1. Solve the following differential equations (you can leave the constant C in your answer):

(a) **(5 pts)** $\frac{dy}{dx} = x - \frac{xy}{y+1}$.

$$\frac{dy}{dx} = x\left(1 - \frac{y}{y+1}\right) = x\left(\frac{y+1-y}{y+1}\right) = x\left(\frac{1}{y+1}\right)$$

separating variables :

$$\int (y+1)\,dy = \int x\,dx$$

$$\boxed{\frac{y^2}{2} + y = \frac{x^2}{2} + C}$$

(b) **(5 pts)** $yy' - 2\frac{ln(x)}{x} = 0$.

$$y\frac{dy}{dx} = \frac{2\ln(x)}{x}$$

separating variables :

$$\int y\,dy = \int \frac{2\ln(x)}{x}\,dx$$

use substitution $u = \ln(x) \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{1}{x}dx$

$$\therefore \frac{y^2}{2} = \int 2u\,du = u^2 = (\ln x)^2 + C$$

$$\therefore \boxed{y^2 = 2(\ln x)^2 + C}$$

2. **(10 pts)** Find the solution of the differential equation

$$x^2y' + 2xy = \frac{1}{x},$$

with the initial conditions that when $x = 1, y = 2$.

$$x^2\frac{dy}{dx} + 2xy = \frac{1}{x}$$

Dividing through by x^2

$$\frac{dy}{dx} + \frac{2xy}{x^2} = \frac{1}{x}\cdot\frac{1}{x^2}$$

$$\therefore \frac{dy}{dx} + \frac{2y}{x} = \frac{1}{x^3}$$

FOLDE form with $P(x) = \frac{2}{x}$, $Q(x) = \frac{1}{x^3}$

1. $\int P(x)\,dx = \int \frac{2}{x}\,dx = 2\ln|x|.$

2. $u(x) = e^{\int P(x)dx} = e^{2\ln|x|} = e^{\ln|x|^2} = |x|^2 = x^2$

3. $\int Q(x)u(x)\,dx = \int \frac{1}{x^3}\cdot x^2\,dx = \int \frac{1}{x}\,dx = \ln|x|$

Final solution : $y = \frac{1}{x^2}\left[\ln|x| + C\right] = \frac{\ln|x|}{x^2} + \frac{C}{x^2}$

Using $x = 1, y = 2$:

$$2 = \frac{\ln|1|}{1^2} + \frac{C}{1^2} = C \implies C = 2.$$

$$\boxed{y = \frac{\ln|x| + 2}{x^2}}$$

3. Let $S(t)$ be the amount (in pounds) of sugar in a sugar solution kept in a tank at time t (in minutes). A solution containing 2 pounds of sugar per gallon enters the tank at the rate of 5 gallons per minute. The well-stirred solution is drained from the tank at the rate of 3 gallons per minute. Initially, at $t = 0$, the tank holds 100 gallons with 20 pounds of sugar.

(a) **(3 pts)** How much solution (in gallons) is present in the tank at time t?

5 gallons enter and 3 gallons leave per minute

After t minutes, we have

$100 + 2t$ gallons.

(b) **(7 pts)** Set up (but do not solve) a differential equation for the quantity $S(t)$ (in pounds) of sugar in the tank as a function of time t.

$$\frac{dS}{dt} = \text{Rate in} - \text{Rate out}$$

Rate in : 5 gallons/minute enters containing 2 pounds of sugar per gallon.

=> $5 \times 2 = 10$ pounds of sugar enters/minute.

Rate out : 3 gallons of solution leaves every minute

How many pounds of sugar in 3 gallons?

At any time t, $100 + 2t$ gallons contains $S(t)$ pounds

∴ 3 gallons contain $3 \times \frac{S(t)}{100+2t}$ pounds.

$\frac{3S}{100+2t}$ ~~gallons~~ pounds of sugar leaves every minute.

$$\Rightarrow \boxed{\frac{dS}{dt} = 10 - \frac{3S}{100+2t}}$$

(c) **(8 pts)** Solve the differential equation you set up in Part (b). Your function should not have any generic constants like C.

Write in FOLDE form :

$$\frac{dS}{dt} + \frac{3S}{100+2t} = 10$$

$$P(t) = \frac{3}{100+2t}, \quad Q(t) = 10$$

1. $\int P(t)\,dt = \int \frac{3}{100+2t}\,dt = \frac{3}{2}\ln|100+2t|$

2. $u(t) = e^{\int P(t)dt} = e^{\frac{3}{2}\ln|100+2t|} = e^{\ln|100+2t|^{3/2}} = (100+2t)^{3/2}$

3. $\int Q(t)u(t)\,dt = \int 10(100+2t)^{3/2}\,dt$

Substitute $u = 100+2t \Rightarrow \frac{du}{2} = dt$

$$\Rightarrow \int 10\,u^{3/2}\,\frac{du}{2} = 5 \times \frac{2}{5}u^{5/2} = 2u^{5/2} = 2(100+2t)^{5/2}$$

Final Solution : $\frac{1}{(100+2t)^{3/2}}\left(2(100+2t)^{5/2} + C\right) = \boxed{2(100+2t) + \frac{C}{(100+2t)^{3/2}}}$

At $t=0$, $S(0) = 20$. $\therefore$ $20 = 2(100) + \frac{C}{100^{3/2}} \Rightarrow 20-200 = \frac{C}{1000}$

$\Rightarrow \boxed{C = (-180)\times(1000)}$

$\therefore$ $S(t) = 2(100+2t) + \frac{(-180)(1000)}{(100+2t)^{3/2}}$

(d) **(2 pts)** Find the amount of sugar after 30 minutes.

After 30 minutes, $t = 30$

$$S(30) = 2(100+2(30)) + \frac{(-180)(1000)}{(100+2(30))^{3/2}}$$

$$= \boxed{2\times 160 - \frac{180\times 1000}{(160)^{3/2}}}$$

4. **(5 pts)** Determine the equilibrium points for the differential equation

$$\frac{dN}{dt} = 5N - N^2$$

and discuss the stability of the *largest* equilibrium point.

$0 = 5N - N^2 = N(5-N)$

$\Rightarrow$ Equilibrium points : $\underline{N=0, \quad N=5}$

$\underline{N=5 \text{ is largest}}$

$g'(N) = 5 - 2N$

At $N=5$

we get $5 - 2(5) = -5 < 0$

$\therefore$ $\underline{\text{locally stable equilibrium}}$

5. **(5 pts)** Find the Taylor polynomial of degree $n = 3$ for the function $f(x) = ln(x)$ about the point $a = 1$.

$f(a) = f(1) = \ln(1) = 0.$

$f'(a) = \left.\frac{1}{x}\right|_{x=1} = 1$

$f''(a) = \left.-\frac{1}{x^2}\right|_{x=1} = -1$

$f'''(a) = \left.\frac{2}{x^3}\right|_{x=1} = 2$

$\therefore \quad P(x) = \boxed{0 + \frac{1(x-1)}{1!} + \frac{(-1)(x-1)^2}{2!} + \frac{2(x-1)^3}{3!}}$

6. **(10 pts)** In a chemical reaction, a compound A changes into compound B at a rate proportional to the square of the unchanged amount of compound A. Initially, at $t = 0$, we have 45 grams of compound A. At $t = 2$ hours, we have 4 grams of the compound A left. Write the amount of unchanged compound A as a function of time t (your answer should not have any generic constants like C or k).

Let $A(t)$ = amount of unchanged A (in grams).

$$\frac{dA}{dt} \propto A^2$$

$$\Rightarrow \frac{dA}{dt} = kA^2$$

Use separation of variables.

$$\int \frac{1}{A^2}\,dA = \int k\,dt$$

$$\Rightarrow -\frac{1}{A} = kt + C$$

$$\Rightarrow A = -\frac{1}{kt + C}$$

At $t = 0$, $A(0) = 45$.

$$\Rightarrow 45 = \frac{-1}{k(0) + C} \Rightarrow C = -\frac{1}{45}.$$

At $t = 2$, $A(0) = 4$

$$\Rightarrow 4 = \frac{-1}{2k - 1/45}$$

$$\Rightarrow 2k - \frac{1}{45} = \frac{-1}{4} \Rightarrow k = \frac{1}{2}\left(\frac{1}{45} - \frac{1}{4}\right) = \frac{-41}{360}$$

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$$A(t) = \frac{-1}{-\frac{41}{360}t - 1/45} = \boxed{\frac{360}{41t + 8}}$$