) = h(\mathbf{u}+\mathbf{v})$ for all $(\mathbf{u}, \mathbf{v}) \in F = U \times V$. Let L be a linear subspace of $\mathbb{R}^k$ such that $(L+U) \times (L+V) = (L \times L) + F \subseteq \operatorname{aff}(F) = \operatorname{aff}(U) \times \operatorname{aff}(V)$. Then there exists a vector $\mathbf{c} \in \mathbb{R}^k$ such that f, g and h are affine with respect to L over $p_1(F) = U$, $p_2(F) = V$ and $p_3(F) = U+V$ respectively, with gradient $\mathbf{c}$.*

Theorem 4.2 follows when $L = \mathbb{R}^k$.

Definition 4.7 For a linear space $L \subseteq \mathbb{R}^k$ and a set $U \subseteq \mathbb{R}^k$ such that for some $\mathbf{u} \in \mathbb{R}^k$ we have $\operatorname{aff}(U) \subseteq L + \mathbf{u}$, we will denote by $\operatorname{int}_L(U)$ the interior of U in the relative topology of $L + \mathbf{u}$.

Note that $\operatorname{int}_L(U)$ is well defined because either $\operatorname{aff}(U) = L+\mathbf{u}$, or $\operatorname{int}_L(U) = \emptyset$. We now state our most general theorem relating to equation (4.15) on a convex domain.

Theorem 4.8 (Convex additivity domain lemma; [19, Theorem 2.11]) *Let $f, g, h\colon \mathbb{R}^k \to \mathbb{R}$ be bounded functions. Let $F \subseteq \mathbb{R}^k \times \mathbb{R}^k$ be a convex set such that $f(\mathbf{u}) + g(\mathbf{v}) = h(\mathbf{u}+\mathbf{v})$ for all $(\mathbf{u}, \mathbf{v}) \in F$. Let L be a linear subspace of $\mathbb{R}^k$ such that $(L \times L) + F \subseteq \operatorname{aff}(F)$. Let $(\mathbf{u}^0, \mathbf{v}^0) \in \operatorname{rel\,int}(F)$. Then there exists a vector $\mathbf{c} \in \mathbb{R}^k$ such that f, g and h are affine with gradient $\mathbf{c}$ over $\operatorname{int}_L((\mathbf{u}^0 + L) \cap p_1(F))$, $\operatorname{int}_L((\mathbf{v}^0 + L) \cap p_2(F))$ and $\operatorname{int}_L((\mathbf{u}^0 + \mathbf{v}^0 + L) \cap p_3(F))$, respectively.*

Theorem 4.3 follows when $L = \mathbb{R}^k$.

4.6 Continuity at the boundary

The one-dimensional Interval Lemma, Lemma 4.1, includes affine properties on the boundaries. Using this, it is easy to prove that a similar Interval Lemma holds on all non-degenerate intervals $U, V \subseteq \mathbb{R}$ that are any of open, half-open, or closed. Only in special cases in higher dimensions is it possible to extend affine properties in Theorem 4.8 to the boundary; in general this is not possible (see [19, Remark 2.12]).

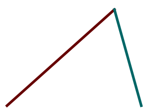
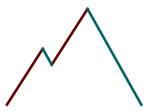
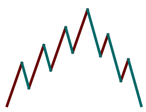
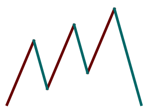
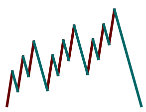
Of course, if we use the stronger regularity assumption that f, g, and h are continuous functions (rather than merely bounded functions), then the affine properties extend to the boundary as well.

Corollary 4.9 (Convex additivity domain lemma for continuous functions; [19, Corollary 2.14])) *Let $f, g, h\colon \mathbb{R}^k \to \mathbb{R}$ be continuous functions. Let $F \subseteq \mathbb{R}^k \times \mathbb{R}^k$ be a convex set such that $f(\mathbf{u}) + g(\mathbf{v}) = h(\mathbf{u} + \mathbf{v})$ for all $(\mathbf{u}, \mathbf{v}) \in F$. Let L be a linear subspace of $\mathbb{R}^k$ such that $L \times L + F \subseteq \operatorname{aff}(F)$. Let $(\mathbf{u}^0, \mathbf{v}^0) \in \operatorname{rel\,int}(F)$. Then there exists a vector $\mathbf{c} \in \mathbb{R}^k$ such that f, g and h are affine with gradient $\mathbf{c}$ over $(\mathbf{u}^0 + L) \cap p_1(F)$, $(\mathbf{v}^0 + L) \cap p_2(F)$ and $(\mathbf{u}^0 + \mathbf{v}^0 + L) \cap p_3(F)$, respectively.*

A Updated compendium of extreme functions

The following tables contain the updated compendium of extreme functions.

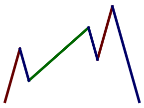
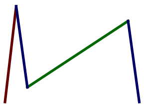
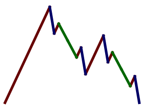
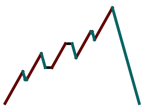
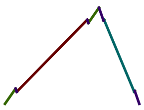
Table 1 An updated compendium of known extreme functions for the infinite group problem I. Parametrized classes of continuous functions for the 1-dimensional case with up to two slopes.

Function[a]	Graph	Slopes	Cont.	Notes
`gmic`		2	C	The famous Gomory mixed integer cut, going back to Gomory's 1960 paper [42]. Dominates the `gomory_fractional` cut, which is not minimal (Figure 1).
`gj_2_slope`		2	C	Two families of continuous extreme functions with 2 slopes, from Gomory–Johnson [49]. By the Gomory–Johnson 2-Slope Theorem (Theorem 2.13), all continuous piecewise linear minimal valid functions with 2 slopes are extreme.
`gj_2_slope_repeat`		2	C	
`dg_2_step_mir`		2	C	Described by Dash–Günlük [35]. Extremality follows from the 2-Slope Theorem (Theorem 2.13).
`kf_n_step_mir`		2	C	Described by Kianfar–Fathi [59]. Extremality follows from the 2-Slope Theorem (Theorem 2.13).
`bccz_counterexample`		1–2[b]	C	Limit of `kf_n_step_mir` for $n \to \infty$; not a piecewise linear function. Described by Basu–Conforti-Cornuéjols–Zambelli [15]; see § **6.4**.

[a] A function name shown in typewriter font is the name of the constructor of this function in the accompanying Sage program.

[b] The function is not piecewise linear. In one case ($\mu^- < 1$) [15], it is absolutely continuous and thus Lebesgue–almost everywhere differentiable; the derivatives take one of two values where they exist. In a second case ($\mu^- = 1$) [60], it is merely continuous (but not absolutely continuous) and Lebesgue–almost everywhere differentiable; the derivatives take only one value where they exist. See subsection 6.4 for more details.

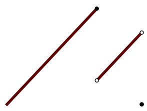
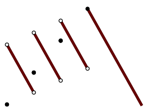
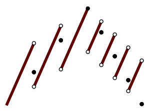
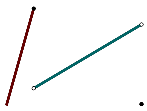
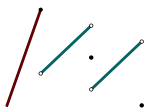
Table 2 An updated compendium of known extreme functions for the infinite group problem II. Parametrized classes of continuous functions for the 1-dimensional case with at least three slopes.

Function[a]	Graph	Slopes	Cont.	Notes
`gj_forward_3_slope`		3	C	Described by Gomory–Johnson [49].
`drlm_backward_3_slope`		3	C	Described by Dey–Richard–Li–Miller [39] based on Aráoz–Evans–Gomory–Johnson [4].
`dr_projected_sequential_merge_3_slope`		3	C	Described by Dey–Richard [38], using their `projected_sequential_merge` procedure; see Table 5 and § **5.2**.
`bhk_irrational`		3	C	Only extreme when certain parameters are $\mathbb{Q}$-linearly independent. Described by Basu–Hildebrand–Köppe [18]; see § **6.2**.
`chen_4_slope`[b]		4	C	Described by Chen [24].

[a] A function name shown in typewriter font is the name of the constructor of this function in the accompanying Sage program.

[b] Chen [24] also constructs a family of 3-slope functions, which he claims to be extreme. However, his proof for this class is flawed, and none of the functions in the described family appear to be extreme, as pointed out in [60]. The functions are available as `chen_3_slope_not_extreme`.

Table 3 An updated compendium of known extreme functions for the infinite group problem III. Parametrized families of discontinuous functions for the 1-dimensional case.

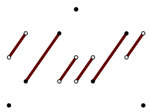
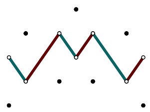
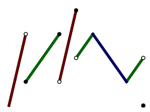
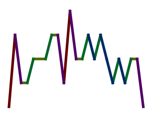
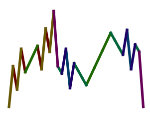
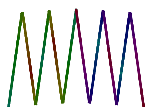
Function[a]	Graph	Slopes	Cont.	Notes
`ll_strong_fractional`[b]		1	D	Described by Letchford–Lodi [62][c]; dominates the `gomory_fractional` cut (Figure 1). Extreme only if $f \geq \frac{1}{2}$; then special case of `dg_2_step_mir_limit`, `drlm_2_slope_limit` (below).
`dg_2_step_mir_limit`		1	D	Described by Dash–Günlük [35] ("extended 2-step MIR"). Special case of `drlm_2_slope_limit` (below). Defined as a limit of `dg_2_step_mir` functions; see § **6** for a discussion of limits.
`drlm_2_slope_limit`		1	D	From Dey–Richard–Li–Miller [39], generalizing `dg_2_step_mir_limit` (above). Defined as a limit; see § **6** for a discussion of limits.
`drlm_3_slope_limit`		2	D	Described by Dey–Richard–Li–Miller [39]. Defined as the limit of `drlm_backward_3_slope` functions; see § **6** for a discussion of limits.
`rlm_dpl1_extreme_3a`		2	D	A DPL_1-extreme function from Richard–Li–Miller [66, case 3a]. Proved extreme in [60]. (All other DPL_1-extreme functions from [66] are known to be special cases of `drlm_2_slope_limit` and `drlm_3_slope_limit`.)

[a] A function name shown in typewriter font is the name of the constructor of this function in the accompanying Sage program.

[b] In the survey [65, Table 19.4], this is called "Improved GFC."

[c] Note that there is a mistake in [62, Figure 3]. The correct figure appears here.

Table 4 An updated compendium of known extreme functions for the infinite group problem IV. "Sporadic" functions for the 1-dimensional case. These functions were found by computer experiments. They have not been described in the literature as a member of a parametrized family; but there is no reason to assume this could not be done.

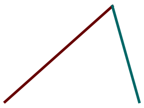
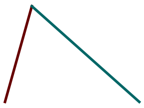
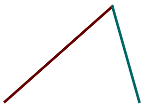
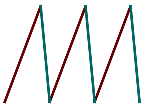
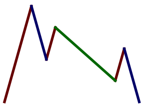
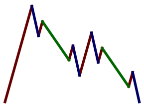
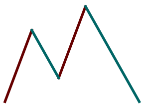
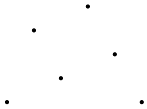
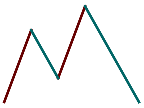
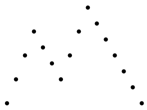
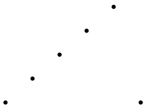
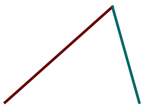
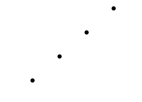
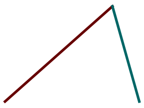
Function[a]	Graph	Slopes	Cont.	Notes
`hildebrand_2_sided_discont_1_slope_1`		1	D	An extreme function that is discontinuous on both sides of the origin, from Hildebrand (2013, unpublished). *Previously unpublished* ♣
`hildebrand_2_sided_discont_2_slope_1`		2	D	An extreme function that is discontinuous on both sides of the origin, from Hildebrand (2013, unpublished). *Previously unpublished* ♣
`hildebrand_discont_3_slope_1`		3	D	A discontinuous extreme function with 3 slopes, from Hildebrand (2013, unpublished). *Previously unpublished* ♣
`hildebrand_5_slope_22_1`[b]		5	C	An extreme function with 5 slopes, from Hildebrand (2013, unpublished). Several examples are known. *Previously unpublished* ♣
`kzh_7_slope_1`[c]		7	C	An extreme function with 7 slopes, from Köppe–Zhou [61]. Several examples are known. *Previously unpublished* ♣
`kzh_28_slope_1`		28	C	An extreme function with 28 slopes, from Köppe–Zhou [61]. The shown graph does not convey the complexity of this function, which has 395 breakpoints in $[0, 1]$ sampled from $\frac{1}{778}\mathbb{Z}$. *Previously unpublished* ♣

[a] A function name shown in typewriter font is the name of the constructor of this function in the accompanying Sage program.

[b] Several examples are known. Use autocompletion in Sage to obtain a list, by typing `hildebrand_5_slope` and pressing the TAB key.

[c] Several examples are known. Use autocompletion in Sage to obtain a list, by typing `kzh_` and pressing the TAB key.

Table 5 An updated compendium of known extreme functions for the infinite group problem V. Procedures.

	Graphs		
Procedure[a]	From	To	Notes
`automorphism`			From Johnson [58]; see [65, section 19.5.2.1].
`multiplicative_homomorphism`			See [65, sections 19.4.1, 19.5.2.1].
`projected_sequential_merge`			Operation $\Diamond^1_n$ from Dey–Richard [38]; see § **5.2**.
`restrict_to_finite_group`			Restrictions to finite group problems $R_f(\frac{1}{q}\mathbb{Z},\mathbb{Z})$ preserve extremality if f and all breakpoints lie in $\frac{1}{q}\mathbb{Z}$. See § **8.2**.
`restrict_to_finite_group (oversampling=3)`			If `oversampling` by a factor $m \geq 3$, the restriction is extreme for $R_f(\frac{1}{mq}\mathbb{Z},\mathbb{Z})$ if and only if the original function is extreme. See § **8.2**.
`interpolate_to_infinite_group`			Interpolation from finite group problems $R_f(\frac{1}{q}\mathbb{Z},\mathbb{Z})$ preserves minimality, but in general not extremality. See § **8.2**.
`two_slope_fill_in`			Described by Gomory–Johnson [48], Johnson [58]. For $k = 1$, if minimal, equal to `interpolate_to_infinite_group` (above). For $k > 1$, see [65, section 19.5.2.3] and [12, 16] for recent developments.

[a] A procedure name shown in typewriter font is the name of the corresponding function in the accompanying Sage program.

Table 6 List of notation in the infinite group problem literature

	Gomory–Johnson		Dey et al.		Basu et al.			Surveys		
Concept	[47, 48]	[49]	[37, 40]	[39]	[15]	[21]	[18, 19]	[65]	[27]	this
Additive group of reals mod 1	I	G	I	I				I		
Mapping from reals to group elements	$u = \mathcal{F}(x)$		$\mathbb{P}(u)$	$\mathbb{P}(u)$				$u = \mathcal{F}(v)$		
Mapping from group elements to canonical reals	$x = \lvert u \rvert$			$\mathbb{P}^{-1}(u)$				$v = \mathcal{F}^{-1}(u)$		
Number of rows of the group problem	1	1	m	m	1	k	k	m	q	k
Group (domain of solutions, valid functions)	$U = I$	G	I^m	I^m		$\mathbb{R}^k$	$G = \mathbb{R}^k$	$G = I^m$	$\mathbb{R}^q$	$G = \mathbb{R}^k$
Subgroup (periodicity)						$\mathbb{Z}^k$	$S = \mathbb{Z}^k$		$\mathbb{Z}^q$	$S = \mathbb{Z}^k$
Right-hand side	u_0	u_0	r	r	f	$-f$	$\mathbf{f}$	r	$-f$	$\mathbf{f}$
Group problem	$\mathrm{P}(U, u_0)$		(mDIIGP)			(IR)				(2.6)
Solutions to the group problem	$t(u)$	$\{t(u)\}$	$t(u)$	$t(u)$	s_r	s_r	$s(\mathbf{r})$	$t(u)$	x_r	$y(\mathbf{r})$
Solution set of the group problem	$T(U, u_0)$		mDIIGP	$PI(r, m)$				$\mathrm{MG}(G, \emptyset, r)$	G_f	
Its convex hull							$R_{\mathbf{f}}(G, S)$			$R_{\mathbf{f}}(G, S)$
Its enclosing space							$\mathcal{V}$			$\mathbb{R}^{(G)}$
Valid functions	$\pi(u)$	$\pi(u)$	$\phi(u)$	$\phi(u)$	$\pi(r)$	$\pi(r)$	$\pi(\mathbf{r})$	$\phi(u)$	$\pi(r)$	$\pi(\mathbf{r})$
Set of tight solutions for a valid function		$P(\pi)$	$P(\phi)$		$P(\pi)$	$S(\pi)$				$P(\pi)$
Subadditivity slack	$\nabla(u, v)$						$\Delta\pi(\mathbf{u}, \mathbf{v})$			$\Delta\pi(\mathbf{u}, \mathbf{v})$
Additivity domain (equality set)		$E(\pi)$	$E(\phi)$		$E(\pi)$	$E(\pi)$	$E(\pi)$			$E(\pi)$

B List of notation in the literature

Table 6 (on page 38) compares the notation in the present survey with that in selected original articles on the infinite group problem and the surveys [27, 65].

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