1 Basic notions and tools from lattice theory

This lecture surveys the basic definitions and important tools from lattice theory that we will need in this course. Throughout this class, we will be concerned with the n-dimensional Euclidean space $\mathbb{R}^n$. For any real number $r \in \mathbb{R}$, we denote the greatest integer smaller than or equal to r by $\lfloor r \rfloor$, the smallest integer greater than or equal to r by $\lceil r \rceil$ and $\{r\}$ will be used to denote $r - \lfloor r \rfloor$.

Definition 1 *Given a set $S \subseteq \mathbb{R}^n$, $span(S)$ is the linear space spanned by the vectors in S. $span(S)$ is equivalent to the set of all finite linear combinations of vectors from S.*

Definition 2 *A subset $\Lambda \subseteq \mathbb{R}^n$ is a subgroup under addition if the following three properties hold :*

i $0 \in \Lambda$.

ii For any $x, y \in \Lambda$, $x + y \in \Lambda$.

iii For any $x \in \Lambda$ then $-x \in \Lambda$.

A subgroup Λ is called a discrete subgroup *if there exists $\epsilon > 0$, such that $B(0, \epsilon) \cap \Lambda = \{0\}$.*

Definition 3 *Given a linear subspace L of $\mathbb{R}^n$, a discrete subgroup Λ is called a* lattice of *L, if $span(\Lambda) = L$. A basis for a lattice Λ in L is a set of linearly independent vectors $a_1, \ldots, a_k \in L$ such that $\Lambda = \{\mu_1 a_1 + \ldots + \mu_k a_k \mid \mu_1, \ldots \mu_k \in \mathbb{Z}\}$. Λ is then said to be* generated *by $a_1, \ldots, a_k$.*

Note that it is not *a priori* obvious that every lattice Λ of a linear subspace L will have a basis. However, note that if Λ does have a basis $A = \{a_1, \ldots, a_k\}$ then $span(A) = L$. We give some examples of lattices.

Example 4

1. *The set $\mathbb{Z}^n$, i.e. all points in $\mathbb{R}^n$ with integral coordinates, is a lattice of $\mathbb{R}^n$.*
2. *Consider the hyperplane $H = \{(x_1, \ldots, x_n) \mid x_1 + \ldots + x_n = 0\}$ in $\mathbb{R}^n$. Then $H \cap \mathbb{Z}^n$ is a lattice of H. Try to draw this lattice for $n = 3$.*
3. *Consider any set of linearly independent vectors $\{b_1, \ldots, b_k\}$ in $\mathbb{R}^n$. Let $L = span(b_1, \ldots, b_k)$ be the linear subspace spanned by these vectors. Then the set $\Lambda = \{\mu_1 b_1 + \ldots + \mu_k b_k \mid \mu_1, \ldots \mu_k \in \mathbb{Z}\}$ is a lattice of L. Prove this !*

The following fact will be useful, we leave the proof for the reader.

Proposition 5 *Let Λ be a discrete subgroup. Consider any bounded set A. Then $|A \cap \Lambda|$ is finite, i.e. there are finitely many points from a discrete subgroup in any bounded set.*

We define $dist(x, y) = \|x - y\|_2$ for any two points $x, y \in \mathbb{R}^n$. Given a set $A \subseteq \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ define $dist(x, A) = \inf\{dist(x, y) \mid y \in A\}$. The following result from finite dimensional analysis will be used (it follows from Weierstrauss' theorem on minimizing continuous functions over compact sets).

Proposition 6 *If $A \subseteq \mathbb{R}^n$ is a compact set (which is equivalent to saying it is closed and bounded), and $x \notin A$ is a point in $\mathbb{R}^n$, $dist(x, A) > 0$ and there exists a $y \in A$ such that $dist(x, A) = dist(x, y)$.*

The following lemma is an important property of lattices. It says that if we consider a lattice Λ of a subspace W and a subspace $L \subsetneq W$, such that L is spanned by lattice vectors from Λ, then there is a non-zero distance $\epsilon > 0$ between L and any lattice point $y \in \Lambda \setminus L$.

Lemma 7 *Let Λ be a lattice of a linear subspace W. Let $b_1, \ldots, b_k \in \Lambda$, $k < dim(W)$ be linearly independent vectors in the lattice Λ and let $L = span(b_1, \ldots, b_k)$. Then there exists $v \in \Lambda \setminus L$ such that $dist(v, L) \leq dist(w, L)$ for all $w \in \Lambda \setminus L$.*

Note that the crucial property of the subspace is that it is spanned by vectors from the lattice. For example, consider the lattice $\mathbb{Z}^2$ of $\mathbb{R}^2$ and the subspace L spanned by the ray $(1, \sqrt{2})$. The only lattice point in L is $\{0\}$. Then for any $v \in \mathbb{Z}^2 \setminus \{0\}$, there exists a $w \in \mathbb{Z}^2 \setminus \{0\}$ such that $dist(w, L) < dist(v, L)$. (Check this !)

Proof:[Proof of Lemma 7]

Consider the set $\Pi = \{\sum_{i=1}^{k} \mu_i b_i \mid 0 \leq \mu_i \leq 1 \text{ for all } i\}$. Consider any $\rho \in \Lambda \setminus \Pi$ (why does such a ρ exist ?) and by Proposition 6, $d = dist(\rho, \Pi) > 0$ since Π is closed and bounded.

Consider the bounded set $\Pi_d = \{p \in \mathbb{R}^n \mid dist(p, \Pi) \leq d\}$. Note that $\Pi \subseteq \Pi_d$ and $\rho \in \Pi_d$. Moreover, Fact 5 shows that there are finitely many points from Λ in Π_d. In particular, there are finitely many points in $\Lambda \setminus L$ in Π_d (ρ is one such point). Therefore, there exists $v \in \Lambda \setminus L$ such that

$$dist(v, \Pi) \leq dist(w, \Pi) \text{ for all } w \in \Lambda \setminus L \text{ (why ?).} \tag{1}$$

We now claim that this implies

$$dist(v, L) \leq dist(w, L) \quad \text{for all } w \in \Lambda \setminus L$$

Consider any $w \in \Lambda \setminus L$ and consider any $y \in L$. Since $y \in L$, one can express y as a linear combination of $b_1, \ldots, b_k : y = \sum_{i=1}^{k} \lambda_i b_i$. Let $z = \sum_{i=1}^{k} \lfloor \lambda_i \rfloor b_i$. Notice that since $b_i \in \Lambda$, an integral combination of the b_i's are also in Λ by the subgroup property (ii) and hence $z \in \Lambda$. Then, $y - z \in \Pi$ and $w - z \in \Lambda \setminus L$ (why ?). We then have the following relations,

$$dist(w, y) = dist(w - z, y - z) \geq dist(w - z, \Pi) \geq dist(v, \Pi) \geq dist(v, L)$$

The first equality is simply the invariance of distance under translation. The second inequality follows because $y - z \in \Pi$. The third inequality follows from (1). The fourth inequality follows simply because $\Pi \subseteq L$. This shows that $dist(w, y) \geq dist(v, L)$ for any $w \in \Lambda \setminus L$ and any $y \in L$. Taking an infimum over all $y \in L$, we get that $dist(w, L) \geq dist(v, L)$ for any $w \in \Lambda \setminus L$. $\square$

We now prove the basis theorem for lattices.

Theorem 8 *Every lattice Λ of a linear subspace L has a basis.*

Proof: We will iteratively build up a basis for Λ. At iteration $k \geq 0$, we will have a set $B_k = \{b_1, \ldots b_k\}$ of linearly independent vectors in L and we will work with the subspace $L_k \subseteq L$ spanned by B_k. We set $B_0 = \{\}$ and so $L_0 = \{0\}$. The property that we will satisfy at every iteration step $k \geq 0$ is that

$$L_k \cap \Lambda = \{\mu_1 b_1 + \ldots + \mu_k b_k \mid \mu_1, \ldots \mu_k \in \mathbb{Z}\}. \tag{2}$$

Hence, if we can show that we reach iteration $k = dim(L)$ then we will be done.

Suppose we are at iteration $k < dim(L)$. So we have a set $B_k = \{b_1, \ldots b_k\}$ with the above property (2) satisfied. We now show how to find b_{k+1}. Consider $\bar{w} \in \Lambda \setminus L_k$ (why does such a $\bar{w}$ exist ?). Let $L' = span(b_1, \ldots, b_k, \bar{w})$ and consider the lattice $\Lambda' = L' \cap \Lambda$ of L' (why is this a lattice of L' ?). By Lemma 7, there exists $v \in \Lambda' \setminus L_k$ such that $dist(v, L_k) \leq dist(w, L_k)$ for all $w \in \Lambda' \setminus L_k$. Note that since $v \in \Lambda' \setminus L_k$, $span(b_1, \ldots, b_k, \bar{w}) = span(b_1, \ldots, b_k, v)$. So $L' = span(b_1, \ldots, b_k, v)$ and we will set $b_{k+1} := v$ and $L_{k+1} = L'$. Consequently, we have that $\Lambda' = L' \cap \Lambda = L_{k+1} \cap \Lambda$. We then need to show the following claim.

Claim 1 $L_{k+1} \cap \Lambda = \Lambda' = \{\mu_1 b_1 + \ldots + \mu_k b_k + \mu_{k+1} v \,|\, \mu_1, \ldots \mu_{k+1} \in \mathbb{Z}\}$.

Consider any $p \in \Lambda' = L_{k+1} \cap \Lambda$. Since L_{k+1} is the span of $\{b_1, \ldots, b_k, v\}$, we can write $p = \sum_{i=1}^{k} \lambda_i b_i + \lambda v$. Then $\bar{p} = p - \lfloor \lambda \rfloor v$ belongs to Λ' (since p and v are both in Λ', the subgroup property (ii) implies that $p - \lfloor \lambda \rfloor v$ is also in Λ'). Note that $\bar{p} = \sum_{i=1}^{k} \lambda_i b_i + \{\lambda\} v$ and $\sum_{i=1}^{k} \lambda_i b_i$ is in L_k. Therefore, $dist(\bar{p}, L_k) = dist(\{\lambda\} v, L_k) = \{\lambda\} dist(v, L_k) < dist(v, L_k)$, since $\{\lambda\} < 1$. This implies that $\bar{p}$ is in L_k, because $dist(v, L_k) \leq dist(w, L_k)$ for all $w \in \Lambda' \setminus L_k$ and therefore $\{\lambda\} = 0$. Therefore, λ is an integer. Moreover, $\bar{p} = \sum_{i=1}^{k} \lambda_i b_i$ and $\bar{p} \in \Lambda' \subseteq \Lambda$. So in fact, $\bar{p} \in \Lambda \cap L_k$. But then property (2) holds for iteration k and therefore λ_i are all integer (note that this uses the standard fact from linear algebra that there is a unique representation of any vector using linearly independent vectors).

We have thus shown that p can be expressed as an integral combination of $\{b_1, \ldots, b_k, v\}$ and the claim is proved (actually this proves only one inclusion, the other inclusion is trivial to prove - check this !).

$\square$

Combined with Example 3 above, this theorem says that the concept of a lattice and the concept of taking integral combinations of linearly independent vectors are equivalent.

Question 1 *The proof for Theorem 8 is constructive. Does it suggest a reasonable algorithm to construct a basis from an implicit definition of a lattice (say, Example 2 above)? Why or why not ?*

2 Maximal Lattice-free Convex Sets

We explore the connections between lattices of $\mathbb{R}^n$ and convex sets in $\mathbb{R}^n$. The structures will prove to be useful for developing a general theory of cutting planes for general Mixed-Integer Linear Programs (MILPs).

Consider some lattice $\Lambda$of $\mathbb{R}^n$.

Definition 9 *Let Λ be a lattice of $\mathbb{R}^n$. A set $S \subset \mathbb{R}^n$ is said to be a* Λ-free convex set of $\mathbb{R}^n$ *if S is convex, $\Lambda \cap \mathbf{int}(S) = \emptyset$, and S is said to be a* maximal Λ-free convex set of $\mathbb{R}^n$ *if it is not properly contained in any Λ-free convex set.*

When the lattice is clear from the context, we will often use the term maximal lattice-free convex sets. A characterization of maximal lattice-free convex sets, is given by the following.

Theorem 10 *Let Λ be a lattice of a linear space V of $\mathbb{R}^n$. A set $S \subset \mathbb{R}^n$ is a maximal Λ-free convex set of V if and only if one of the following holds:*

(i) S *is a polyhedron of the form* $S = P + L$ *where* P *is a polytope,* L *is a lattice-subspace of* $\mathbb{R}^n$*,* $\dim(S) = \dim(P) + \dim(L) = n$*,* S *does not contain any point of* Λ *in its interior and there is a point of* Λ *in the relative interior of each facet of* S*;*

(ii) $\dim(S) < n$*,* S *is an affine hyperplane of* $\mathbb{R}^n$ *of the form* $S = v + L$ *where* $v \in S$ *and* L *is not a lattice-subspace of* $\mathbb{R}^n$*.*

The following Lemma is a corollary of Lemma 7, we leave the proof for the reader.

Lemma 11 *Given a lattice-subspace* L *of* V*, the orthogonal projection of* Λ *onto* $L^\perp$ *is a lattice of* $L^\perp \cap V$*.*

Lemma 12 *Given* $y \in \Lambda$ *and* $r \in V$*, then for every* $\varepsilon > 0$ *and* $\bar{\lambda} \geq 0$*, there exists a point of* Λ *at distance less than* ε *from the half line* $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$*.*

Proof: First we show that, if the statement holds for $\bar{\lambda} = 0$, then it holds for arbitrary $\bar{\lambda}$. Given $\varepsilon > 0$, let Z be the set of points of Λ at distance less than ε from $\{y+\lambda r \mid \lambda \geq 0\}$. Suppose, by contradiction, that no point in Z has distance less than ε from $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$. Then Z is contained in $B_\varepsilon(0) + \{y + \lambda r \mid 0 \leq \lambda \leq \bar{\lambda}\}$. By Theorem 5, Z is finite, thus there exists an $\bar{\varepsilon} > 0$ such that every point in Z has distance greater than $\bar{\varepsilon}$ from $\{y + \lambda r \mid \lambda \geq 0\}$, a contradiction. So we only need to show that, given $\bar{\varepsilon}$, there exists at least one point of Λ at distance at most $\bar{\varepsilon}$ from $\{y + \lambda r \mid \lambda \geq 0\}$.

Let $m = \dim(V)$ and $a_1, \ldots, a_m$ be a basis of Λ. Then there exists $\alpha \in \mathbb{R}^m$ such that $r = \alpha_1 a_1 + \ldots, \alpha_m a_m$. Denote by A the matrix with columns $a_1, \ldots, a_m$, and define $\|A\| = \sup_{x:\, \|x\| \leq 1} \|Ax\|$ where, for a vector v, $\|v\|$ denotes the Euclidean norm of v. Choose $\varepsilon > 0$ such that $\varepsilon < 1$ and $\varepsilon \leq \bar{\varepsilon}/(\|A\|\sqrt{m})$. By Dirichlet's theorem, there exist $p \in \mathbb{Z}^m$ and $\lambda > 0$ such that

$$\|\alpha - \frac{p}{\lambda}\| = \sqrt{\sum_{i=1}^{m} \left|\alpha_i - \frac{p_i}{\lambda}\right|^2} \leq \frac{\varepsilon\sqrt{m}}{\lambda} \leq \frac{\bar{\varepsilon}}{\|A\|\lambda}.$$

Let $z = Ap + y$. Since $p \in \mathbb{Z}^m$, then $z \in \Lambda$. Furthermore

$$\|(y + \lambda r) - z\| = \|\lambda r - Ap\| = \|A(\lambda\alpha - p)\| \leq \|A\|\|\lambda\alpha - p\| \leq \bar{\varepsilon}.$$

□

The following is a converse to Lemma 7.

Lemma 13 *If a linear subspace* L *of* V *is not a lattice-subspace of* V*, then for every* $\varepsilon > 0$ *there exists* $y \in \Lambda \setminus L$ *at distance less than* ε *from* L*.*

Proof: The proof is by induction on $k = \dim(L)$. Assume L is a linear subspace of V that is not a lattice-subspace, and let $\varepsilon > 0$. If $k = 1$, then, since the origin 0 is contained in Λ, by Lemma 12 there exists $y \in \Lambda$ at distance less than ε from L. If $y \in L$, then $L = \langle y \rangle$, thus L is a lattice-subspace of V, contradicting our assumption.

Hence we may assume that $k \geq 2$ and the statement holds for spaces of dimension $k - 1$. Suppose L contains a nonzero vector $r \in \Lambda$. Let

$$L' = \mathrm{proj}_{\langle r \rangle^\perp}(L), \quad \Lambda' = \mathrm{proj}_{\langle r \rangle^\perp}(\Lambda).$$

By Lemma 11, Λ' is a lattice of $\langle r\rangle^\perp \cap V$. Also, L' is not a lattice subspace of $\langle r\rangle^\perp \cap V$ with respect to Λ', because if there exists a basis $a_1, \ldots, a_{k-1}$ of L' contained in Λ', then there exist scalars $\mu_1, \ldots, \mu_{k-1}$ such that $a_1 + \mu_1 r, \ldots, a_{k-1} + \mu_{k-1} r \in \Lambda$, but then $r, a_1 + \mu_1 r, \ldots, a_{k-1} + \mu_{k-1} r$ is a basis of L contained in Λ, a contradiction. By induction, there exists a point $y' \in \Lambda' \setminus L'$ at distance less than ε from L'. Since $y' \in \Lambda'$, there exists a scalar μ such that $y = y' + \mu r \in \Lambda$, and y has distance less than ε from L.
Thus $L \cap \Lambda = \{0\}$. By Lemma 12, there exists a nonzero vector $y \in \Lambda$ at distance less than ε from L. Since L does not contain any point in Λ other than the origin, $y \notin L$. $\square$

The following lemma proves the "only if" part of Theorem 10 when S is bounded and full-dimensional.

Lemma 14 *Let Λ be a lattice of a linear space V of $\mathbb{R}^n$. Let $S \subset V$ be a bounded maximal Λ-free convex set with $\dim(S) = \dim(V)$. Then S is a polytope with a point of Λ in the relative interior of each of its facets.*

Proof: Since S is bounded, there exist integers L, U such that S is contained in the box $B = \{x \in \mathbb{R}^d \,|\, L \leq x_i \leq U\}$. For each $y \in \Lambda \cap B$, since S is convex there exists a closed half-space H^y of V such that $S \subseteq H^y$ and $y \notin \mathbf{int}(H^y)$. By Proposition 5, $B \cap \Lambda$ is finite, therefore $\bigcap_{y \in B\cap\Lambda} H^y$ is a polyhedron. Thus $P = \bigcap_{y \in B\cap\Lambda} H^y \cap B$ is a polytope and by construction $\Lambda \cap \mathbf{int}(P) = \emptyset$. Since $S \subseteq B$ and $S \subseteq H^y$ for every $y \in B \cap \Lambda$, it follows that $S \subseteq P$. By maximality of S, $S = P$, therefore S is a polytope. We only need to show that S has a point of Λ in the relative interior of each of its facets. Let $F_1, \ldots, F_t$ be the facets of S, and let $H_i = \{x \in V \,|\, \alpha_i x \leq \beta_i\}$ be the closed half-space defining F_i, $i = 1, \ldots, t$. Then $S = \bigcap_{i=1}^t H_i$. Suppose, by contradiction, that one of the facets of S, say F_t, does not contain a point of Λ in its relative interior. Given $\varepsilon > 0$, the polyhedron $S' = \{x \in V \,|\, \alpha_i x \leq \beta_i,\, i = 1, \ldots, t-1,\, \alpha_t x \leq \beta_t + \varepsilon\}$ contains points of Λ in its interior by the maximality of S. By Proposition 5, $\mathbf{int}(S')$ has a finite number of points in Λ, hence there exists one minimizing $\alpha_t x$, say z. By construction, the polytope $S' = \{x \in V \,|\, \alpha_i x \leq \beta_i,\, i = 1, \ldots, t-1,\, \alpha_t x \leq \alpha_t z\}$ does not contain any point of Λ in its interior and properly contains S, contradicting the maximality of S. $\square$

Lemma 15 *Let L be a linear subspace of V with $\dim(L) = \dim(V) - 1$, and let $v \in V$. Then $v + L$ is a maximal Λ-free convex set if and only if L is not a lattice subspace of V.*

Proof: ($\Rightarrow$) Let $S = v + L$ and assume that S is a maximal Λ-free convex set. Suppose by contradiction that L is a lattice-subspace. Then there exists a basis $a_1, \ldots, a_m$ of Λ such that $a_1, \ldots, a_{m-1}$ is a basis of L. Thus $S = \{\sum_{i=1}^m x_i a_i \,|\, x_m = \beta\}$ for some $\beta \in \mathbb{R}$. Then, $K = \{\sum_{i=1}^m x_i a_i \,|\, \lceil \beta - 1 \rceil \leq x_m \leq \lceil \beta \rceil\}$ strictly contains S and $\mathbf{int}(K) \cap \Lambda = \emptyset$, contradicting the maximality of S.

($\Leftarrow$) Assume L is not a lattice-subspace of V. Since $S = v + L$ is an affine hyperplane of V, $\mathbf{int}(S) = \emptyset$, thus $\mathbf{int}(S) \cap \Lambda = \emptyset$, hence we only need to prove that S is maximal with such property. Suppose not, and let K be a maximal convex set in V such that $\mathbf{int}(K) \cap \Lambda = \emptyset$ and $S \subset K$. Then by maximality K is closed. Let $w \in K \setminus S$. Since K is convex and closed, then $K \supseteq \mathrm{conv}(\{v, w\}) + L$. Let ε be the distance between $v + L$ and $w + L$, and δ be the distance of $\mathrm{conv}(\{v, w\}) + L$ from the origin. By Lemma 13, since L is not a lattice-subspace of V, there exists a vector $y \in \Lambda \setminus L$ at distance $\bar{\varepsilon} < \varepsilon$ from L. Let $z = (\lfloor \frac{\delta}{\bar{\varepsilon}} \rfloor + 1)y$. By definition, z is strictly between $v + L$ and $w + L$, hence $z \in \mathbf{int}(K)$. Since z is an integer multiple of $y \in \Lambda$, then $z \in \Lambda$, a contradiction. $\square$

We are now ready to prove Lovász's Theorem.

Proof:[Proof of Theorem 10.] ($\Leftarrow$) If S satisfies (ii), then by Lemma 15, S is a maximal Λ-free convex set. If S satisfies (i), then, since $\mathbf{int}(S) \cap \Lambda = \emptyset$, we only need to show that S is maximal. Suppose not, and let K be a convex set in V such that $\mathbf{int}(K) \cap \Lambda = \emptyset$ and $S \subset K$. Given $y \in K \setminus S$, there exists a hyperplane H separating y from S such that $F = S \cap H$ is a facet of S. Since K is convex and $S \subset K$, then $\mathrm{conv}(S \cup \{y\}) \subseteq K$. Since $\dim(S) = \dim(V)$, $F \subset S$ hence the $\mathbf{relint}(F) \subset \mathbf{int}(K)$. By assumption, there exists $x \in \Lambda \cap \mathbf{relint}(F)$, so $x \in \mathbf{int}(K)$, a contradiction.

($\Rightarrow$) Let S be a maximal Λ-free convex set. We show that S satisfies either (i) or (ii). Observe that, by maximality, S must be closed.

If $\dim(S) < \dim(V)$, then S is contained in some affine hyperplane H. Since $\mathbf{int}(H) = \emptyset$, we have $S = H$ by maximality of S, therefore $S = v + L$ where $v \in S$ and L is a hyperplane in V. By Lemma 15, (ii) holds.

Therefore we may assume that $\dim(S) = \dim(V)$. In particular, since S is convex, $\mathbf{int}(S) \neq \emptyset$. By Lemma 14, if S is bounded, (i) holds. Hence we may assume that S is unbounded. Let C be the recession cone of S and L the lineality space of S. Since S is unbounded, $C \neq 0$.

Claim 1. $L = C$.

By the definition of L and C, $L \subseteq C$. We show the opposite inclusion. Let $r \in C$, $r \neq 0$. We only need to show that $S + \langle r \rangle$ is Λ-free; by maximality of S this will imply that $S = S + \langle r \rangle$. Suppose there exists $y \in \mathbf{int}(S + \langle r \rangle) \cap \Lambda$. We show that $y \in \mathbf{int}(S) + \langle r \rangle$. Suppose not. Then $(y + \langle r \rangle) \cap \mathbf{int}(S) = \emptyset$, which implies that there is a hyperplane H separating the line $y + \langle r \rangle$ and $S + \langle r \rangle$. This contradicts $y \in \mathbf{int}(S + \langle r \rangle)$. This shows $y \in \mathbf{int}(S) + \langle r \rangle$. Thus there exists $\bar{\lambda}$ such that $\bar{y} = y + \bar{\lambda} r \in \mathbf{int}(S)$, i.e. there exists $\varepsilon > 0$ such that $B_\varepsilon(\bar{y}) \cap V \subset S$. Since $y \in \Lambda$, then $y \notin \mathbf{int}(S)$, and thus, since $\bar{y} \in \mathbf{int}(S)$ and $r \in C$, we must have $\bar{\lambda} > 0$. Since $r \in C$, then $B_\varepsilon(\bar{y}) + \{\lambda r \mid \lambda \geq 0\} \subset S$. Since $y \in \Lambda$, by Lemma 12 there exists $z \in \Lambda$ at distance less than ε from the half line $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$. Thus $z \in B_\varepsilon(\bar{y}) + \{\lambda r \mid \lambda \geq 0\}$, hence $z \in \mathbf{int}(S)$, a contradiction. $\diamond$

Let $P = \mathrm{proj}_{L^\perp}(S)$ and $\Lambda' = \mathrm{proj}_{L^\perp}(\Lambda)$. By Claim 1, $S = P + L$ and $P \subset L^\perp \cap V$ is a bounded set. Furthermore, $\dim(S) = \dim(P) + \dim(L) = \dim(V)$ and $\dim(P) = \dim(L^\perp \cap V)$. Notice that $\mathbf{int}(S) = \mathbf{relint}(P) + L$, hence $\mathbf{relint}(P) \cap \Lambda' = \emptyset$. Furthermore P is inclusionwise maximal among the convex sets of $L^\perp \cap V$ without points of Λ' in the relative interior: if not, given a convex set $K \subseteq L^\perp \cap V$ strictly containing P and with no point of Λ' in its relative interior, we have $S = P + L \subset K + L$, and $K + L$ does not contain any point of Λ in its interior, contradicting the maximality of S.

Claim 2. *L is a lattice-subspace of V.*

By contradiction, suppose L is not a lattice-subspace of V. Then, by Lemma 13, for every $\varepsilon > 0$ there exists $y \in \Lambda' \setminus \{0\}$ such that $\|y\| < \varepsilon$. Let V_ε be the linear subspace of $L^\perp \cap V$ generated by the points in $\{y \in \Lambda' \mid \|y\| < \varepsilon\}$. Then $\dim(V_\varepsilon) > 0$.

Notice that, given $\varepsilon' > \varepsilon'' > 0$, then $V_{\varepsilon'} \supseteq V_{\varepsilon''} \supset \{0\}$, hence there exists $\varepsilon_0 > 0$ such that $V_\varepsilon = V_{\varepsilon_0}$ for every $\varepsilon < \varepsilon_0$. Let $U = V_{\varepsilon_0}$.

By definition, Λ' is dense in U (i.e. for every $\varepsilon > 0$ and every $x \in U$ there exists $y \in \Lambda'$ such that $\|x - y\| < \varepsilon$). Thus, since $\mathbf{relint}(P) \cap \Lambda' = \emptyset$, we also have $\mathbf{relint}(P) \cap U = \emptyset$. Since $\dim(P) = \dim(L^\perp \cap V)$, it follows that $\mathbf{relint}(P) \cap (L^\perp \cap V) \neq \emptyset$, so in particular U is a proper subspace of $L^\perp \cap V$.

Let $Q = \text{proj}_{(L+U)^\perp}(P)$ and $\Lambda'' = \text{proj}_{(L+U)^\perp}(\Lambda')$. We show that $\textbf{relint}(Q) \cap \Lambda'' = \emptyset$. Suppose not, and let $y \in \textbf{relint}(Q) \cap \Lambda''$. Then, $y + w \in \Lambda'$ for some $w \in U$. Furthermore, we claim that $y + w' \in \textbf{relint}(P)$ for some $w' \in U$. Indeed, suppose no such w' exists. Then $(y + U) \cap (\textbf{relint}(P) + U) = \emptyset$. So there exists a hyperplane H in $L^\perp \cap V$ separating $y + U$ and $P + U$. Therefore the projection of H onto $(L + U)^\perp$ separates y and Q, contradicting $y \in \textbf{relint}(Q)$. Thus $z = y + w' \in \textbf{relint}(P)$ for some $w' \in U$. Since $z \in \textbf{relint}(P)$, there exists $\bar{\varepsilon} > 0$ such that $B_{\bar{\varepsilon}}(z) \cap (L^\perp \cap V) \subset \textbf{relint}(P)$. Since Λ' is dense in U and $y + w \in \Lambda'$, it follows that Λ' is dense in $y + U$. Hence, since $z \in y + U$, there exists $\bar{x} \in \Lambda'$ such that $\|\bar{x} - z\| < \bar{\varepsilon}$, hence $\bar{x} \in \textbf{relint}(P)$, a contradiction. This shows $\textbf{relint}(Q) \cap \Lambda'' = \emptyset$.

Finally, since $\textbf{relint}(Q) \cap \Lambda'' = \emptyset$, then $\textbf{int}(Q + L + U) \cap \Lambda = \emptyset$. Furthermore $P \subseteq Q + U$, therefore $S \subseteq Q + L + U$. By the maximality of S, $S = Q + L + U$ hence the lineality space of S contains $L + U$, contradicting the fact that L is the lineality space of S and $U \neq \{0\}$. $\diamond$

Since L is a lattice-subspace of V, Λ' is a lattice of $L^\perp \cap V$ by Lemma 11. Since P is a bounded maximal Λ'-free convex set, it follows from Lemma 14 that P is a polytope with a point of Λ' in the relative interior of each of its facets, therefore $S = P + L$ has a point of Λ in the relative interior of each of its facets, and (i) holds. $\square$

Exercise 2 *Prove rigorously that any lattice-free convex set is contained in a* maximal *lattice-free convex set. There exists both a non-constructive and a constructive proof of this fact.*

We now show a bound on the number of facets for a maximal lattice-free convex set.

Theorem 16 *Any maximal $\mathbb{Z}^n$-free convex set B in $\mathbb{R}^n$ has at most 2^n facets.*

Proof: If B is an affine hyperplane, then we are done. So we can assume it is full dimensional with nonempty interior by Theorem 10. Suppose to the contrary that B has at least $2^n + 1$ facets. By Theorem 10, every facet has a point from $\mathbb{Z}^n$ in its relative interior. We denote this set of integral points by I, so that $|I| \geq 2^n + 1$. Consider the set of vectors $\bar{I} = \{p \mod 2 \,|\, p \in I\}$, where $p \mod 2$ represents reducing each coordinate of p modulo 2. Note that each element in $\bar{I}$ is a vector in $\{0, 1\}^n$. Since $|I| \geq 2^n + 1$, this implies that there exist $p, q \in I$ with $p \neq q$ and $p \mod 2 = q \mod 2$. This implies that $\frac{p+q}{2}$ is in fact integral and moreover lies in the interior of B, because p and q lie in distinct facets of B. This is a contradiction. $\square$

3 Corner Polyhedron and Cutting Planes

3.1 Corner Polyhedron

Corner polyhedra were introduced by Gomory as a generalization of his original cutting plane method and this structure was studied extensively by Gomory and Johnson. We introduce this concept and the notation in this section.

Consider a general MILP in the following standard form. Let A be a rational $m \times n$ matrix, b be a rational $m \times 1$ column vector. We are interested in $x \in \mathbb{R}^n$ such that

$$\begin{array}{rcl} Ax & = & b, \\ x & \geq & 0, \quad x_j \in \mathbb{Z} \quad \text{for } j \in I \subseteq \{1, \ldots n\} \end{array} \tag{3}$$

We assume, without loss of generality, that A has full row rank. A basis D is a maximal linearly independent set of columns in A; so D is an invertible $m \times m$ matrix. Corresponding to any basis D, we define the corner polyhedron with respect to D, denoted by $corner(D)$ as the convex hull of all $x \in \mathbb{R}^n$ such that

$$\begin{array}{rcl} x_{\mathcal{D}} & = & D^{-1}b - D^{-1}Nx_{\mathcal{N}}, \\ x_{\mathcal{N}} & \geq & 0, \quad x_j \in \mathbb{Z} \text{ for } j \in I \subseteq \{1, \ldots n\} \end{array} \tag{4}$$

where $x_{\mathcal{D}}$ refers to the variables corresponding to the columns of D (a.k.a the basic variables) and $x_{\mathcal{N}}$ refers to the remaining variables (a.k.a the non-basic variables). To simplify notation, we will denote the set of indices of the basic variables by $\mathcal{D}$, and the non-basic variables by $\mathcal{N}$. The point $\bar{x} = (D^{-1}b, 0)$ is called the *basic solution* corresponding to the basis D. Let $LP(corner(D))$ be the *relaxation* of $corner(D)$ where the integrality constraints are dropped on all the variables in (4). It is well-known that $corner(D) \neq LP(corner(D))$ if and only if $\bar{x}$ does not satisfy the integrality constraints. Furthermore, $corner(D)$ is a *relaxation* of (3) because the non-negativity constraints on the basic variables have been dropped. This implies that valid inequalities for $corner(D)$ are also valid inequalities for (3).

The significance of this corner polyhedron lies in the fact that most cutting planes for general MILPs (e.g. GMI cuts, Split cuts, Reduce-and-Split cuts, MIR cuts and so forth) are obtained by solving the linear relaxation using linear programming methods and then using integrality arguments to find an inequality which cuts off the fractional solution of the linear relaxation. Many families of such cutting planes, including the four examples listed above, are valid inequalities for $corner(D)$ for some appropriate basis D, which cut off the corresponding basic solution $\bar{x}$. D usually corresponds to the optimal basis in the standard simplex method for solving the LP relaxation. Moreover, for the derivation of these cutting planes, one needs to only consider a relaxation of the corner polyhedra obtained by retaining only one equality constraint. Therefore, having a characterization of polyhedra in $\mathbb{R}^n$ defined using constraints like (4) would be an automatic generalization of most cutting planes used in practice. **In addition, corner polyhedra have facets which cannot be obtained by applying integrality arguments to a single constraint from the system**. So the analysis of corner polyhedra provides us with new families of cutting planes, derived from multiple constraints, which may be useful for solving general MILPs.

3.2 A Reformulation

Andersen et al. [1] introduced a relaxation of the corner polyhedron, where the integrality conditions on the non-basic variables are dropped. We can formulate this relaxation in the following notation, which provides a little more geometric intuition.

$$\begin{array}{rcl} x & = & f + \sum_{i=1}^{k} r^i s_i \\ x & \in & \mathbb{Z}^m \\ s_i & \geq & 0 \quad \text{for all } i. \end{array} \tag{5}$$

In the above, $f \in \mathbb{R}^m$ corresponds to $D^{-1}b$ in (4) and r^i's are rays in $\mathbb{R}^m$ which correspond to the non-basic columns $(-D^{-1}N_i)$ in (4) (N_i denotes the i-th column of the matrix N). x corresponds to $x_{\mathcal{D}}$ and s corresponds to $x_{\mathcal{N}}$. Note that to describe the solutions of (5), one only needs to record the values of the s_i variables. We use $R_f(r^1, \ldots, r^k)$ to denote the set of all points s such that (5) is satisfied.

Since $\operatorname{conv}(R_f(r^1, \ldots, r^k))$ is a relaxation of the corner polyhedron, any valid inequality for $\operatorname{conv}(R_f(r^1, \ldots, r^k))$ is valid for the corner polyhedron and hence for our initial MILP problem from which the corner polyhedron is derived.

However, it turns out that a simple trick shows that this formulation is as powerful as the original corner polyhderon relaxation. One can retain the integrality constraints on the non-basic variables in the corner polyhderon and still have a system like (5). For every non-basic variable s_i (coming from $x_{\mathcal{N}}$) with an integrality constaint, we introduce an auxiliary variable y_i and include the constraint $y_i = s_i$ in the system (5), and one imposes the integrality constraint on y_i. This new system with these additional equations, one corresponding to each non-basic integer variable, is still in the same form as (5).

3.3 Intersection Cuts and the Minkowski functional

In this section, we will develop the ideas for deriving valid inequalities for $\text{conv}(R_f(r^1, \ldots, r^k))$. As discussed before, this will provide cutting planes for our original MILP, derived using multiple constraints. The key observation is that one can use lattice-free convex sets to derive valid inequalities for $\text{conv}(R_f(r^1, \ldots, r^k))$ using the following classical tool from convex analysis.

Definition 17 *Let $K \subset \mathbb{R}^n$ be a closed convex set containing the origin in its interior. The* gauge *or the Minkowski functional, is the function γ_K defined by*

$$\gamma_K(x) = \inf\{t > 0 \,|\, t^{-1}x \in K\} \quad \textit{for all } x \in \mathbb{R}^n.$$

By definition γ_K is nonnegative.

The assumption that $0 \in \textbf{int}(K)$ implies that the Minkowski function is real valued. The following lemma gives a little more intuition about the Minkowski functional.

Lemma 18 *Let $K \subset \mathbb{R}^n$ be a closed convex set containing the origin in its interior. If $\gamma_K(x) > 0$, then $\frac{x}{\gamma_K(x)} \in \textbf{bd}(K)$, where $\textbf{bd}(K)$ is the boundary of K. If $\gamma_K(x) = 0$ then $x \in \text{rec}(K)$, where $\text{rec}(K)$ denotes the recession cone of K.*

Proof: Suppose $\gamma_K(x) > 0$. Then by definition, there exists a sequence $\{t_n\}$ such that $t_n \to \gamma_K(x)$, such that $\frac{x}{t_n} \in K$. Since K is closed, $\lim_{n\to\infty} \frac{x}{t_n} \in K$, and so $\frac{x}{\gamma_K(x)} \in K$. Moreover, for any $t < \gamma_K(x)$, $\frac{x}{t} \notin K$ which shows that $\frac{x}{\gamma_K(x)}$ is not an interior point of K and hence must lie on the boundary. $\gamma_K(x) = 0$ implies that $\frac{x}{t} \in K$ for all $t > 0$, which is precisely the definition of the recession cone. □

We now collect some key properties of Minkowski functionals.

Proposition 19 *Let $K \subseteq \mathbb{R}^n$ be a closed, convex set containing the origin in its interior. Then the Minkowski functional γ_K satisfies :*

i $\gamma_K(r_1 + r_2) \leq \gamma_K(r_1) + \gamma_K(r_2)$, *for all* $r_1, r_2 \in \mathbb{R}^n$.

ii $\gamma_K(sr) = s\gamma_K(r)$ *for every* $s \in \mathbb{R}$ *and* $r \in \mathbb{R}^n$.

iii $K = \{r \,|\, \gamma_K(r) \leq 1\}$.

Proof: We will prove only the subadditivity condition (condition i) here. Let $s_1 > 0$ be any real number such that $s_1^{-1}r_1 \in K$ and $s_2 > 0$ be a real number such that $s_2^{-1}r_2 \in K$. Since K is convex, the point

$$\frac{s_1}{s_1 + s_2}(s_1^{-1}r_1) + \frac{s_2}{s_1 + s_2}(s_2^{-1}r_2) \in K.$$

Therefore $(s_1 + s_2)^{-1}(r_1 + r_2) \in K$. Hence, $\gamma_K(r_1 + r_2) \le s_1 + s_2$. Taking an infimum over all s_1 on both sides of the inequality, we obtain that $\gamma_K(r_1 + r_2) \le \gamma_K(r_1) + s_2$ and then taking an infimum over s_2, we obtain $\gamma_K(r_1 + r_2) \le \gamma_K(r_1) + \gamma_K(r_2)$. □

We now show how to derive valid inequalities for $\mathrm{conv}(R_f(r^1, \dots, r^k))$ using the Minkowski functional. This idea was first presented by Egon Balas under the name of "Intersection Cuts".

Theorem 20 (Intersection Cuts) *Consider any closed, convex set B containing the point f in its interior, but no integer point in its interior. Let $K = B - f$. Then the inequality $\sum_{i=1}^k \gamma_K(r^i)s_i \ge 1$ is valid for* $\mathrm{conv}(R_f(r^1, \dots, r^k))$.

Proof: Consider any point s that satisfies (5). Let $\bar{x} = f + \sum_{i=1}^k r^i s_i$. Note that since $\bar{x} \in \mathbb{Z}^n$, $\bar{x}$ is not in the interior of B and therefore $\bar{x} - f$ is not in the interior of K. Therefore, $\gamma_K(\bar{x} - f) \ge 1$ by condition iii) in Proposition 19. Therefore,

$$1 \le \gamma_K(\bar{x} - f) = \gamma_K(\sum_{i=1}^k r^i s_i) \le \sum_{i=1}^k \gamma_K(r^i s_i) \le \sum_{i=1}^k \gamma_K(r^i)s_i$$

where the second and third inequalities follow from conditions i) and ii) of Proposition 19 respectively. □

The following proposition follows directly from the definition of the Minkowski functional.

Proposition 21 *Consider two closed convex sets C_1 and C_2 containing the origin in their interiors, such that $C_1 \subseteq C_2$. Show that the associated Minkowski functionals satisfy the inequality $\gamma_{C_2}(r) \le \gamma_{C_1}(r)$ for all $r \in \mathbb{R}^n$.*

Note that if we have two valid inequalities $\sum_{i=1}^k \gamma_i s_i \ge 1$ and $\sum_{i=1}^k \gamma'_i s_i \ge 1$ such that $\gamma_i \le \gamma'_i$, then the inequality $\sum_{i=1}^k \gamma'_i s_i \ge 1$ dominates the inequality $\sum_{i=1}^k \gamma_i s_i \ge 1$, i.e. every point in $\mathbb{R}^n_+$ satisfying $\sum_{i=1}^k \gamma'_i s_i \ge 1$ also satisfies $\sum_{i=1}^k \gamma_i s_i \ge 1$. Hence, the inequality $\sum_{i=1}^k \gamma_i s_i \ge 1$ is redundant. Combining this simple observation with Proposition 21, we conclude that we only need to consider *maximal* lattice-free convex sets to derive our valid inequalities. In other words, the inequalities derived from non-maximal lattice-free convex sets are redundant and are dominated by inequalities from maximal lattice-free convex sets.

One can further show that *every* valid inequality for $\mathrm{conv}(R_f(r^1, \dots, r^k))$ comes from an intersection cut. We say that a valid inequality is nontrivial if it is not implied by the non-negativity constraints, i.e., it is not a positive combination of the inequalities $s_i \ge 0$.

Theorem 22 *Assume that all data in the problem (5) is rational and* $\mathrm{conv}(R_f(r^1, \dots, r^k))$ *is nonempty. Then every non-trivial valid inequality for* $\mathrm{conv}(R_f(r^1, \dots, r^k))$ *is dominated by an intersection cut obtained from a closed, convex set B with f in its interior, and no integer point in its interior.*

Proof:[Sketch of Proof]

Claim 2 *Every non-trivial valid inequality for* $\mathrm{conv}(R_f(r^1, \dots, r^k))$ *is of the form $\sum_{i=1}^k \gamma_i s_i \ge 1$, with $\gamma_i \ge 0$ for all i.*

Proof: Any valid inequality is of the form $\sum_{i=1}^k \gamma_i s_i \geq \delta$. Assume to the contrary that for some j, $\gamma_j < 0$. Since r_j is rational, there exists a positive integer D such that Dr_j has integer components. Let M be any positive integer. Let s be some point in $R_f(r^1, \ldots, r^k)$. Consider the point $\bar{s}$ defined as :

$$\bar{s}_i = \begin{cases} MD + s_j & i = j \\ s_i & i \neq j \end{cases}$$

Note that $\bar{s}$ also belongs to $R_f(r^1, \ldots, r^k)$ for every positive integer M. But then, if let M grow large, then the inequality $\sum_{i=1}^k \gamma_i \bar{s}_i = \sum_{i \neq j}^k \gamma_i s_i + \gamma_j MD$ which can be made less than δ since $\gamma_j < 0$. Therefore, for all i $gamma_i \geq 0$. This implies that if the inequality $\sum_{i=1}^k \gamma_i s_i \geq \delta$ is non-trivial, $delta > 0$. Dividing the equation by δ, we get the desired form.

□

We next consider the following set in $\mathbb{R}^n$.

$$B = \text{conv}\{f + \frac{r^j}{\gamma_j} \mid \gamma_j \neq 0\} + \text{cone}\{r^j \mid \gamma_j = 0\}$$

. Observe that $\gamma_j = \gamma_K(r^j)$, where γ_K is the Minkowski functional of $B - f$.

Claim 3 *No face of B containing f contains an integer point in its relative interior.*

Proof: Suppose to the contrary that a face F containing f contains an integer point $\bar{x}$ in its relative interior. Then $\bar{x} = \alpha f + (1-\alpha)p$ for some p in F and $\alpha > 0$ (since $\bar{x}$ is in the relative interior of F). Now $p \in C$ and therefore p can be expressed as $p = \beta f + \sum_{j:\gamma_j \neq 0}^n \alpha_j (f + \frac{r^j}{\gamma_j}) + \sum_{j:\gamma_j = 0} \beta_j r_j$ with $\beta + \sum \alpha_j = 1$. Therefore,

$$\begin{aligned} \bar{x} &= \alpha f + (1-\alpha)p \\ &= \alpha f + (1-\alpha)(\beta f + \textstyle\sum_{j:\gamma_j \neq 0}^n \alpha_j (f + \frac{r^j}{\gamma_j}) + \sum_{j:\gamma_j = 0} \beta_j r_j) \\ &= f + \textstyle\sum_{j:\gamma_j \neq 0}^n (1-\alpha)\alpha_j \frac{r^j}{\gamma_j} + \sum_{j:\gamma_j = 0} (1-\alpha)\beta_j r_j \end{aligned} \tag{6}$$

One observes that $\sum \alpha_j \leq 1$ and $1 - \alpha < 1$. Therefore $(1-\alpha)\sum \alpha_j < 1$. We consider the solution

$$s_j = \begin{cases} (1-\alpha)\frac{\alpha_j}{\gamma_j} & \gamma_j \neq 0 \\ (1-\alpha)\beta_j & \gamma_j = 0 \end{cases}$$

From the last equation in (6), one sees that s_j is a valid solution to (5). But $\sum \gamma_j s_j < 1$ because $(1-\alpha)\sum \alpha_j < 1$. This is a contradiction to the fact that $\sum \gamma_j s_j \geq 1$ is a valid inequality. □

We finally need to have f in the interior of the convex set. Let B be defined by $Ax \leq b$. Since all the data is rational, we can assume that A and b have integer entries. If $Af < b$, then f is in the interior and we are done. Suppose the first h inequalities are satisfied by f at equality. So $a_i f = b_i$ for $i = 1, \ldots h$ and $a_i f < b_i$ for $i > h$,where a_i is the $i-th$ row of the matrix A and b_i is the $i-th$ entry in the column vector b. Consider the vector b' where $b'_i = b_i + 1$ for $i = 1, \ldots, h$ and $b'_i = b_i$ for $i > h$. Now clearly, $Af < b'$. We prove that the convex polyhderon B' defined by $Ax \leq b'$ has no integer point in its interior. If $\bar{x}$ is such a point, then $A\bar{x} < b'$. By the integrality of A, b, this implies that $a_i \bar{x} \leq b_i$ for $i = 1, \ldots, h$ and $a_i \bar{x} < b_i$

for $i > h$. Let J be the set of indices such that $a_i\bar{x} = b_i$. Then these set of equalities indexed by J define a face of B containing f and containing $\bar{x}$ in its relative interior. This contradicts Claim 3.

Now, B' contains f in its interior and no integer point in its interior. Moreover, $B \subseteq B'$ and so Proposition 21 shows that the inequality derived from B' dominates B. One finally observes that $\gamma_B(r_i) = \gamma_i$.

□

In conclusion, we studied how one can use lattice-free convex sets to derive cutting planes for general MILPs. In this context, the maximal lattice-free convex sets are the "important" ones because all valid inequalities for $\text{conv}(R_f(r^1, \ldots, r^k))$ are dominated by valid inequalities derived using the Minkowski functional of maximal lattice-free convex sets. Note further that the Minkowski functional is a function that depends only on the point f and the convex set used. It does not "see" the actual rays in the system (5). In this sense, it gives generic "formulas" dependent only on f, such that one can simply plug in the actual rays of the problem and derive the inequalities.

Theorem 22 can be reformulated in the following manner. Let $\mathcal{C}_f$ be the family of all maximal integer point-free convex sets with f in their interior.

Corollary 23 $\text{conv}(R_f(r^1, \ldots, r^k)) = \cap_{C \in \mathcal{C}_f} \sum_{i=1}^k \gamma_{C-f}(r^i)s_i \geq 1$

3.4 Minkowski Functionals of Maximal Lattice-free Convex Sets

In this section, we will derive simple formulas for Minkowski functionals of polyhedra with a certain property. In particular, we prove

Theorem 24 *Let P be a polyhedra given by $P = \{a_i x \leq 1 \,|\, i \in I\}$, where I is the index set for the constraints of P (the particular form implies that $0 \in \mathbf{int}(P)$). Suppose that the recession cone of P is not full dimensional (i.e., $a_i r \leq 0$ has no solution satisfying all constraints at strict inequality). Then,*

$$\gamma_P(r) = \max_{i \in I} a_i \cdot r \tag{7}$$

Proof: For any $t > \gamma_P(r)$, we have that $\frac{r}{t} \in P$ and therefore $a_i r \leq t$ for all $i \in I$. Taking an infimum over all such t, we obtain that $a_i r \leq \gamma_P(x)$ for all $i \in I$ and therefore $\max_{i \in I} a_i r \leq \gamma_P(r)$. We now show that for some $i \in I$, $a_i r = \gamma_P(r)$. If $\gamma_P(r) > 0$, then by Lemma 18 we know that $\frac{r}{\gamma_P(r)} \in \mathbf{bd}(P)$ and so there exists $i \in I$ such that $a_i \frac{r}{\gamma_P(r)} = 1$ and so $a_i r = \gamma_P(r)$. If $\gamma_P(r) = 0$ then Lemma 18 tells us that $r \in \text{rec}(P)$. Since $\text{rec}(P)$ is not full-dimensional, there exists some $i \in I$ such that $a_i r = 0 = \gamma_P(r)$. □

Remark 25

1. *The above formula suggests that for polyhedra whose recession cone is not full dimensional, the Minkowski functional is piecewise linear. Moreover, in order to compute the Minkowski functional for such polyhedra, instead of trying to find a scaling factor to reach the boundary of P, one can simply take the maximum of a finite number of real numbers. This turns out to be useful both practically and theoretically, when dealing with these cutting planes.*

2. *For maximal lattice-free convex sets, we know from the results in Section 2 that they are polyhedra and that their recession cone is the same as their lineality space. The second condition implies that the recession cone is in fact not full dimensional. Therefore, for such sets, we can apply (7) for computing*

the intersection cuts. Hence, the coefficients of the cutting planes we want to derive from these sets can be computed easily once we fix the facet description of the set.

3. *Theorem 16 also says that the number of facets for maximal lattice-free convex sets are bounded by 2^m, where m is the number of x variables in our system (5). Hence, we have a bound on the complexity of computing the maximum in formula (7).*

References

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