Edmonds' Matching Polytope Theorem

\[ G = (V, E) \text{ undirected, } w_e : e \in E \]

\[ \max \left\{ \sum_{e \in M} w_e : M \text{ matching} \right\} \]

\[ \text{Embedding} \]

\[ 0/1 \text{ vectors in } R^{|E|} \]

\[ \text{LP formulation} \]

\[ \sum_{e \in \delta(v)} x_e \leq 1 \quad \forall v \in V \]

\[ 0 \leq x_e \leq 1 \quad \forall e \in E \]

\[ \sum_{e \in E(U)} x_e \leq \left\lfloor \frac{|U|}{2} \right\rfloor \quad \forall U \subseteq V, |U| \text{ odd} \]
variables of the LP $n = |E|$

constraints $m = 2^{|V|} - 1$

$O(2^{|V|})$

Simpler method / Interior method

$O(m^{1.5} n \times \text{size}(A, b, c))$

Ellipsoid method

Given a "description" of closed convex set $\{A x \leq b\}$

\[ x^2 + x_2^2 \leq 1 \]

$C = \emptyset$? [Feasibility problem]

We will assume we have access to $C$

(or its description) via a separation oracle:

**INPUT:** $x \in \mathbb{R}^d$

**OUTPUT:** "YES" if $x \in C$
"NO" if $\bar{x} \notin C$ and outputs $(a, b) \in \mathbb{R}^d \times \mathbb{R}$ s.t.

$$C = \{ x : \langle a, x \rangle \leq b \}$$
and $\bar{x} \notin \{ x : \langle a, x \rangle > b \}$

**Sc:** Suppose $C = \{ x : Ax \leq b \}$.

Given $\bar{x}$: test all inequalities

If all satisfied $\rightarrow$ YES

If any inequality is not satisfied then report NO and this inequality as the separating hyperplane/halfspace.

**Ex:**

$\Phi$-matching ($G$) = $\Sigma e \in R^L$:

$$\sum_{e \in S(v)} \Sigma x_e = 1 \forall v$$

$$\sum_{e \in S(v)} x_e \geq 1 \forall v$$

$$0 \leq x_e \leq 1$$

Can implement a separation oracle in the following way: INPUT $\rightarrow \bar{x}$.
for \( \forall e \in E \)
\[ O(|V| + |E|) \]

\[
\sum_{e \in E} x_e \geq 1 \\
\forall e \in E
\]

compute min-cut.

There is a version of min-cut that returns the minimum odd-sized cut.
that runs in time \( \text{poly}(|V|, |E|) \).

\[
\max \sum_{c \in C} \langle c, x \rangle : x \in C
\]

and \( C \) is given via a separation oracle.
We can reduce the optimization problem to the feasibility problem as follows.

\[ C_y := C \cup \{ x : \langle c, x \rangle = \gamma \} \]

Guess for the optimal value:

\[ C_0 = \emptyset ? \]

\[ \langle c, x \rangle \geq 0 \]

No: \[ C_1 = \emptyset ? \]

\[ \langle c, x \rangle \geq 1 \]

No: \[ C_2 = \emptyset ? \]

\[ \langle c, x \rangle \geq 2 \]

No: \[ C_2' = \emptyset ? \]

\[ \langle c, x \rangle \geq 2' \]

Optimal value is OPT > 0

Then in log OPT steps we find \( M \) s.t.

\[ \text{OPT} \leq M. \]

Yes
\[
\langle c, x \rangle = -1
\]

\[
\langle c, x \rangle \geq -2
\]

will find an interval 
\[
[L, U]
\]
which contains \( \text{OPT} \) in at most \( O(\log \text{OPT}) \) steps.

Now, we do a binary search:

\[
\frac{L + U}{2}
\]

\[
\langle c, x \rangle \geq \frac{L + U}{2}
\]

\[
\frac{\log_2 \left( \frac{U - L}{c} \right)}{2}
\]

queries.
we will be left with an interval of length $\leq \epsilon$

$\rightarrow$ can solve the optimization problem
up to $\epsilon$-accuracy by making

$O\left(\log \frac{0.87}{\epsilon}\right)$

queries to the feasibility problem.

$\overline{C} \rightarrow$ with a separation oracle.

$C = \emptyset$?

Assume that if $C \neq \emptyset$ then

$C \cap B(0, R) \neq \emptyset$

and $\exists z \in \mathbb{R}^d$ s.t.

$B(z, R) \subseteq C$
Def: An ellipsoid in $\mathbb{R}^d$ is an affine transformation of $B(0,1)$; i.e., there exists a matrix $M \in \mathbb{R}^{d \times d}$ and $c \in \mathbb{R}^d$ such that:

$$E := \{c + Mx : x \in B(0,1)\}$$

where $c$ is the center of $E$. 
Iteration invariant: if $C \neq \emptyset$, then $E \cap C \neq \emptyset$.

Query the separation oracle with center $c \in E$.

If oracle returns separating halfspace $H$, then construct $E'$ s.t.

$$H \cap E \subseteq E'.$$

with guarantee that

$$\text{vol}(E') \leq e^{-\frac{1}{2d}} \text{vol}(E) < 1.$$
\[ \text{vol}(B(0, R)) \leq (2R)^d \]

\[ \text{vol} \geq R \]

After \( k \) iterations of constructing these ellipsoids,

\[ p^n \leq \text{vol}(E_k) \leq e^{-\frac{k}{2d}} (2R)^d \]

\[ \Rightarrow e^k \leq \frac{(2R)^d}{p} \]

\[ \Rightarrow k \leq (2d^2) \log \frac{2R}{p} \]
Conclusion: One of these iterations will visit a feasible center, or we can conclude that \( C = \emptyset \).

Polytope \[ A \mathbf{x} \leq b \]

\[ L = \text{size}(A, b) \]

If \( A \mathbf{x} \leq b \) has a solution then

\[ \{ \mathbf{x} : A \mathbf{x} \leq b \} \subset B(0, 2^{4d^2L}) \]

and if \( A \mathbf{x} \leq b \) is full dimensional then \( \text{vol}(P) > 2^{-8d^3L} > 0 \)

We know the radius to begin the ellipsoid iteration and if the volume of the ellipsoid falls below then we know the polytope is empty.
dimensional and we can recurse on

dimension.

\[ O(d^4 L) \] iterations before

realizing the problem is

lower dimensional or

finding a feasible center.