

1. (4 pts) Given two polynomials $f, g \in \mathbb{K}[x_1, \dots, x_n]$ and some monomial order, define the S-polynomial $S(f, g)$.

$$\text{Let } x^\delta = \text{LCM}(\text{LM}(f), \text{LM}(g))$$

$$S(f, g) = \frac{x^\delta}{\text{LT}(f)} \cdot f - \frac{x^\delta}{\text{LT}(g)} \cdot g$$

2. (6 pts) Let I be an ideal. Define the following three terms : Groebner basis for I , a *minimal* Groebner basis for I and a *reduced* Groebner basis for I .

1). Groebner basis : $G = \{g_1, \dots, g_s\}$ is a Groebner basis for I if

$$\langle \text{LM}(I) \rangle = \langle \text{LM}(g_1), \text{LM}(g_2), \dots, \text{LM}(g_s) \rangle$$

(or. $\langle \text{LT}(I) \rangle = \langle \text{LT}(g_1), \dots, \text{LT}(g_s) \rangle$).

2. G is a ~~minimal~~ Groebner basis if

1) $\text{LC}(p) = 1 \quad \forall p \in G$.

2) ~~$\forall p \in G$~~ , $\text{LT}(p)$ is not divisible by any $\text{LT}(g)$, $g \in G \setminus \{p\}$

3. G is a reduced Groebner basis if

1) $\text{LC}(p) = 1 \quad \forall p \in G$.

2) ~~$\forall p \in G$~~ , no term of p is divisible by $\text{LT}(g)$, $g \in G \setminus \{p\}$

3. (10 pts) Prove that an ideal I has a unique reduced Groebner basis. You can assume that every minimal Groebner basis of an ideal I has the same set of leading monomials, and therefore, the same number of elements (so you can use this fact without proof).

Every reduced Groebner basis is a minimal Groebner basis. Consider G_1, G_2 to be 2 reduced Groebner bases. Since they are minimal $\text{LM}(G_1) = \text{LM}(G_2)$.

We show that if $p_1 \in G_1$ & $p_2 \in G_2$ such that $\text{LM}(p_1) = \text{LM}(p_2)$ then $p_1 = p_2$.

Now, $p_1 - p_2 \in I$. If $p_1 - p_2 \neq 0$,

then $\text{LM}(p_1 - p_2)$ is divisible by some $\text{LM}(g)$ for $g \in G_1$, non-zero. Moreover, $\text{LM}(p_1 - p_2)$ is either a term of p_1 or a term in p_2 and is not equal to $\text{LM}(p_1) = \text{LM}(p_2)$ because the leading terms cancel.

But in either case, $\text{LM}(p_1 - p_2)$ is divisible by $\text{LM}(g)$ for some $g \in G_1$ or some $g \in G_2$.

This contradicts the definition of a reduced Groebner basis. Thus, $p_1 = p_2$ and we are done.

4. You can use MAPLE to answer the following questions. Clearly state which MAPLE command was used, the output you got and how that led to your conclusions. Recall that in the package *with(Groebner)*, *Basis()* gives you a Groebner basis, *SPolynomial()* computes S-polynomials and *NormalForm()* computes remainders.

(a) (8 pts) Is the polynomial $p = xy^3 - z^2 + y^5 - z^3$ in the ideal

$$I = \langle -x^3 + y, x^2y - z \rangle?$$

~~B := Basis~~ Basis($[x^3 + y, x^2y - z]$, plex(x, y, z));

NormalForm($xy^3 - z^2 + y^5 - z^3$, B, plex(x, y, z));

$$= 0$$

\Rightarrow $P \in I$

(b) (12 pts) Check if $\{x - z^4, y - z^5, xz - y\}$ is a Groebner basis for $\langle x - z^4, y - z^5, xz - y \rangle$ with respect to the lexicographic order where $x > y > z$.

$$f1 := x - z^4; f2 := y - z^5; f3 := xz - y;$$

NormalForm(SPolynomial(f1, f2, plex(x, y, z)), [f1, f2, f3],
plex(x, y, z));

$$= 0$$

NormalForm(SPolynomial(f1, f3, plex(x, y, z)), [f1, f2, f3],
plex(x, y, z));

NormalForm(SPolynomial(f2, f3, plex(x, y, z)), [f1, f2, f3],
plex(x, y, z));

\Rightarrow It is a Groebner basis $= 0$

5. (10 pts) Recall that we can define an ideal generated by infinitely many polynomials : Given any set $F \subseteq \mathbb{K}[x_1, \dots, x_n]$ of polynomials (possibly infinite), define

$$I = \langle F \rangle = \left\{ \sum_{\text{finite sum}} h_\alpha f_\alpha : h_\alpha \in \mathbb{K}[x_1, \dots, x_m], f_\alpha \in F \right\}$$

Suppose $F = \{f_1, f_2, f_3, \dots\}$ is an infinite sequence of polynomials. Let $I = \langle F \rangle$. Show that there exists a natural number $N \in \mathbb{N}$ such that $I = \langle f_1, \dots, f_N \rangle$. (Be careful : the Hilbert Basis theorem shows that I is finitely generated; however, it does not give any guarantee that the basis will come from the elements in F).

Consider the sequence of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$$

where $I_k = \langle f_1, \dots, f_k \rangle$

Since this is an ascending chain of ideals, we have by the Ascending Chain Theorem that $\exists N \in \mathbb{N}$ s.t.

$$I_N = I_{N+1} = I_{N+2} = \dots$$

Now $I_N = \langle f_1, \dots, f_N \rangle$ by definition.

Thus $\forall f_i \in I_N \quad \forall i \geq N$.

Hence $f_i \in I_N \quad \forall i \in \mathbb{N}$

But then $I = \langle f_1, f_2, \dots \rangle \subseteq I_N$.

Obviously $I_N \subseteq I$. Thus $I = I_N = \langle f_1, \dots, f_N \rangle$ □

6. (5 pts) [Extra Credit] Find a polynomial $g \in \mathbb{K}[x, y]$ in the ideal $\langle f_1, f_2 \rangle = \langle 2xy^2 - x, 3x^2y - y - 1 \rangle$ whose remainder on division by (f_1, f_2) is nonzero. Use the lex order with $x > y$. You can use any MAPLE command for this problem. If you do use MAPLE, state clearly which command you use, what the output was and how it led to your answer.

Consider the S-polynomial

$$S(f_1, f_2) = -3x^2 + 2y^2 + 2y.$$

Dividing $S(f_1, f_2)$ by (f_1, f_2)
we get $S(f_1, f_2) \neq 0$.

$S(f_1, f_2) \in I$ and is the example.