1. (a) **(5 pts)** State Descartes' Rule of Signs for estimating the number of positive real roots of a univariate polynomial.

Let $f \in \mathbb{R}[x]$ be a univariate polynomial.

Let P be the # of positive real roots.

Let V be the # of sign variations in the coefficient sequence of f.

Then $P \leq V$ and $V-P$ is even.

(b) **(5 pts)** State the definition of the Greatest Common Divisor (GCD) of a set of univariate polynomials $f_1, f_2, \ldots, f_k \in \mathbb{R}[x]$.

$h \in \mathbb{R}[x]$ is a GCD of $f_1, \ldots, f_k$ if the following conditions hold:

1) $h \mid f_i \quad \forall i = 1, \ldots, k$.

2) For any $p \in \mathbb{R}[x]$ s.t. $p \mid f_i \ \forall i = 1, \ldots, k$, then $p \mid h$.

2. Using the algorithms we learned in class, answer the following questions.

   (a) **(5 pts)** Determine the number (including multiplicity) of real roots of $x^4 + x^2 - x - 1$. How many of these are positive and how many are negative ? (Do NOT use MAPLE for this problem. State clearly how you arrived at your conclusions - "I used *solve*() on MAPLE and counted the real roots" is NOT a good answer and will not receive any credit)

\# of sign changes in $f(x) = 1$.

$\Rightarrow$ \# positive real roots ≤ 1

and same parity as 1.

$\therefore$ \# positive real roots $= 1$.

For negative real roots, count sign changes in $f(-x) = x^4 + x^2 + x - 1$

$= 1$.

$\therefore$ \# of negative real roots is also 1.

$\therefore$ 2 real roots, 1 +ve, 1 -ve.

(b) **(10 pts)** Find two intervals of length 0.5 that each contain a real root of $x^4 + x^2 - x - 1$.

For this problem you can use MAPLE. However, DO NOT use the *solve*(), *sturmseq*() or *sturm*() commands; you can use any other MAPLE command that you want, like *diff*(), *rem*(), *eval*(). Note that you can input a list of expressions into *eval*() and it will evaluate all the expressions in the list at the prescribed x value. For example $eval([x, x^2, x^3], x = 2)$ will return the list $[2, 4, 8]$. State clearly the commands you used in MAPLE, the output you got and how that led to your final answer.

Solution 1: Compute Sturm sequence using MAPLE:

$p_0 := x^4 + x^2 - x - 1$

$p_1 := diff(p_0, x) = 4x^3 + 2x - 1$; $p_2 := -rem(p_0, p_1, x) = 1 - \frac{1}{2}x^2 + \frac{3}{4}x$

$p_3 := -rem(p_1, p_2) = -11 - 19x$

$p_4 := -rem(p_2, p_3) = -\frac{575}{1444}$

$eval([p_0, p_1, p_2, p_3, p_4], x=0) = [-1, -1, 1, -11, -\frac{575}{1444}]$

$eval([p_0, p_1, p_2, p_3, p_4], x=1) = [0, 5, \frac{5}{4}, -30, -\frac{575}{1444}]$

$\therefore 2 - 1 = 1$ real root in $[0, 1]$ by Sturm's theorem.

$eval([p_0, p_1, p_2, p_3, p_4], x=\frac{1}{2}) = [-\frac{19}{16}, \frac{1}{2}, \frac{5}{4}, -\frac{41}{2}, -\frac{575}{1444}]$

$\therefore 2 - 1 = 1$ real root in $[\frac{1}{2}, 1]$

$$\text{eval}\left([p_0, p_1, p_2, p_3, p_4], x=-1\right) = \left[2, -7, -\frac{1}{4}, 8, \frac{-575}{1444}\right]$$

$\Rightarrow 3-2=1$ real root in $[-1, 0]$.

$$\text{eval}\left([p_0, p_1, p_2, p_3, p_4], x=-\tfrac{1}{2}\right) = \left[-\frac{3}{16}, -\frac{5}{2}, \frac{1}{2}, -\frac{3}{2}, \frac{-575}{1444}\right]$$

$\Rightarrow 3-2=1$ real root in $[-1, -\frac{1}{2}]$.

<u>Solution 2</u> :

$$\text{eval}\left(x^4+x^2-x-1, x=1\right) = 0$$

$$\text{eval}\left(x^4+x^2-x-1, x=\tfrac{1}{2}\right) = \frac{-19}{16}$$

$$\text{eval}\left(x^4+x^2-x-1, x=0\right) = -1$$

$$\text{eval}\left(x^4+x^2-x-1, x=-\tfrac{1}{2}\right) = -\frac{3}{16}$$

$$\text{eval}\left(x^4+x^2-x-1, x=-1\right) = 2.$$

Now this means by the intermediate value theorem.

there is a root in $(-1, -\frac{1}{2})$

and

there is a root in $[\frac{1}{2}, 1]$.

(c) **(15 pts)** Show that

$$\langle -2x^3 - 2x^2 + 1, -x^4 + x^3 + x^2 + x - 1, 5x^5 - 5x^4 + x^2 + x - 3 \rangle = \mathbb{R}[x],$$

i.e., *every* univariate polynomial is in the ideal generated by these 3 polynomials. You can use any function from MAPLE for this problem (you might find *gcd*() useful which computes the GCD of two polynomials). Clearly state which commands you used, the output given by MAPLE and how that led to your conclusion.

Compute $GCD(-2x^3-2x^2+1, -x^4+x^3+x^2+x-1, 5x^5-5x^4+x^2+x-3)$

in MAPLE :

$gcd(-2x^3-2x^2+1, gcd(-x^4+x^3+x^2+x-1, 5x^5-5x^4+x^2+x-3))$

$= 1$

Now any univariate polynomial f is divisible by 1 : $f = 1 \cdot f$.

$\therefore$ f in the ideal.

3. **(15 pts)** Let $I \subseteq \mathbb{K}[x]$ be *any* ideal in $\mathbb{K}[x]$ (recall that $\mathbb{K}[x]$ is the set of all univariate polynomials over a field $\mathbb{K}$. Show that there exists $f \in \mathbb{K}[x]$ such that $I = \langle f \rangle$.

Let f be the polynomial with smallest degree in I.

For any $p \in I$, divide p by f.

$p = q \cdot f + r$.

If $r \neq 0$, degree(r) < degree(f).

and $r = p - qf$, $\therefore r \in I$.

This is a contradiction because f has the smallest degree in I.

$\therefore$ $r = 0$ and so $p = q \cdot f$.

$\therefore$ $I = \langle f \rangle$.

4. **(7 pts)** Prove that $\langle x + xy, y + xy, x^2, y^2\rangle = \langle x, y\rangle$.

Easy to check that $\langle x+xy, y+xy, x^2, y^2\rangle \subseteq \langle x, y\rangle$

Now $x = (1+x)(x+xy) + (-1)x^2 + (-x)(y+xy)$

and $y = (1+y)(y+xy) + (-1)y^2 + (-y)(x+xy)$

$\therefore$ $\langle x, y\rangle \subseteq \langle x+xy, y+xy, x^2, y^2\rangle$

5. **(8 pts)** Let $S \subseteq \mathbb{R}^n$ be a subset of points in n-dimensional Euclidean space. Recall that the set of polynomials

$$I(S) = \{f \in \mathbb{R}[x_1, \ldots, x_n] \colon f(s) = 0 \quad \forall s \in S\}$$

is an ideal. Show that if $f^m \in I(S)$ for some $m \geq 1$, then $f \in I(S)$, i.e., if the m-th power of a polynomial is in $I(S)$, then that polynomial is also in $I(S)$.

If $f^m \in I(S)$

$\Rightarrow f^m(s) = 0 \quad \forall s \in S.$

$\Rightarrow (f(s))^m = 0 \quad \forall s \in S.$

$\Rightarrow f(s) = 0 \quad \forall s \in S.$

$\Rightarrow f \in I(S).$