Maximal Lattice-free Convex sets in 3 Dimensions

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1 Preliminaries

1.1 Definitions

Given a lattice Λ, a *lattice polyhedra* is a convex polyhedra P, such that all vertices of P are lattice points and no other other point in P is a lattice point. By *lattice-free convex set* we mean a convex set with no lattice point from Λ in its *strict* interior. However lattice points are allowed on the boundary. We will usually work with the standard lattice in $\mathbb{R}^3$, i.e. points which have all three coordinates integral. We study maximal lattice-free convex sets in three dimensions in this paper.

We now define some three dimensional polyhedra that we will need. A *tetrahedron* is a simplex, i.e. an affine transformation of the convex hull of $\{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\}$. Given a two dimensional convex polygon P and a vector d in $\mathbb{R}^3$, a *cylinder* over P in the direction d is the polyhedron $\{x + \gamma d : x \in P, \gamma \in \mathbb{R}\}$.

A cone C is *pointed* if $C \cap -C = \{0\}$. An *unbounded pyramid* is a polyhedron of the form $\{v + C\}$, where v is a point in $\mathbb{R}^3$ and C is a pointed cone generated by 4 distinct rays in $\mathbb{R}^3$.

A *valley* is the intersection of two half-spaces $a_1 \cdot x \leq 1, a_2 \cdot x \leq 1$. Note that a valley might be empty, or be one of the original halfspaces or be equivalent to a split if a_1, a_2 are not linearly independent. (Note that, in this paper, a split is the region between two parallel

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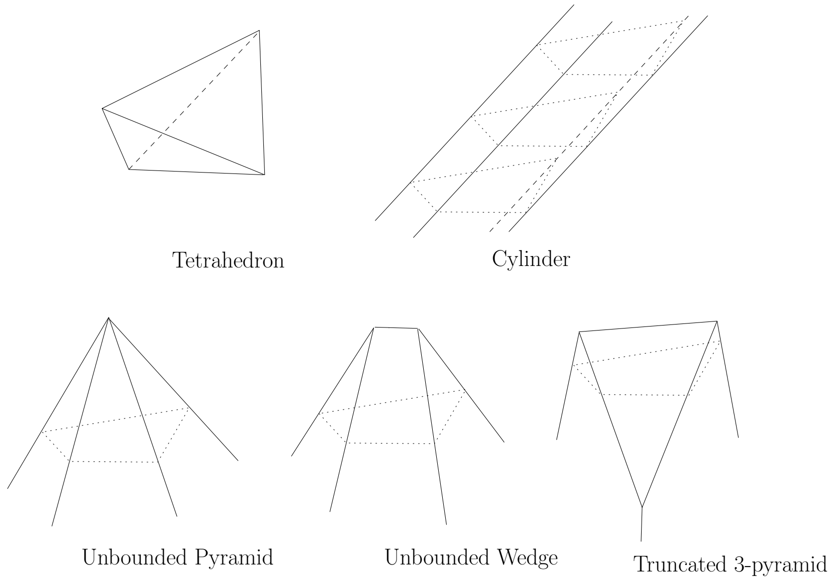

Figure 1: Polyhedra in 3-space with four facets

planes and may contain lattice points in it. This notion is different from the standard notion of a split which is a lattice-free set.) If a_1, a_2 are linearly independent, then a valley V has an associated $cleft(V)$ which is the intersection line of the two planes $a_1 \cdot x = 1, a_2 \cdot x = 1$. An *unbounded wedge* is a polyhedron formed by the intersection of two valleys V_1, V_2, at least one of which has a cleft and the other is a split or has a cleft. Moreover, if both V_1 and V_2 have a cleft, then $cleft(V_1) \cap V_2 = \phi$ and the two clefts are *not* parallel. If one valley (say V_1) is a split and V_2 has a cleft, then the $cleft(V_2) \cap V_1 \neq \phi$. This implies that an unbounded wedge has four facets, all of which are unbounded. Moreover, two of these facets are two-dimensional cones generated by two rays. The other two facets are two dimensional polyhedrons with 3 edges, one bounded and two unbounded. The bounded edge is the same for these two facets. Also an unbounded wedge has exactly one bounded edge and 4 unbounded edges, and 2 vertices. The bounded edge in an unbounded wedge W will be denoted by $edge(W)$.

A *3-pyramid* is of the form $\{v + C\}$, where v is any point in $\mathbb{R}^3$ and C is a cone generated by 3 linearly independent rays. A *truncated 3-pyramid* is formed by intersecting a half-space with a cone generated by three linearly independent rays, such that the resulting polyhedron is unbounded and the plane corresponding to the half-space intersects all three rays. This plane is called the *truncating plane* for the truncated 3-pyramid. Note that the facet corresponding to the truncating plane is a triangle.

1.2 Lattice Polyhedra in Three dimensions

The following theorem is from Scarf [1].

Theorem 1.1 (Howe's Theorem). *A lattice polyhedron with eight vertices in 3-space can, by a unimodular transformation, be brought into the form where the vertices are given by the columns of the matrix :*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & \beta & \beta' & p \\ 0 & 0 & 1 & 1 & 0 & \gamma & \gamma' & q \end{pmatrix}$$

with p, q positive integers which are prime to each other, and with $(\beta, \gamma), (\beta', \gamma')$ nonnegative integers satisfying $\beta\gamma' - \gamma\beta' = 1, \beta + \beta' = p, \gamma + \gamma' = q$. Moreover, a lattice polyhedron with fewer than eight vertices is a subset of an integral polyhedron with eight vertices.

A plane in 3-space is defined as a *lattice plane* if it passes through three non-collinear lattice points. Two parallel lattice planes are *adjacent* if there are no lattice points between them. The lattice points on a lattice plane P form a lattice, which will referred to by $\Lambda(P)$. Theorem 1.2 can be shown to be equivalent to the following statement which will be more useful for this paper.

Theorem 1.2 (Alternative form of Howe's Theorem)**.** *The vertices of a lattice polyhedron in 3-space lie on two adjacent lattice planes.*

1.3 Tools

The following technical lemmas will also be useful.

Observation 1.3. *Five lattice-points whose convex hull is a lattice polyhedron cannot be coplanar in $\mathbb{R}^3$.*

Proof. Assume to the contrary that there is a lattice plane P containing these five lattice points. Consider a basis v_1, v_2 for $\Lambda(P)$. Let the coordinates of the five lattice points in this basis be (x_1^i, x_2^i), $i = 1, 2, 3, 4, 5$ where x_1^i, x_2^i are integers. We now look at these coordinates modulo 2. Since there are only four possibilities, two of these five points p_1, p_2 have the same coordinates modulo 2. But then the mid-point $(p_1 + p_2)/2$ is a lattice point and is contained in the convex hull, contradicting the fact that it was a lattice polyhedron. □

Lemma 1.4. *Consider a maximal lattice-free quadrilateral X, with the lattice points on its boundary forming a parallelogram Y. The opposite sides of Y are on adjacent split lines. Let these two splits be in directions d_1 and d_2. Then for one of d_1 or d_2, any translation of X intersects at most 3 split lines in that direction.Moreover, if it intersects 3 lines, then two of the lines pass through the vertices of X.*

Proof. We make an affine transformation so that the vertices of Y are $(0, 0), (0, 1), (1, 1), (1, 0)$. Therefore, the splits considered in the statement of the theorem are $S_1 = 0 \leq x_1 \leq 1$ and $S_2 = 0 \leq x_2 \leq 1$. We label the vertices of X as A, B, C, D in clockwise order, with A as the vertex with $x_1 \leq 0$. We denote the ith coordinate of a vertex v by $x_i(v)$. The following claim gives us the theorem.

Claim 1.5. $(x_1(C) - x_1(A) - 1)(x_2(B) - x_2(D) - 1) \leq 1$.

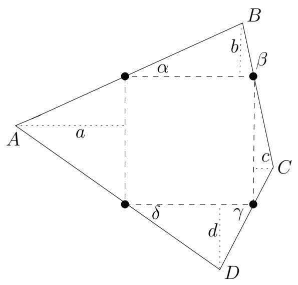

Figure 2: Maximal Lattice-free Quadrilateral

Proof. See Figure 2 for the labelings of angles $\alpha, \beta, \gamma, \delta$. Note that these angles can *independently* take values between $0 \leq \frac{\pi}{2}$ and we would still have a maximal lattice-free quadrilateral. So we have to analyze all possible values of these four angles.

We wish to argue that $(a+c)(b+d) \leq 1$. These four values are the following.

$$a = \frac{1}{\tan\alpha + \tan\delta}, b = \frac{\tan\alpha + \tan\beta}{\tan\alpha + \tan\beta}, c = \frac{1}{\tan\beta + \tan\gamma}, d = \frac{\tan\gamma + \tan\delta}{\tan\gamma + \tan\delta}$$

For the ranges of the four angles, the tan values can take all positive values. So we need to show that

$$(\frac{1}{x_1 + x_4} + \frac{1}{x_2 + x_3})(\frac{x_1x_2}{x_1 + x_2} + \frac{x_3x_4}{x_3 + x_4}) \leq 1$$

where $x_i \geq 0$ for $i = 1, 2, 3, 4$.

This is equivalent to showing that

$$(x_1+x_2+x_3+x_4)(x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4)-(x_1+x_4)(x_2+x_3)(x_1+x_2)(x_3+x_4) \leq 0$$

The lhs of the above inequality simplifies to $-(x_1x_3 - x_2x_4)^2$, which is always less than or equal to 0.

□

□

Lemma 1.6. *Consider a maximal lattice-free quadrilateral X in the plane, with the lattice points on the boundary forming a parallelogram Y. If any translation X' of X by the vector v contains four lattice points forming a parallelogram of area 1, then these four points are on the boundary of X' and the parallelogram Y' formed by them is a translation of Y by the same vector v.*

Proof. Consider two opposite sides of Y'. These two sides lie on two adjacent lattice lines in the Euclidean plane, because Y' is of area 1. These lattice lines are of the form $c \cdot x = p$, where p is an integer and $c = (c_1, c_2)$ with c_1, c_2 relatively prime. Consider the intersections of all lattice lines $c \cdot x = p, \forall$integers p with X. These will be line segments parallel to each other. Let this set of line segments be L. Translating X with respect to the lattice is equivalent to translating the lattice with respect to X. This is therefore equivalent to moving the lattice lines $c \cdot x = p$ parallel to themselves by an amount $0 \le \epsilon < \frac{1}{||c||}$. Intersecting these new lattice lines with X, we obtain the set of parallel line segments L'. Now the vertices of Y' lie on two adjacent line segments s_1 and s_2 from L'. Since $0 \le \epsilon < \frac{1}{||c||}$, there exists some segment s from L which lies between s_1 and s_2 (possibly s coincides with s_1 or s_2, with $\epsilon = 0$). We know that s contains at most 2 lattice points. Suppose $\epsilon > 0$. Consider the two sides of X passing through the end-points of s. If they are not parallel, they meet at some point p when extended (p might be outside X). Consider the part of X on the same side of s as p. Any line segment parallel to s in this part has length *strictly* less than s. Since one of s_1 and s_2 falls in this part, that segment cannot contain two lattice points (since s had at most 2). Since this would be a contradiction, these two sides of X must be parallel and hence opposite sides of X. But these are also opposite sides of Y' and hence form a split. But opposite sides of X which are parallel can never be a split as they contain the two lattice points on the other sides of X. So $\epsilon > 0$ is impossible. If $\epsilon = 0$, then s coincides with s_1 or s_2. But then s contains exactly two lattice points and both have to be at its end-points. Hence Y' is a translation of Y by the same vector v. □

Remark 1.7. *Lemma 1.4 gives an alternative proof for Lemma 1.6.*

Lemma 1.8. *Given any maximal lattice-free Type 3 triangle T, there does not exist a translation of T which contains four lattice points forming a parallelogram of area 1.*

Proof. Denote the parallelogram formed by the lattice points after the translation by Y. Consider the split lines corresponding to one pair of opposite sides of Y. These are lines of the form $c \cdot x = p$ for integers p, where $c = (c_1, c_2)$ with c_1, c_2 relatively prime. The intersection of T with these lines is a set of parallel line segments which we refer to by L. Now a translation of T with respect to the lattice is equivalent to a translation of the lattice with respect to T. After this translation, the lines $c \cdot x = p$ would have moved parallel to themselves by an amount $0 \le \epsilon < \frac{1}{||c||}$. These new set of lines intersect T in the set of line segments L'. By assumption, the four vertices of Y lie on two adjacent line segments s_1 and s_2 from L'. Now since the distance between s_1 and s_2 is $\frac{1}{||c||}$, there exists a line segment s from L which lies between s_1 and s_2. s divides T into two parts one of which is a triangle. One of s_1 or s_2 is in this triangle and hence has length less than or equal to s, say s_1. But we know that s has at most 1 lattice point on it, which implies s_1 can have at most 1 lattice point, contradicting the assumption that it contained two lattice points. □

Claim 1.9. *Consider a maximal lattice-free quadrilateral X with the lattice points on its boundary forming a parallelogram Y. For any translation X' of X over the lattice, if X' contains 3 or less lattice points in its interior so that no three are collinear, then this set of lattice points are a translation of a subset of the vertices of Y.*

Proof. Translating X is equivalent to translating the lattice with respect to X. After this lattice translation, there is exactly one lattice point in the interior of Y, say point p_1. If X contains *four* lattice points, then by Lemma 1.6, these four must be on the boundary and the same translation of Y. Therefore X contains three or less lattice points in its interior and p_1 is one of them.

Moreover, by Lemma 1.4 one of the split directions of Y can intersect X in at most 3 lines. Let this split direction be d. The lattice points which are contained in X after the translation have to lie on these 3 lines along direction d. Before the translation of the lattice, the lines along direction d partitioned X into two triangles and a 5-sided polygon. The two triangles share opposite edges of parallelogram Y. Let these sides be s_1 and s_2 and the corresponding triangles be t_1 and t_2. After the translation of the lattice, at most one of the lattice lines in the direction d intersects Y. Since all three lattice points are not on the same line, at least one of them, say p_2 is in t_1 or t_2. Say we have a point in t_1. Translate t_1 so that one of its sides now corresponds to s_2. Then, p_2 would be in the interior of Y and hence must coincide with p_1. This implies that p_1p_2 is a translation of an edge of Y. If there is a third point p_3 in the interior of X and it is on the same split line as p_1, then we have the theorem. If not, the third point is in triangle t_2. Then we have the three lines along the direction d intersecting X after translation, and therefore Lemma 1.4 states that two of these pass through the vertices of X. Which means that p_2 and p_3 are on the vertices, and when we translate the triangles t_1, t_2 to the opposite sides, both p_2, p_3 coincide with p_1. This implies that these three are collinear, a contradiction. □

Lemma 1.10. *Consider a maximal lattice-free quadrilateral X with the lattice points on its boundary forming a parallelogram Y. For any homothetic copy X' of X, if X' contains 2 or more lattice points, then the segment formed by some two of these points is a translation of an edge of Y.*

Proof. Without loss of generality, the the vertices of Y be $(0,0),(1,0),(1,1),(0,1)$. Let the vertex of X in the region $x_1 \leq 0$ be A. We consider a homothetic version X'' of X such that one vertex of X'' coincides with A. Let the points in X'' corresponding to the vertices of Y be named y_1, y_2, y_3, y_4 in the same order. Now translate the lattice with respect to X''. We now claim that if two or more lattice points are in the X'', then the segment formed by some two of these is a translation of an edge of Y. This would then prove the lemma.

If the translation superimposed the lattice on itself, then it is trivial (for instance the lattice points coinciding with y_2 and y_3 are in X'' are a translation of that edge). If not, some lattice point p now is in the interior of the square y_1, y_2, y_3, y_4. From the hypothesis of the theorem, there is some lattice point q other than p in X''. The horizontal line h passing through p divides X'' into two polygons. q lies in one of these polygons and the vertical line through q therefore intersects h at some point s. Now the horizontal segment joining p and s is contained in X'' and also contains two lattice points at distance 1 (since both p and s are lattice points). So we have our claim.

□

Consider any lattice plane P in 3-space. The lattice points on this plane form a lattice $\Lambda(P)$. Consider the family $\mathcal{F}(P)$ of three dimensional polyhedra with exactly four facets, such that the intersection of this polyhedron with P is a maximal lattice-free quadrilateral in the lattice $\Lambda(P)$. In the next proposition, we characterize this family $\mathcal{F}(P)$.

Proposition 1.11. *Any polyhedron Q in the family $\mathcal{F}(P)$ defined above is one of the following polyhedra : tetrahedron, cylinder over a maximal lattice-free quadrilateral in $\Lambda(P)$, unbounded pyramid, unbounded wedge, truncated 3-pyramid. Moreover, if it is an unbounded wedge, then both vertices of Q lie on the same side of P.*

Proof. Consider the maximal lattice-free quadrilateral X obtained from the intersection of the plane P and the polyhedron Q. Let the four facet-defining planes of Q corresponding to the sides of X be named A, B, C, D in topological order. Denote by $hs(A)$ (similarly $hs(B), hs(C), hs(D)$) the halfspace defined by A (resp. B, C, D) and containing the quadrilateral. Consider two opposite sides of this quadrilateral formed by facets A and C. $hs(A) \cap hs(C)$ is by definition a valley which is either a split or has a cleft; denote it by V_1. Similarly, for the planes B and D we have valley V_2, and $Q = V_1 \cap V_2$.

If both V_1 and V_2 are splits, then the intersection is a cylinder over X.

If V_2 has a cleft and V_1 is a split, then note that since V_1 and V_2 are formed using opposite sides of X, $cleft(V_2) \cap V_1$ is non-empty. By definition, we have an unbounded wedge. Moreover, note that the point of intersection of $cleft(V_2)$ and the plane P is cut off from X by one of the sides of V_1. This implies that $cleft(V_2) \cap V_1$ is entirely on one side of P. So both vertices of this unbounded wedge are on the same side of P.

The above argument holds by symmetry if V_1 has a cleft and V_2 is a split.

It remains to analyze the case when both V_1 and V_2 have clefts. Note in this case that the two clefts cannot be parallel since V_1 and V_2 were defined using opposite sides of a quadrilateral.

Case 1 : $cleft(V_1) \cap V_2 = \phi$. In this case, by definition we have an unbounded wedge. Moreover, the point $cleft(V_2) \cap P$ is cut off from X by a side of V_1; which implies that $cleft(V_2) \cap V_1$ lies on one side of P and hence both vertices of the wedge are on the same side of P.

By symmetry, the above argument also takes care of the case $cleft(V_2) \cap V_1 = \phi$.

Case 2 : Both $cleft(V_1) \cap V_2$ and $cleft(V_2) \cap V_1$ are non-empty. Consider the point $cleft(V_1) \cap P$. Some side of V_2 cuts off this point from X on P. Without loss of generality let this side be B. B intersects $cleft(V_1)$ on one side of P, say at point b. Call this side 1 of P and call the other side of P side 2. Consider the point of intersection d of plane D and $cleft(V_1)$.

If d is in side 1 of P, then there are two cases. If d is the same point as b, then we have a pyramid. Otherwise, look at the two dimensional polyhedrons $B \cap V_1$ and $D \cap V_1$. These are two dimensional cones. Since $cleft(V_2) \cap V_1$ is non-empty, $cleft(V_2) \cap V_1$ has to be the intersection of these two cones. If $cleft(V_2) \cap V_1$ is bounded, then $V_1 \cap V_2$ is a tetrahedron. Note that in this case, both $cleft(V_1) \cap V_2$ and $cleft(V_2) \cap V_1$ are bounded and on opposite sides of P. If $cleft(V_2) \cap V_1$ is not bounded, then $cleft(V_2) \cap V_1$ intersects only one of A or C. Without loss of generality, let this be A. Then $V_1 \cap V_2$ is a truncated 3-pyramid, where A is the truncating plane and the cone of the 3-pyramid has B, C, D as facets.

Finally, we have the case that d is in side 2 of P. Note that since this point is on $cleft(V_1)$ and on side 2 of P, this point is cut off from $V_1 \cap V_2$ by B because B intersected $cleft(V_1)$ on side 1. Then $V_1 \cap V_2$ is a truncated 3-pyramid, where B is the truncating plane and the cone of the 3-pyramid has A, C, D as facets.

□

The next lemma is useful in reducing the analysis of unbounded pyramids, wedges and truncated 3-pyramids to the analysis of cylinders in the rest of the paper.

Lemma 1.12. *Consider three parallel, adjacent lattice planes. Without loss of generality, let these be $P_1 : x_3 = -1$, $P_2 : x_3 = 0$ and $P_3 : x_3 = 1$. Consider a polyhedron Q which is* *either an unbounded pyramid, an unbounded wedge or a truncated 3-pyramid, such that its intersection with $x_3 = 0$ is a quadrilateral, $Q \cap \{x_3 \geq 0\}$ is bounded and $Q \cap P_3 \neq \phi$. Then there exists a cylinder C with the following properties :*

1. $C \cap P_1 \subset Q \cap P_1$
2. $C \cap P_2 = Q \cap P_2$
3. $C \cap P_3 \supseteq Q \cap P_3$

Proof. We consider each case in turn.

Case 1 : Q is an unbounded pyramid. Since $Q \cap \{x_3 \geq 0\}$ is bounded, $Q \cap P_3$ is a homothetic copy of $Q \cap P_2$ with dilation less than 1. This implies that there is a *relaxation* X of $Q \cap P_3$ which is congruent to $Q \cap P_2$. Let C be the cylinder passing through X and $Q \cap P_2$. By construction, $C \cap P_3 \supseteq Q \cap P_3$ and $C \cap P_2 = Q \cap P_2$. Corresponding to any facet f of Q, consider the half-space $hs(f)$ associated with the plane defining f. In our construction, $hs(f) \cap P_3$ is a relaxation and $hs(f) \cap P_2$ remains the same. This implies $hs(f) \cap P_1$ is a strengthening, which in turn implies that $C \cap P_1 \subset Q \cap P_1$.

Case 2 : Q is an unbounded wedge. By Proposition 1.11, we know that $edge(Q)$ lies completely on one side of P_2. Since we know that $Q \cap \{x_3 \geq 0\}$ is bounded, $edge(Q)$ must be in the region $x_3 \geq 0$. This implies that the intersection of each facet f of Q with P_3 is a homothetic copy of $f \cap P_2$ with dilation less than 1 (the dilation might be 0 if $f \cap P_3 = \phi$). Therefore, there exists a *relaxation* of X of $Q \cap P_3$ which is congruent to $Q \cap P_2$. Using the same arguments as above, the cylinder C passing through X and $Q \cap P_2$ works.

Case 3 : Q is a truncated 3-pyramid Let the triangle facet of Q be denoted by t. Since P_2 intersects t, two of the edges of t will be intersected by P_2 and the third edge will be entirely on one side of P_2. Moreover, since $Q \cap \{x_3 \geq 0\}$ is bounded, the third edge will be in the region $\{x_3 \geq 0\}$. Let the other facet of Q incident on this third edge be denoted by f. Denote the unbounded edge of Q_T which is *not* incident on f by r. Now start tilting t about its line of intersection with P_2 such that its angle with f becomes smaller. This movement *relaxes* $Q \cap P_3$ and *strengthens* $Q \cap P_1$. As we keep tilting t, at some point it will become parallel to r but continue to intersect f (this is because r does not intersect f). At this point, we would have an unbounded wedge Q' with $Q' \cap P_1 \subseteq Q \cap P_1$, $Q' \cap P_2 = Q \cap P_2$ and $Q' \cap P_3 \supseteq Q \cap P_3$. From Case 2 above, we know there exists a cylinder C for the wedge Q' satisfying the properties stated in the lemma and hence C will also work for Q.

□

2 Maximal lattice-free polyhedra with 8 facets

In this section, we provide a complete characterization of maximal lattice free polyhedra with 8 facets. It can be shown that for such polyhedra Q, every facet will contain *exactly* 1 lattice point. Let this set of eight lattice points be V. Note that the convex hull of V, denoted by $conv(V)$, is a lattice polytope with V as its vertices. Therefore, by Theorem 1.2, these eight lattice points lie on two adjacent lattice planes, with four points on each plane and the four vertices on a plane form a parallelogram of area 1. Moreover, Theorem 1.2 also states that we can choose these planes to be $x_3 = 0$ and $x_3 = 1$, upto unimodular transformations. We will refer to $x_3 = 1$ as the top plane TP and $x_3 = 0$ as the bottom plane BP. Consider the four planes corresponding to the facets of Q which are tangent to the lattice points in TP. Clearly, the intersection of these four planes is a polyhedron from the family $\mathcal{F}(TP)$, because the intersection with TP is a maximal lattice-free quadrilateral. Similarly for the facets tangent to the lattice points from BP. Therefore $Q = Q_T \cap Q_B$, where $Q_T \in \mathcal{F}(TP)$ and $Q_B \in \mathcal{F}(BP)$. The next theorem exactly characterizes which are the possible pairs of polyhedrons from each family.

Theorem 2.1. *Any maximal lattice-free polytope Q in 3-space with 8 facets can be expressed as the intersection of two polyhedra Q_T and Q_B, where Q_T is either a tetrahedron, an unbounded wedge, an unbounded pyramid or a truncated 3-pyramid, and similarly for Q_B. Upto unimodular transformations, the two planes containing the eight lattice points are $x_3 = 0$ and $x_3 = 1$. Moreover, if Q_T is an unbounded wedge, then edge(Q_T) lies in the half-space $x_3 \geq 1$. If Q_T is a unbounded pyramid, then the vertex of Q_T lies in the region $x_3 \geq 1$. If Q_T is a truncated 3-pyramid, then one of its bounded edges is in the region $x_3 \geq 1$. Analogously, for Q_B these sets will be in the region $x_3 \leq 0$.*

Conversely, for any pair from the above types of four polyhedrons, there exists examples of maximal lattice-free polytopes Q which are intersections of two polyhedrons of these two types.

Proof. From the discussion preceding the statement of the theorem, we know that $Q = Q_T \cap Q_B$ with $Q_T \in \mathcal{F}(TP)$ and $Q_B \in \mathcal{F}(BP)$. We first show that neither Q_T nor Q_B can be a cylinder. Consider Q_B. We know that Q_B contains the four lattice points in TP, so $Q_B \cap TP$ contains these four lattice points. Since Q_B is a cylinder, $Q_B \cap TP$ is a translation of $Q_B \cap BP$ over the lattice $\Lambda(BP)$ $(= \Lambda(TP))$. But then by Lemma 1.6, we know that the four lattice points on TP are on the boundary of $Q_B \cap TP$. But then the facets of Q corresponding to the four planes of Q_B contain two lattice points, which is a contradiction to the fact that each facet of Q has exactly 1 lattice point on it.

If Q_B is an unbounded pyramid, unbounded wedge or a truncated 3-pyramid, first note that the vertex or the bounded edge must be on side of BP by Proposition 1.11. Then it has the property stated in the theorem. This is because otherwise Q_B satisfies the hypothesis of Lemma 1.12 and there would exist a cylinder Q'_B with $Q'_B \cap BP = Q_B \cap BP$ and $Q'_B \cap TP \supseteq Q_B \cap TP$ and hence Q'_B would contain the four lattice points from TP. This would contradict the arguments in the previous paragraph.

All arguments hold by symmetry for Q_T.

To show the second part of the theorem, observe that if we can construct examples of Q_T (resp. Q_B) of each type such that the intersection of Q_T and the half-space $x_3 \geq 1$ (resp. the intersection of Q_B and $x_3 \leq 0$) is lattice-free, then $Q_T \cap Q_B$ is going to be lattice-free. This is because Q_T is the disjoint union of Q_T^1, Q_T^2 and Q_T^3, where Q_T^1 is the intersection of Q_T and $x_3 \geq 1$, Q_T^2 is the intersection of Q_T and $0 \leq x_3 \leq 1$ and Q_T^3 is the intersection of Q_T and $x_3 \leq 0$. Similarly, Q_B is the intersection of Q_B^1, Q_B^2 and Q_B^3, where Q_B^1 is the intersection of Q_B and $x_3 \leq 0$, Q_B^2 is the intersection of Q_B and $0 \leq x_3 \leq 1$ and Q_B^3 is the intersection of Q_B and $x_3 \geq 1$. Then $Q_T \cap Q_B = (Q_T^1 \cap Q_B^3) \cup (Q_T^2 \cap Q_B^2) \cup (Q_T^3 \cap Q_B^1)$. If Q_T^1 and Q_B^1 are lattice-free, then $Q_T \cap Q_B$ will be lattice-free, since $(Q_T^2 \cap Q_B^2)$ is lattice free.

To get an unbounded wedge for Q_T, take a line segment in the region $1 \leq x_3 \leq 2$ and construct an unbounded wedge W intersecting TP with this line segment as *edge*(W).

Clearly, $W \cap \{x_3 \geq 1\}$ is lattice-free. A similar construction can be made for a truncated 3-pyramid. For a pyramid, take any point in the region $1 \leq x_3 \leq 2$ and construct a pyramid intersecting TP . This will again have the desired property. For a bounded tetrahedron, we give the following equations for the four planes.

$$\begin{array}{lcl} -x_1 - x_2 + 2x_3 & \leq & 2 \\ -10x_1 + 10x_2 + x_3 & \leq & 11 \\ x_1 + x_2 + 4x_3 & \leq & 6 \\ 2x_1 - 2x_2 - x_3 & \leq & 1 \end{array}$$

The first plane has $(0,0,1)$ on it, the second one has $(0,1,1)$, the third has $(1,1,1)$ and the fourth has $(1,0,1)$ on it. On the $x_3 = 0$ plane, the four lattice points included in Q_T are $(0,0,0),(0,1,0),(1,2,0),(1,1,0)$. The intersection with $x_3 \geq 1$ is a subset of the polyhedron Z formed by the first and third inequalities and $x_3 \geq 1$, and the intersection of the first and third inequalities is a line parallel to $x_3 = 0$ plane and at a height between 1 and 2. Therefore Z is lattice-free.

All constructions can be made for Q_B with a reflection about the $x_3 = 0.5$ plane. □

3 Polytopes with seven or less number of facets

In the remaining sections, we characterize maximal lattice-free polytopes with less than eight facets. In this analysis, we make the assumption that every facet has *exactly* one lattice point on it. As mentioned earlier, this is automatically true for polytopes with eight facets. This helps to weed out a whole slew of degenerate cases and makes the characterization of these polytopes much cleaner. This assumption can be argued to be not too restrictive because we can always tilt facets with more than one lattice point very slightly so that we get a non-degenerate maximal lattice-free polytope. Such a modification helps to approximate *any* lattice-free polytope (in the limit) by a sequence of non-degenerate polytopes, in the context of deriving valid inequalities for MILPs from these polytopes.

4 Polytopes with 7 facets

In this section, we analyze the set of maximal lattice-free sets with exactly 7 facets. As mentioned in Section 3, we give a complete characterization all maximal lattice-free polytopes with 7 facets which have *exactly* 1 lattice point on each facet. Theorem 1.2 implies that these 7 lattice points lie on two adjacent lattice planes. We make a unimodular transformation so that these two planes are $x_3 = 1$ and $x_3 = 0$. As in the previous section, we refer to these two planes as TP and BP respectively. Each plane contains at most 4 of the 7 lattice points by Lemma 1.3. So we have 3 points on one plane and 4 points on another.

Without loss of generality, let TP have 3 points. Any maximal lattice-free polytope Q with these 7 points on its facets intersects TP in a maximal lattice-free Type 3 triangle and intersects BP in a maximal lattice-free quadrilateral. Let Q_T be the polyhedron formed by the three planes corresponding to the facets of Q intersecting TP, and Q_B by the polyhedron formed by the four planes intersecting BP. Proposition 1.11 characterizes all the possible forms Q_B can have. Since Q_T is the intersection of only three planes, it can be either a cylinder over the Type 3 triangle or a 3-pyramid. So $Q = Q_T \cap Q_B$ where Q_T and Q_B can have the possible types described above. The next theorem exactly specifies which pairs are possible.

Theorem 4.1. *Any maximal lattice-free polytope with seven facets is the intersection of two polyhedra Q_T and Q_B, where Q_T is a 3-pyramid or a cylinder and Q_B is either a tetrahedron, an unbounded pyramid, an unbounded wedge or a truncated 3-pyramid. Upto unimodular transformations, the two lattice planes containing the seven lattice points are $x_3 = 0$ and $x_3 = 1$. If Q_B is an unbounded pyramid, then its vertex lies in the region $x_3 \leq 0$. If Q_B is an unbounded wedge, then edge(Q_B) lies in the region $x_3 \leq 0$. If Q_B is a truncated 3-pyramid, then one of the bounded edges of Q_B is in the region $x_3 \leq 0$ and the other two bounded edges are intersected by $x_3 = 0$. Moreover, if $Q_T \cap TP$ is lattice-free, then Q_T is a 3-pyramid with the vertex in the region $\{x_3 \geq 1\}$.*

Conversely, for any pair of polyhedra allowed in the statement above, there exist examples of maximal lattice-free polytopes with 7 facets which can be represented as the intersection of this pair of polyhedra.

Proof. We first show that Q_B cannot be a cylinder. $Q_B \cap TP$ is a translation of the quadrilateral $Q_B \cap BP$. Moreover, $Q_B \cap TP$ contains the three lattice points incident on the facets of Q_T.

Lemma 1.9 implies that $Q_B \cap TP$ contains a translation of three lattice points p_1, p_2, p_3 from $Q_B \cap BP$. Consider one of these three lattice points p_1 in BP and its translated copy p'_1 in TP. We claim that the vector $v = p'_1 - p_1$ is in the recession cone of Q_T. Let the 3 planes defining Q_T be $a^i x \leq b_i$, $i = 1, 2, 3$. Also, let the numbering be such that $a^i x \leq b_i$ passes through p'_i, where p'_i is the copy of p_i in TP. So $a^i p'_i = b_i$. Recall that Q_T *strictly* contains p_1, p_2, p_3 and also the fourth lattice point p_4 incident on a facet of Q_B. Since p_i belongs to Q_T, $a^i \cdot (p'_i + v) < b_i$, for $i = 1, 2, 3$. Therefore $a^i \cdot v < 0$ for $i = 1, 2, 3$; thus proving that v is in the recession cone of Q_T. Now note that $p_4 + v$ is in the cylinder Q_B (since $p_i - v$ was in the cylinder for $i = 1, 2, 3$). Moreover, p_4 is contained in Q_T and so $p_4 + v$ is contained in Q_T. So the lattice point $p_4 + v$ is contained in $Q_T \cap Q_B$, which is a contradiction.

If Q_B is an unbounded pyramid, an unbounded wedge or a truncated 3-pyramid, first note that the vertex or the bounded edge must be on one side of BP by Proposition 1.11. Then it must satisfy the conditions stated in the theorem. Otherwise it would satisfy the hypothesis of Lemma 1.12 and there would exist a cylinder Q'_B with $Q'_B \cap BP = Q_B \cap BP$, $Q'_B \cap TP \supseteq Q_B \cap TP$ and $Q'_B \cap \{x_3 = -1\} \subseteq Q_B \cap \{x_3 = -1\}$. But from the previous paragraph, we know that $Q'_B \cap \{x_3 = -1\}$ would contain a lattice point also contained in Q_T. So this lattice point would also be in $Q_B \cap Q_T$, a contradiction.

We now show that if $Q_T \cap TP$ is lattice-free then Q_T cannot be a cylinder. $Q_T \cap TP$ is a Type 3 triangle T. $Q_T \cap BP$ is a translation of T over the lattice $\Lambda(BP)(= \Lambda(TP)$. But Lemma 1.8 says that this translated version of T cannot contain four lattice points from BP forming a parallelogram of area 1. This is a contradiction, because we know that Q_T contains the four lattice point incident on the facets of Q_B. So, in this case, Q_T is a 3-pyramid. Since TP intersects all three facets and BP is parallel to TP and intersects Q_T, the vertex must either be in the region $x_3 \geq 1$ or in the region $\{x_3 \leq 0\}$. In the latter case, $Q_T \cap BP$ is a homothetic copy of $Q_T \cap TP$ with dilation less than 1. But then Lemma 1.8 implies it cannot contain four lattice point from BP. Therefore the vertex is in the region $\{x + 3 \geq 1\}$.

We now give an example of a maximal lattice-free polytope Q with seven facets where Q_T is a cylinder. As stated in the theorem, this implies that $Q_T \cap TP$ is not lattice-free. The lattice points in the interior of $Q_T \cap TP$ are cut off by Q_B, so that Q is lattice-free.

Q_T is defined by the following inequalities.

$$\begin{aligned} x_1 + x_2 &\geq 0 \\ -2x_1 + x_2 &\leq 1 \\ x_1 - 2x_2 &\leq 1 \end{aligned}$$

So Q_T is a cylinder with the x_3-axis as its linearity space. Note that $Q_T \cap \{x_3 = 1\}$ is an unbounded polygon which is not lattice-free.

In this example, Q_B is an unbounded pyramid. We describe Q_B by describing its intersection with $\{x_3 = 0\}$ and giving the coordinates of its vertex. This is sufficient for defining the entire pyramid. $Q_B \cap \{x_3 = 0\}$ is a 2-dimensional cross-polytope defined by the following inequalities.

$$\begin{aligned} x_1 + x_2 &\geq 0 \\ -x_1 + x_2 &\leq 1 \\ x_1 + x_2 &\leq 2 \\ x_1 - x_2 &\leq 1 \end{aligned}$$

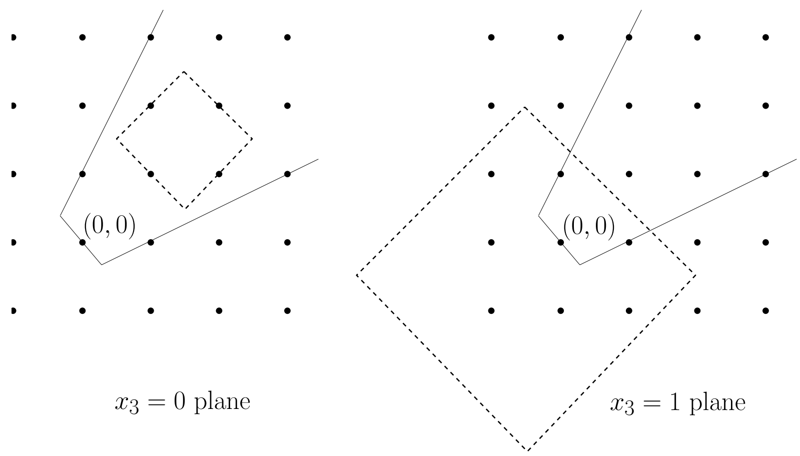

Figure 3: Example of a 7 sided polytope with Q_T as a cylinder

The vertex of Q_B is $(\frac{17}{6}, \frac{17}{6}, \frac{1}{3})$. Figure 3 shows the intersections of Q_T and Q_B with the planes $x_3 = 0$ and $x_3 = 1$. Note that $Q_T \cap Q_B = \phi$ in the region $\{x_3 > 1.6\}$. Also, $Q_B \cap \{x_3 \leq 0\}$ is lattice-free. These two facts imply that $Q = Q_T \cap Q_B$ is lattice-free.

From this example, it is possible to construct examples exhibiting all the other pairs allowed by the statement of the theorem by just slightly tilting some appropriate facet of Q_T or Q_B. For instance, note that the theorem allows examples where Q_T is a 3-pyramid with the vertex in the region $x_3 \leq 0$ (this again implies that $Q_T \cap TP is not lattice-free$). These can also be constructed from our example above by modifying Q_T slightly.

□

5 Polytopes with 6 facets

We remind the reader of the assumption made in Section 3, which says that every facet of our six-sided polytope has exactly 1 lattice point on it. By Theorem 1.2, we know that these six lattice points lie on two adjacent lattice planes. Lemma 1.3 implies that either we have 3 on each plane or we have 4 on one plane and 2 on the other. We look at each of these case in turn in the next two subsections.

Upto unimodular transformations, the adjacent lattice planes are taken to be $x_3 = 0$ and $x_3 = 1$ (referred to as TP and BP respectively, like in the previous section). Further, without loss of generality, in the case with four coplanar lattice points, we assume that BP contains these four and TP contains two. We again consider the polytopes Q_T and Q_B formed by the planes corresponding to the facets intersecting TP and BP respectively.

5.1 Three lattice points on the adjacent planes

Theorem 5.1. *Consider a maximal lattice-free polytope Q with six facets such that the six lattice points can be split into two groups of 3 points each lying on adjacent lattice planes. Without loss of generality, let these two planes be $x_3 = 0$ and $x_3 = 1$ respectively. Then Q is the intersection of two polytopes Q_T and Q_B where both Q_T and Q_B are either cylinders or 3-pyramids. Furthermore, in the case of 3-pyramids, there is no restriction on the position of the vertex like in Theorems 2.1 and 4.1; there exist examples of Q for every pair of these polyhedra.*

Proof. The statement of the theorem is trivial to prove. We exhibit an example of Q where both Q_T and Q_B are cylinders. All other pairs of polyhedra (with vertices of the 3-pyramids as desired) can be constructed from this canonical example by slightly tilting some appropriate facet of Q_T or Q_B.

Figure 4 shows the intersections of Q_T and Q_B with the planes $x_3 = -1, 0, 1, 2$. The lattice points on the three planes are shown, after which point they become periodic. The boxes are lattice points in the $x_3 = 1$ plane, the black dots are the lattice points on the $x_3 = 0$ plane and the crosses are lattice points in the $x_3 = -1$ (and $x_3 = 2$ plane).

□

5.2 Four lattice points on one plane and two on the other

We have $Q = Q_T \cap Q_B$ where Proposition 1.11 gives us the possibilities for Q_B and Q_T is clearly either a split or a valley with a cleft. The next theorem exactly characterizes the possible pairs for these two polytopes.

Theorem 5.2. *Consider a maximal lattice-free polytope Q with six facets such that four of the lattice points on its facets are coplanar and the other two lie on an adjacent lattice plane. Without loss of generality, let these two planes be $x_3 = 0$ and $x_3 = 1$ respectively. Then Q is the intersection of two polytopes Q_T and Q_B where Q_T is a valley with a cleft and Q_B is a tetrahedron, an unbounded pyramid, an unbounded wedge, or a truncated 3-pyramid. Moreover, if Q_B is not a tetrahedron, the vertex or a bounded edge of Q_B is in the region $x_3 \leq 0$.*

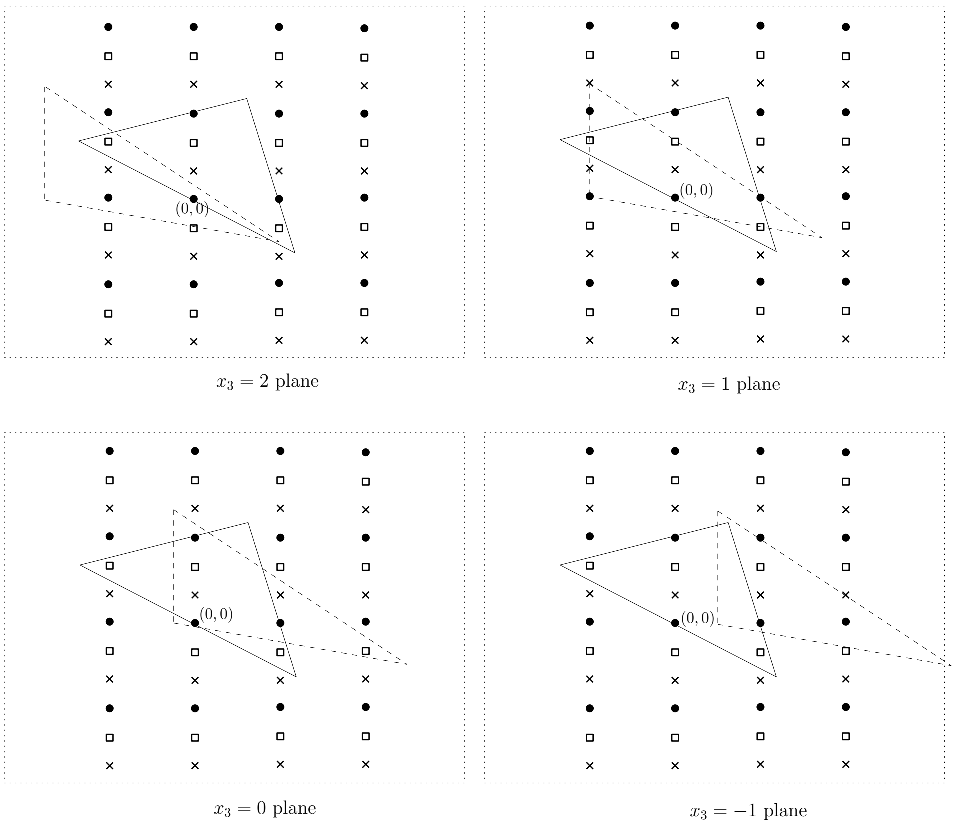

Figure 4: Example of a 6 sided polytope as intersection of two cylinders

Proof. Let the four lattice points on BP which form a parallelogram Y and the two lattice points on TP be q_1, q_2. Without loss of generality, Y can be assumed to be the unit square.

We first show that Q_B cannot be a cylinder. $Q_B \cap TP$ is a translation of $X = Q_B \cap BP$ over the lattice $\Lambda(TP)(= \Lambda(BP))$, and also contains both q_1, q_2. Lemma 1.9 states that q_1q_2 must be a translation of an edge or diagonal of Y. Let it be the translation of the segment joining vertices p_1, p_2 of Y, and let $v = p_1 - q_1 = p_2 - q_2$. Let the two planes defining Q_T be $a^1 \cdot x \leq b_1$ and $a^2 \cdot x \leq b_2$. Since p_1 and p_2 are both strictly contained in Q_T, $a^i \cdot p_i < b_i$ for $i = 1, 2$. Also, $a^i \cdot q_i = b_i$, since q_i's lie on these two planes. Therefore, $a^i \cdot v < 0$, for $i = 1, 2$. Now there exists a vertex p of Y such that $p + v$ is in the cylinder Q_B. And since p is in Q_T, $p + v$ is in Q_T. So the lattice point $p + v$ is in $Q_T \cap Q_B$, contradicting the fact that Q is lattice-free.

Moreover, like in the proofs of Theorems 2.1 and 4.1, this implies the statement about the vertex or the bounded edge of Q_B.

We next show that Q_T cannot be a split. Since Q_B is not a cylinder, $Q_B \cap TP$ is a homothetic copy of $Q_B \cap BP$ and it contains the two lattice points q_1, q_2 from TP in its interior. If there are lattice points other than q_1, q_2 in $Q_B \cap TP$, then Lemma 1.10 tells that some two of these are a translation of an edge of Y. □

References

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