Maximum Matchings in General Graphs

Jack Edmonds 1960

Edmonds' matching algorithm

\[ G = (V, E) \] undirected \[ M \subset E \] matching

Def: An \( M \)-alternating tree \( T = (V', E') \)

where \[ V' \subset V \], \( E' \subset E \) with \( r \in V' \)
called the root of the tree:

1. \( T \) is a tree

2. \( r \) is \( M \)-exposed in \( G \).

3. \( \forall v \in V' \setminus \{r\} \), the unique path
   from \( r \) to \( v \) in \( T \) is an
   \( M \)-alternating path.
\[ A(T) = \{ v \in V' : \text{distance of } v \text{ from } s \text{ is odd} \} \]
\[ B(T) = \{ v \in V' : \text{dist} \ldots \text{ is even} \} \]
\[ (u \in B(T)) \]

"Look at vertices in \( B(T) \) that have edges to a vertex outside \( V' \)"

1. We find such a vertex \( u \in B(T) \) s.t. \( vw \) is an edge with \( w \notin V' \) and \( w \) is \( M \)-exposed.

\[ \Rightarrow s - w \text{ is a } M \text{-augmenting path.} \]

2. We find \( u \in B(T) \) s.t. \( vw \) is an edge to \( w \notin V' \) and \( wu \in M \).
In the situation where all edges on $B(T)$ go back into $T$.

1. All these edges go back to vertices in $A(T)$.

2. If an edge between two vertices in $B(T)$.

In (2) above with an edge $e$ between $v_1, v_2 \in B(T)$, we discover an odd cycle in the graph.
Shrinking operation

\[ G = (V, E) \text{ undirected} \]

\[ C \text{ odd cycle in } G. \]

\[ G \times C = (V', E') \]

\[ V' = V \setminus V(C) \cup \{v^2 \} \]

\[ \text{vertices in cycle } C \]

\[ uv \in E' \text{ if one of the following hold:} \]

\[ \begin{align*}
0 & \quad u, v \notin V(C) \text{ and } uv \in E \\
\end{align*} \]
2. $u \notin V(C)$, $v = n$ and $\exists e \in V(C)$ with $u \in E$.

**Expansion of super vertex**

$G = (V, E)$

$G' = G \times C$

" $(V', E')$"
Define the expanded matching $M$ as follows:

$$E = \{ uv \in M : 0 \text{ if } u \neq n, v \neq n \text{ then } uv \in M \}.$$

2. If $u$ is the super vertex, then let $s \in V$ in $G$ s.t. $vs \in E$ and put $vs \in M$.

3. Match up all the other vertices in $C$ using edges in the cycle.

Expansion Lemma: $\# \text{ of } M'$-exposed vertices in $G'$

$\# \text{ of } M'$-exposed vertices in $G'$
Shrinking Lemma:

\[ G = (V, E) \quad M \subseteq E \text{ matching} \]
\[ T \subseteq M \text{-alternating tree} \]
\[ C \text{ odd cycle because of an edge } e \]
\[ \text{between } v_1, v_2 \in B(T) \]

\[ G' = G \times C \]
\[ M' = M \setminus E(C) \]
\[ T' = (T \cup \{e = v_1v_2\}) \times C \]

0. \( M' \) is still a matching in \( G' \)
2. $T'$ is an $M'$-alternating tree.

3. The upper vertex is in $B(T')$.
max-matching \( (G) \)

\[
G' := G \\
M := \emptyset
\]

Pick \( r \in V(G) \)

\[
T := (\exists x, \emptyset), B(T) := \emptyset, A(T) := \emptyset
\]

While \((\exists w \in E' \text{ s.t. } v \in B(T), w \notin A(T))\)

\[
\text{Case 1: } w \notin V(T), \quad \omega \in M\text{-exposed} \\
\Rightarrow \text{path from } r \text{ to } v \text{ and then to } \omega \text{ using } \omega w \\
\quad \text{so an } M\text{-augmenting path.}
\]

Augment \( M \). Expand \( M \) to a matching \( \hat{M} \) in \( G' \).

If all vertices of \( G' \) are covered, stop.

Else choose \( r \in M\text{-exposed}. \)

\[
G' := G, \quad T := (\exists x, \emptyset)
\]

\[
\text{Case 2: } w \notin V(T), \quad \omega \in M\text{-covered} \quad \text{by } wz \in E'NM
\]

\[
T := (V(T) \cup \exists w, x, z), B(T) \cup \{(wz)\}
\]

\[
\text{Case 3: } w \in B(T), \text{ let } C \text{ be an odd cycle containing } Vw.
\]

\[
G' := G \times C \\
T := T \times C \\
M := M \setminus E(C)
\]

T is frustrated in \( G' \) with respect to \( M \).
$M' = \text{expanded matching from } M$

$M'' = \text{max-matching } (G', T) \text{ removing all vertices and edges of } T \text{ from } G'$

return $M' \cup M''$
Class discussions

Max flow $O(nm^2)$

Bipartite $O(nm)$

Cardinality $O(n^3)$

General $O(n^2m)$

Best $O\left(\frac{nm \log n}{\log\log n + \log m} \right)$

O(\sqrt{nm \log(n^2/m)})$ where $W = \max_{e \in E} |w_e|$

Weighted matchings in general graphs.

Minimum cost flow problem

$G = (V, E)$ directed graph.

$\forall v \in V, \ a_v, b_v \in \mathbb{R}$

$\forall e \in E, \ l_e, u_e \in \mathbb{R}$ ($l_e \leq u_e$)

$\forall e \in E, \ c_e \in \mathbb{R}$ (cost)