1. **(5 pts)** Solve the following differential equations (you can leave the constant C in your answer):

$$\frac{dy}{dx} = \frac{x}{y} - \frac{x}{y+1}$$

$$\frac{dy}{dx} = x\left(\frac{1}{y} - \frac{1}{y+1}\right) = x\left(\frac{y+1-y}{y(y+1)}\right) = \frac{x}{y(y+1)}$$

separating variables :

$$\Rightarrow \int y(y+1)\,dy = \int x\,dx$$

$$\int (y^2+y)\,dy = \int x\,dx$$

$$\boxed{\frac{y^3}{3} + \frac{y^2}{2} = \frac{x^2}{2} + C}$$

2. **(5 pts)** Compute the following improper integral and decide if it is convergent or divergent.

$$\int_0^{27} \frac{5}{x^{1/3}}\,dx$$

$$= \lim_{z \to 0} \int_z^{27} \frac{5}{x^{1/3}}\,dx$$

$$= \lim_{z \to 0} \left[5\frac{x^{2/3}}{(2/3)} \Big|_z^{27} \right] = \lim_{z \to 0} \left[\frac{15(27)^{2/3}}{2} - \frac{15 z^{2/3}}{3} \right]$$

$$= \frac{15 \times 9}{2}$$

3. **(10 pts)** Find the solution of the differential equation

$$xy' + 2y = -x^2,$$

with the initial conditions that when $x = 1, y = 5$.

$$x\frac{dy}{dx} + 2y = -x^2$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2y}{x} = -x \qquad \text{(Dividing through by } x\text{)}$$

FOLDE with $P(x) = \frac{2}{x}$, $Q(x) = -x$

1. $\int P(x)\,dx = \int \frac{2}{x}\,dx = 2\ln|x|$

2. $u(x) = e^{\int P(x)dx} = e^{2\ln|x|} = e^{\ln|x|^2} = x^2$

3. $\int Q(x)\,u(x)\,dx = \int (-x)\,x^2\,dx$

$$= -\frac{x^4}{4}\,dx$$

Final solution : $y = \frac{1}{x^2}\left[-\frac{x^4}{4} + C\right] = -\frac{x^2}{4} + \frac{C}{x^2}$

Using $x=1$, $y=5$

$$\Rightarrow \quad 5 = -\frac{1}{4} + C \quad \Rightarrow \quad C = 5 + \frac{1}{4} = \frac{21}{4}$$

$$\therefore \quad \boxed{y = -\frac{x^2}{4} + \frac{21}{4x^2}}$$

4. Let $S(t)$ be the amount (in pounds) of salt in a salt solution kept in a tank at time t (in minutes). A solution containing 1/2 pound of salt per gallon enters the tank at the rate of 6 gallons per minute. The well-stirred solution is drained from the tank at the rate of 4 gallons per minute. Initially, at $t = 0$, the tank holds 100 gallons with 10 pounds of salt.

(a) **(3 pts)** How much solution (in gallons) is present in the tank at time t ?

$100 + 2t$ gallons. (6 gallons/min enter 4 gallons/min leaves)

(b) **(7 pts)** Set up (but do not solve) a differential equation for the quantity $S(t)$ (in pounds) of salt in the tank as a function of time t.

$$\frac{dS}{dt} = \text{Rate in} - \text{Rate out}$$

pounds per minute

$$= 6 \times \frac{1}{2} - \frac{4S}{100+2t}$$

$$\Rightarrow \boxed{\frac{dS}{dt} = 3 - \frac{4S}{100+2t}}$$

(c) **(8 pts)** Solve the differential equation you set up in Part (b). Your function should not have any generic constants like C.

$$\frac{dS}{dt} + \frac{4S}{100+2t} = 3$$

Using FOLDE with $P(t) = \dfrac{4}{100+2t}$, $Q(t) = 3$.

1. $\displaystyle\int P(t)\,dt = \int \frac{4}{100+2t}\,dt = \frac{4}{2}\ln|100+2t|$

$= 2\ln|100+2t|$

2. $u(t) = e^{\int P(t)dt} = e^{2\ln|100+2t|} = (100+2t)^2$

3. $\displaystyle\int Q(t)\,u(t)\,dt = \int 3(100+2t)^2\,dt$

$u = 100+2t \rightarrow \dfrac{du}{2} = dt$

$\displaystyle\int 3u^2\frac{du}{2} = \frac{3}{2}\frac{u^3}{3} = \frac{u^3}{2}$

$= \dfrac{(100+2t)^3}{2}$

$\Rightarrow S(t) = \dfrac{1}{(100+2t)^2}\left[\dfrac{(100+2t)^3}{2} + C\right]$

Using $S(0) = 10$

$\Rightarrow 10 = \dfrac{1}{100^2}\left[\dfrac{100^3}{2} + C\right]$

$\Rightarrow C = -40 \times 100^2$

$\Rightarrow S(t) = \dfrac{(100+2t)}{2}$ ...

~~(d) **(2 pts)** Find the~~ amount of salt after 10 minutes.

At $t = 10$,

$$\boxed{S(10) = \frac{(100+20)}{2} - \frac{40 \times 100^2}{(100+20)^2}}$$

5. **(5 pts)** Determine the equilibrium points for the differential equation

$$\frac{dN}{dt} = -6N + 5N^2 - N^3$$

and discuss the stability of the *largest* equilibrium point.

$$-6N + 5N^2 - N^3 = 0$$

$$-N(6 - 5N + N^2) = 0$$

$$-N(N-2)(N-3) = 0 \quad \Rightarrow \quad N = 0, 2, 3.$$

$N = 3$ is <u>largest</u>.

$$\frac{d}{dN}(-6N + 5N^2 - N^3) = -6 + 10N - 3N^2$$

at $N = 3$, $\quad -6 + 30 - 27 < 0$

$\Rightarrow$ <u>locally stable equilibrium</u>.

6. **(5 pts)** Find the Taylor polynomial of degree $n = 3$ for the function $f(x) = e^x$ about the point $a = 2$.

$$f(a) = e^2$$

$$f'(a) = e^x\big|_{x=2} = e^2$$

$$f''(a) = e^x\big|_{x=2} = e^2$$

$$f^{(3)}(a) = e^x\big|_{x=2} = e^2$$

$$P(x) = \boxed{e^2 + \frac{e^2(x-2)}{1!} + \frac{e^2(x-2)^2}{2!} + \frac{e^2(x-2)^3}{3!}}$$

7. **(10 pts)** In a chemical reaction, a compound A changes into compound B at a rate proportional to the unchanged amount of compound A. Initially, at $t = 0$, we have 20 grams of compound A. At $t = 1$ hours, we have 16 grams of the compound A left. Write the amount of unchanged compound A as a function of time t (your answer should not have any generic constants like C or k).

Let $A(t)$ = amount of unchanged A in grams.

$$\frac{dA}{dt} \propto A(t) \Rightarrow \frac{dA}{dt} = kA.$$

$$\Rightarrow \int \frac{1}{A}\, dA = \int k\,dt$$

$$\Rightarrow \ln|A| = kt + C$$

$$\Rightarrow A = \pm e^{C} e^{kt} = C_1 e^{kt}$$

at $t=0$, $A(0) = 20$.

$$\Rightarrow 20 = C_1 e^{k(0)} = C_1 \Rightarrow C_1 = 20$$

at $t=1$, $A(1) = 16$

$$\Rightarrow 16 = C_1 e^{k(1)} \Rightarrow 16 = 20e^{k}$$

$$\Rightarrow k = \ln\left(\frac{16}{20}\right)$$

$$\therefore \boxed{A(t) = 20\, e^{\ln\left(\frac{16}{20}\right)t}}$$