Graph Theory:

What is a graph?

\((V, E)\)

vertices/nodes edges

\(f\)

\(\text{unordered pairs from } V\)

\(|V| = n\)

\(|E| = m\)

self loops

\(e_1 \rightarrow \{v_1, v_3\}\)

\(e_2 \rightarrow \{v_2, v_3\}\)

\(e_3 \rightarrow \{v_1, v_2\}\)
Directed $f^D: E \rightarrow \text{ordered pairs from } V \subseteq V \times V$

$e_1 \rightarrow v_1 v_2$
$e_2 \rightarrow v_2 v_1$
$e_3 \rightarrow v_1 v_2$

A (directed) path in a (directed) graph $G = (V, E)$ is a sequence

$\forall v, e_1, e_2, ..., e_k \in V$

such that

$\forall e_k = \{v_{k-1}, v_k\}$ (undirected)

$\Rightarrow v_{k-1} v_k$
$v_1, v_2, v_3, v_4$
directed

Example:

$3 \neq v_2 \neq v_6 \neq v_5 \neq v_7$

and 2) $e_i \neq e_j \neq i \neq j$

and 3) $v_i \neq v_j \neq i \neq j$

Complexity of Algorithms

Algorithm $\rightarrow$ 

\[ \# \text{ of arithmetic operations that we perform:} \]

\[ +, -, *, /, \text{ and comparisons} \]

Notion of problem size $\approx$ roughly, $\# \text{ bits needed to write down problem data.}$

Linear regression: $N$ data pts, $d$ dimension
size of numerical data: $\mathbf{D}$ \\
"dimensional size" / "Numerical size" data: $\mathbf{S}$ \\
$\mathbf{f}(\mathbf{D}, \mathbf{S})$ bound on the running time / complexity \\

sorting: $n$ numbers $a_1, \ldots, a_n$ \\

$O(n^2)$ \\
$O(n \log n)$ \\

Polynomial time algorithms: when $\mathbf{f}$ is a polynomial of $\mathbf{D}$ and $\mathbf{S}$ \\

Strongly polynomial time: where $\mathbf{f}$ is a polynomial of $\mathbf{D}$ and is independent of $\mathbf{S}$.
Maximum Flow Problem:

**Data:** \( G = (V,E) \) directed

- \( u_e : e \in E \) capacity.
- Designated source \( s \in V \), sink \( t \in V \)

**Problem:** Assign flow values

\[ x_e : e \in E \]

s.t.

1. \( 0 \leq x_e \leq u_e \)
2. \( \sum_{e \in V \setminus \{s,t\}} x_e = 0 \)

\[ f(v) = \sum_{e = vv} x_e - \sum_{e = uv} x_e = 0 \]

and we maximize \( f(s) \).

\[ \sum_{e = ws} x_e - \sum_{e = su} x_e \]