1. **(10 pts)** For each of the following pair of statements, circle the correct statement :

(a) $2^x 2^y = 2^{x+y}$ $\qquad$ $2^x 2^y = 2^{xy}$.

(b) $\int f(x)g(x)dx = (\int f(x)dx)(\int g(x)dx)$ $\qquad$ $\int[f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

(c) $log_7 x = \frac{ln(x)}{ln(7)}$ $\qquad$ $log_7 x = ln(7)ln(x)$.

(d) $e^{ln(x)} = x$ $\qquad$ $e^{ln(x)} = x^e$.

(e) $ln(x) - ln(y) = ln(x - y)$ $\qquad$ $ln(x) - ln(y) = ln(x/y)$.

2. Differentiate the following functions.

(a) **(4 pts)** e^{x^2}.

$$\frac{d}{dx}\left(e^{x^2}\right) \xleftarrow[\text{Rule}]{\text{chain}} e^{x^2}\frac{d}{dx}(x^2)$$

$$= e^{x^2}(2x)$$

$$= \boxed{2x\,e^{x^2}}$$

(b) **(4 pts)** $ln(4x\sqrt{1+x^2})$.

$$= \ln 4 + \ln x + \frac{1}{2}\ln(1+x^2)$$

$$\frac{d}{dx}(\ln 4) + \frac{d}{dx}(\ln x) + \frac{d}{dx}\left(\frac{1}{2}\ln(1+x^2)\right)$$

$$= 0 + \frac{1}{x} + \frac{1}{2}\cdot\frac{1}{1+x^2}\frac{d}{dx}(1+x^2)$$

$$= \frac{1}{x} + \frac{1}{\cancel{2}}\frac{1}{1+x^2}(\cancel{2}x) = \boxed{\frac{1}{x} + \frac{x}{1+x^2}}$$

(c) **(4 pts)** $(e^x + x + 5)^2$.

$$\frac{d}{dx}(e^x+x+5)^2 \underset{\text{Rule}}{\overset{\text{chain}}{=}} 2(e^x+x+5)\frac{d}{dx}(e^x+x+5)$$

$$= 2(e^x+x+5)\left[\frac{d}{dx}(e^x) + \frac{d}{dx}(x) + \frac{d}{dx}(5)\right]$$

$$= \boxed{2(e^x+x+5)(e^x+1)}$$

(d) **(8 pts)** $(2x)^{5x}$.

Rewrite $(2x)^{5x} = e^{(5x)\ln(2x)}$

$$\frac{d}{dx}\left(e^{5x\ln(2x)}\right) = e^{5x\ln(2x)}\frac{d}{dx}(5x\ln(2x))$$

$$= e^{5x\ln(2x)}\left[5x\frac{d}{dx}(\ln(2x)) + \ln(2x)\frac{d}{dx}(5x)\right]$$

$$= e^{5x\ln(2x)}\left[5\cancel{x}\cdot\frac{1}{\cancel{2}\cancel{x}}\cdot\cancel{2} + \ln(2x)\cdot 5\right]$$

$$= e^{5x\ln(2x)}\left[5 + \ln(2x)\cdot 5\right]$$

$$= 5e^{5x\ln(2x)}(1+\ln(2x))$$

$$= \boxed{5(2x)^{5x}(1+\ln(2x))}$$

3. A bank offers 1% yearly rate of interest.

(a) **(3 pts)** I deposit $100 at the beginning of the year. The interest in the bank is compounded yearly. How much money do I have at the end of 5 years ?

$$100\left(1+0.01\right)^{5} = \boxed{100\left(1.01\right)^{5}}$$

(b) **(3 pts)** Now suppose the interest in the bank is compounded every 3 months, that is, it is compounded 4 times a year. If I deposit $100 at the beginning of the year, how much will I have after 5 years ?

$$100\left(1+\frac{0.01}{4}\right)^{4\cdot 5} = \boxed{100\left(1.0025\right)^{20}}$$

(c) **(4 pts)** Now suppose the interest in the bank is compounded continuously. If I deposit $100 at the beginning of the year, how much time does it take for the amount to become $500 ?

$$100\, e^{0.01t} = 500$$

$$\Rightarrow \quad e^{0.01t} = 5$$

$$\Rightarrow \quad 0.01t = \ln 5$$

$$\Rightarrow \quad t = \boxed{\frac{\ln 5}{0.01}}$$

4. **(10 pts)** Suppose that a bacterial culture grows according to the exponential growth function

$$y = Ce^{kt},$$

where time t is measured in minutes. We observe that we have 5000 bacteria after 2 minutes, and 25000 bacteria after 3 minutes. How many bacteria will we have after 4 minutes ?

$t=2, y=5000$: $5000 = Ce^{2k}$

$t=3, y=25000$: $25000 = Ce^{3k}$

Using first eq.

$$C = \frac{5000}{e^{2k}}$$

Substituting into second eq.

$$25000 = \frac{5000}{e^{2k}} \cdot e^{3k} = 5000e^{k}$$

$$\Rightarrow \frac{25\cancel{000}}{5\cancel{000}} = e^{k} \Rightarrow \ln\left(\frac{25}{5}\right) = k$$

$$\Rightarrow k = \ln 5.$$

$$\therefore C = \boxed{\frac{5000}{e^{2\ln 5}}} = \frac{5000}{e^{\ln(5^2)}} = \frac{5000}{25} = \boxed{200}$$

$\therefore$ At $t=4$, $y = 200e^{4k}$

$$= 200e^{4\ln 5} = 200e^{\ln(5^4)} = \boxed{125000}$$

$$\boxed{y = \frac{5000}{e^{2\ln 5}} \cdot e^{4\ln 5}} \leftarrow \text{this also OK!}$$

5. **(30 pts)** Compute the following integrals.

(a) $\int (ln(x) + x)^2(\frac{1}{x} + 1)dx$

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Substitute $u = \ln x + x$ $\quad \frac{du}{dx} = \frac{1}{x} + 1 \Rightarrow du = \left(\frac{1}{x} + 1\right)dx$

$$\int (\ln x + x)^2\left(\frac{1}{x}+1\right)dx = \int u^2 du = \underbrace{\frac{u^3}{3} + C}_{2} = \boxed{\frac{(\ln x + x)^3}{3} + C}$$

2 (subst.)

(b) $\int \frac{x^3+5x}{\sqrt{x}}dx = \int \left(\frac{x^3}{\sqrt{x}} + \frac{5x}{\sqrt{x}}\right)dx = \int \frac{x^3}{\sqrt{x}}dx + \int \frac{5x}{\sqrt{x}}dx$

$$= \int x^{3-\frac{1}{2}}dx + \int 5x^{1-\frac{1}{2}}dx$$

$$\boxed{= \frac{x^{3.5}}{3.5} + \frac{5x^{1.5}}{1.5} + C}$$

(c) $\int x^2 e^{(x^3+5)}dx$

$u = x^3 + 5 \quad \frac{du}{dx} = 3x^2 \Rightarrow du = 3x^2 dx$

$$\int x^2 e^{(x^3+5)}dx = \int \frac{1}{3} \cdot 3x^2 e^{(x^3+5)}dx$$

$$= \int \frac{1}{3}e^u du = \frac{1}{3}\int e^u du$$

$$= \frac{1}{3}e^u + C$$

$$\boxed{= \frac{1}{3}e^{(x^3+5)} + C}$$

(d) $\int \frac{x^{10}+10x^9+1}{x+10}dx$ Use long division and rewrite integral.

$$= \int \left(x^9 + \frac{1}{x+10}\right) dx$$

$$= \int x^9\, dx + \int \frac{1}{x+10}\, dx = \frac{x^{10}}{10} + \int \frac{1}{x+10}\, dx$$

Substitute $u = x+10 \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$

$$\frac{x^{10}}{10} + \int \frac{1}{x+10}\, dx = \frac{x^{10}}{10} + \int \frac{1}{u}\, du = \frac{x^{10}}{10} + \ln|u| + C$$

$$= \boxed{\frac{x^{10}}{10} + \ln|x+10| + C}$$

(e) $\int \frac{xe^{x^2}}{1+e^{x^2}}dx$

Substitute $u = 1 + e^{x^2}$

$$\frac{du}{dx} = \frac{d}{dx}\left(1+e^{x^2}\right) = 0 + \frac{d}{dx}\left(e^{x^2}\right)$$

$$= 2xe^{x^2}$$

$$du = 2xe^{x^2}\, dx$$

Rewrite integral :

$$\int \frac{1}{2} \cdot \frac{2xe^{x^2}}{1+e^{x^2}}\, dx = \int \frac{1}{2} \frac{1}{u}\, du$$

$$= \frac{1}{2} \int \frac{1}{u}\, du$$

$$= \frac{1}{2} \ln|u| + C$$

$$= \boxed{\frac{1}{2} \ln\left|1+e^{x^2}\right| + C}.$$