This: $G = (V, E)$ directed with $n$ vertices and $m$ edges. Capacities $u_e, r,s \in V$.

Shortest augmenting path algorithm terminated in $O(nm)$ iterations/augmentations.

Recall: $\mu(G, r, s) = \text{length of shortest directed } r,s \text{ path in } G$

$\alpha(G, r, s) = \text{all edges that appear on some shortest } r,s \text{ path}$

$G' = (V', E')$

$\forall u \in V \cup E \exists \nu : uv \in \alpha(G, r, s)$
Lemma: $G = (V, E)$ directed $\Rightarrow$

$\mu(G', r, s) = \mu(G, r, s)$

$\alpha(G', r, s) = \alpha(G, r, s)$

Pf: suffices to prove that no new edge in $G'$ appears on a shortest $r, s$ path in $G'$.

Suppose to the contrary, $P$ shortest $r, s$ path in $G'$ that uses $e = vu$ s.t. $uv \in \alpha(G, r, s)$.

Assume WLOG that these edges are all in $G$.

$A = \text{path in } G \text{ from } r \text{ to } v$ (and $G'$)
Observe that $AB'$ is a path in $G$.

$\Rightarrow |A'| + 1 + |B'|$

comes from $e$

$\leq |A| + |B'|$

$\Rightarrow |A'| + 1 \leq |A| \Rightarrow |A'| < |A|$

Also observe that $A'B'$ is a path in $G'$.

Since $P = A \cdot e \cdot B'$ is a shortest path in $G'$, we have

$|A| + 1 + |B'| \leq |A'| + |B'|$

$\Rightarrow |A| + 1 \leq |A'| \Rightarrow |A| < |A'|$

Contradiction
Claim: \( G(x') \subseteq G(x)' \)

edges of \( G(x') \) \subseteq edges of \( G(x)' \)

Lemma: \( G = (V,E), r, s \) and let \( x \) be a feasible flow. Suppose we augment on a shortest \(+\)-augmenting path in \( G \) to get \( x' \) a new flow. Then either: \( \mu(G(x'), r, s) > \mu(G(x), r, s) \)
OR: \( \mu(G(x'), r, s) = \mu(G(x), r, s) \)
and \( \alpha(G(x'), r, s) \subset \alpha(G(x), r, s) \)

Since \( G(x') \subseteq G(x) \)
\( \mu(G(x'), r, s) \geq \mu(G(x)', r, s) \)
previous lemma \( \Rightarrow \mu(G(x), r, s) \)

Suppose \( \mu(G(x'), r, s) = \mu(G(x), r, s) \)
\( \Rightarrow \mu(G(x)', r, s) \)
\( \alpha(G(x), r, s) \)
previous lemma
\( \alpha(G(x), r, s) \subset \alpha(G(x)', r, s) \)

Now we show that at least one edge disappears from \( \alpha(G(x), r, s) \).

\( x \)-augmenting with \( P \)

\( \Delta P_x = \min \{ \min \{ \delta \in \partial P \text{ such that } \delta \text{ is an edge} \} \} \).
\[
\min \{ x_e : \text{back edges}\} \\
\Rightarrow x_e \rightarrow x_e + \Delta P_x \\
\Rightarrow \text{The bottleneck edge in } P \text{ disappears.} \]

\[
X^0 = 0 \quad \Rightarrow \quad X^1 \quad \Rightarrow \quad X^2 \quad \Rightarrow \quad \text{max flow} \]

\[\leq n-1\]

\[\text{augmentations}\]

\[\text{where shortest path length increases}\]

\[\downarrow\]

\[\text{the shortest paths in}\]

\[\text{the auxiliary graphs are}\]

\[\text{the same length.}\]

\[\leq m \text{ augmentations in these intervals}\]

\[\Rightarrow \leq O(nm) \text{ augmentations in all.}\]
Comments:
1. The shortest augmenting path algorithm always terminates in \( \leq 0(nm) \) iterations.
   So real valued capacities are OK.
2. If \( u_e \in \mathbb{Z} \Rightarrow \forall e \in E \), then
   \( \exists \) a maximum flow with all \( x_e \in \mathbb{R} \).
   and the shortest augmenting path
   algorithm finds such an integral soln.
3. \[
\begin{align*}
\max_{x_e} & \quad f(x(s)) \\
\text{s.t.} & \quad \sum_{e=s}^{e=v} x_e - \sum_{e=v}^{e=s} x_e = 0, \quad \forall v \neq s, \forall s, v \\
\text{min} & \quad 0 \leq x_e \leq c_e
\end{align*}
\]
\[
\text{minimize} & \quad \sum_{e \in E} c_e x_e
\]
Maximum matching in undirected graphs.

\[ G = (V, E) \text{ undirected} \]

\[ \omega_e : e \in E. \]

**Def:** A subset \( M \subseteq E \) is a matching in \( G \) if \( e_1 \cap e_2 = \emptyset \) for all \( e_1, e_2 \in M \).

\[
\max \left\{ \sum_{e \in M} \omega_e : M \text{ matching in } G \right\}
\]

\[ M \]

\[ m \]
Unweighted/Cardinality version \( w_e = 1 \)

Application in astronomy:
\[ w_e = 1 \]
Def: $G = (V, E)$ undirected

$S \subseteq V$ is called a vertex cover

if $\forall e \in E : e \cap S \neq \emptyset$

Observation: $M \subseteq E$ matching

$S \subseteq V$ vertex cover.

then $|M| \leq |S|$

$\Rightarrow$ if we find $M^* \subseteq E, S^* \subseteq V$

s.t. $|M^*| = |S^*|$
Then \( M^* \) must be a maximum matching, \( S^* \) is a minimum size vertex cover.