Combinatorial Optimization
February 23, 2021

\[ E : S \rightarrow \mathbb{R}^n \]

\[ c \in \mathbb{R}^n \]

**Linear Programming formulation:** \( A, b \)

\[ \text{conv} (E(S)) = \{ x : A x \leq b \} \]

**Integer Programming formulation:** \( A, b \)

\[ E(S) = \{ x \in \mathbb{Z}^n : A x \leq b \} \]

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**Matching in non-bipartite graphs**

\[ \begin{align*}
    x_{e_1} + x_{e_2} & \leq 1 \\
    + x_{e_2} + x_{e_3} & \leq 1 \\
    x_{e_1} + x_{e_3} & \leq 1 \\
    0 & \leq x_{e_i} \leq 1
\end{align*} \]

**Graph**

\[ G = (V, E) \]

\[ \sum_{e \in E(i)} x_e \leq 1 \]

\[ 0 \leq x_e \leq 1 \]
\( x_{e_1} + x_{e_2} + x_{e_3} \leq 1 \)

\( G = (V, E) \)

\[
\sum_{e \in E[V]} x_e = \frac{|U| - 1}{2} \leq \frac{|U|}{2}, \quad \text{odd}
\]

\( E[U] = \{ e \in E : \text{both ends of } e \text{ are in } U \} \)

\[0 \leq x_e \leq 1\]

\( \emptyset \text{ matching}(G) \)

\( \text{matching}(G) = \text{convex hull of all 0/1 matching vectors for } G. \)

**Edmonds' Matching Theorem:** If \( G = (V, E) \)

\[ \text{matching}(G) \leq \emptyset \text{ matching}(G). \]

**Def.** A perfect matching in \( G \) is a matching where all vertices are covered.

\[ \text{p-perfect matching}(G) = \text{convex hull of all 0/1 perfect matchings in } G. \]
\[ Q_{p,\text{matching}}(G) = S \rightarrow \sum_{e \in \delta(V)} x_e = 1 \]

\[ \sum_{e \in E[V]} x_e \leq \left\lfloor \frac{|V|}{2} \right\rfloor \]

Edmonds' Perfect Matching Form: \( \forall G = (V, E) \),
\[ P_{p,\text{matching}}(G) = Q_{p,\text{matching}}(G) \]

Lemma 1: \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \)
- All vertices \( x \in \mathbb{R}^n : Ax \leq b \) are rational.
- Characterization of vertices:
  \[ x = A^{-1}b \]

Lemma 2: \( P_1, \ldots, P_k \in \mathbb{R}^n \) rational
- Suppose \( \forall x \in \text{conv}(\{P_1, \ldots, P_k\}) \) and \( \nu \) is rational

\[ 0 = \nu P_1 + \cdots + \nu P_k \]
- for \( 0 \leq \nu_i \leq 1 \) with \( \sum \nu_i = 1 \)
- If \( \nu \) follows from applying Lemma 1 to the following polytope
\[ 0 \leq \nu_i \leq 1 \]

\[ \sum \nu_i = 1 \]
If a Perfect Matching then:

\[ \sum_{u \in U} \left( \sum_{e \in \delta(u)} x_e \right) = |U| \]

\[ \sum_{e \in \delta(U)} x_e + (|U| - 1) \geq \sum_{e \in \delta(U)} x_e + 2 \sum_{e \in \delta(U)} x_e - \frac{|U| - 1}{2} \]

Let \( e : \) one endpoint in \( U \) and another outside \( U \).

\[ \sum_{e \in \delta(U)} x_e = 1 \]

We have shown: \( p\text{-matching}(G) = 0 \)

We will show: \( p\text{-matching}(G) = \mathcal{O}(G) \)

\( \subseteq \) easy.

Suffices to show that every vector of \( \mathcal{O}(G) \) is a convex combination of 0/1 perfect matching vectors.

Proof by induction on \(|V| + |E|\).
Base Case: \( e \) \( \notin \mathcal{E}(G) \)

Consider any graph with \( |V| + |E| \geq 3 \).

Let \( \tilde{x} \in \mathcal{Q}(\tilde{G}) \) vertex.

Case 1: \( \exists e \in E \) s.t. \( \tilde{x}_e = 0 \)

Let \( \tilde{G} = G \setminus \{e^*\} \) and define \( \bar{x}_e \)

\[
\bar{x}_e \begin{cases} 
\tilde{x}_e & e \in e^* \\
0 & e \notin e^* \end{cases}
\]

Claim: \( \bar{x} \in \mathcal{Q}(\tilde{G}) \)

\[ \Rightarrow \text{by I.H., } \tilde{x} \text{ is a convex combination of } 0/1 \text{ perfect matchings in } \tilde{G} \]

\( \Rightarrow \) consider them as \( 0/1 \) perfect matchings in \( G \).

\( \tilde{x}_{e^*} = 0 \) \( \Rightarrow \) and all these \( 0/1 \) perfect matchings in \( G \) have \( 0 \) on \( e^* \).

Case 2: \( \exists e^* \text{uv s.t. } \bar{x}_{e^*} = 1 \)

\[ \sum_{e \in \mathcal{E}(u)} \bar{x}_e = 1 \]

\( \Rightarrow \bar{x}_e = 0 \) \( e \in \mathcal{E}(u) \setminus e^* \)

Similarly for \( v \).
\[ G = G \setminus \{u,v\} \]

and define \( \tilde{x} \) to be \( x \) restricted to edges of \( \tilde{G} \).

Claim: \( \tilde{x} \in Q(\tilde{G}) \) \( \sum_{e \in S(v)} \tilde{x}_e = 1 \) \( \forall v \in V(\tilde{G}) \)

\[ \sum_{e \in S(v)} \tilde{x}_e > 1 \] \( \forall v \in V(\tilde{G}) \) \( \text{odd} \).

\[ \sum_{e \in S(v)} \tilde{x}_e > 0 \] \( \forall v \in V(\tilde{G}) \)

so by Ind. \( \tilde{x} \) is a cons. comb. of \( \emptyset \) perfect matching vectors in \( \tilde{G} \).

Extend these perfect matchings to perfect matchings in \( G \) by including \( e^* \).

\( \tilde{x} \) is a cons. comb. of these new perfect matchings in \( \tilde{G} \).

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**Case 3:** \( 0 < \tilde{x}_e < 1 \) \( \forall e \in E \).

\(
\Rightarrow \) degree of every vertex \( \geq 2 \)

(Hard shaking)

\[ \sum_{v \in V} \text{degree of } v = 2 |E| \] for any \( G = (V,E) \)

\[ \sum_{v \in V} \text{degree of } v = 2 |E| \]

\[ \Rightarrow 2 |E| = 2 |V| \]

\[ \Rightarrow |E| = |V| \]
Case 3a: \( |E| = |V| \Rightarrow \) each vertex has degree 2.

\[
\overline{x} e \times M_1 + (1 - \overline{x}) e \times M_2 = \overline{x} \quad \text{restricted to this cycle.}
\]

\[
\text{Case 3b: } |E| > |V| \quad \text{since } x \text{ is a vertex of } G, \text{ the tight constraint must have rank } |E|.
\]

\[
\Rightarrow \text{ must have at least } |E| \text{ tight constraints.}
\]

\[
\sum_{e \in E(V)} x_e = 1 \quad \text{for } e \in E(V)
\]

\[
\Rightarrow \exists \quad V^* \subseteq V \text{ with } |V^*| \text{ odd s.t. } \sum_{e \in E(V^*)} x_e = 1 \text{ and } |V^*| \geq 3
\]
[Assume w.l.o.g. $G$ has even # vertices]

$\phi \leftrightarrow \text{P-matching } (G) = \emptyset$

and $Q(G) = \emptyset \quad \forall \in V$

Construct $\tilde{G}$,

\begin{align*}
\tilde{G}_1 & \\
\tilde{G}_2 &
\end{align*}

Can apply 1.4 to $\tilde{x}_1$ and $\tilde{G}_1$ since $|U^*| = 3$.

Claim: $\tilde{x}_i \in \mathcal{Q}(\tilde{G}_i)$ for $i = 1, 2$.

\[ \tilde{x}_1 + \ldots + \tilde{x}_k \]

where $M_1', \ldots, M_k'$ are perfect matchings in $\tilde{G}_1$.

$\tilde{x}_i \in \mathcal{Q}(G_i)$

(\text{the 0/1 vector corresponding to these matchings})
Similarly \[ \tilde{X}_2 = \frac{\alpha^2}{1} X_{M_1} + \ldots + \frac{\alpha^2}{k_2} X_{M_{k_2}} \]

for perfect matchings \( \tilde{G}_2 \).

By Lemmas 1 and 2:

\[ \alpha^1 = \frac{p_1}{q_1} \ldots \frac{p_{k_1}}{q_{k_1}} \]

and

\[ \alpha^2 = \frac{p_1}{q_1} \ldots \frac{p_k}{q_k} \]

Can further assume all \( q_j \)'s are the same natural number \( Q \).

\[ \alpha^1 = \frac{p_1}{Q} X_{M_1} + \ldots + \frac{p_{k_1}}{Q} X_{M_{k_1}} \]

and

\[ \alpha^2 = \frac{p_1}{Q} X_{M_1} + \ldots + \frac{p_{k_2}}{Q} X_{M_{k_2}} \]

Then

\[ QX_1 = \frac{p_1}{Q} X_{M_1} + \ldots + \frac{p_{k_1}}{Q} X_{M_{k_1}} \]

and

\[ QX_2 = \frac{p_1}{Q} X_{M_1} + \ldots + \frac{p_{k_2}}{Q} X_{M_{k_2}} \]
We must have $K$ perfect matchings within $M^1_1, \ldots, M^1_k, M^2_1, \ldots, M^2_k$ that have a 1 on edge $e$.

For every edge, we can pair up these matchings.
to create perfect matchings in the original graph
and then using the fact that
\[ Q X_1 = P_1 X M_1 + \ldots + P_{k_1} X M_{k_1}, \]
and
\[ Q X_2 = P_1 X M_2 + \ldots + P_{k_2} X M_{k_2}, \]
\( \tilde{X}_1 \) and \( \tilde{X}_2 \) equal \( \tilde{X} \) on all edges,
we can express \( \tilde{X} \) as a convex combination
of these new perfect matchings in \( G \).

Matching from perfect matchings
\[ G : (V, E) \quad \max \sum_{e \in E} w_e x_e = \langle w, \tilde{X} \rangle \]
Matching \( G \) = \{ \( x \in \mathbb{R}^E : \sum_{e \in E} x_e = 1 \land \sum_{e \in E(U)} x_e \leq |U| - 1 \land 0 \leq x_e \leq 1 \) \},
\( G \) is odd \( U \).

\( G \) matching = \{ \( \sum_{e \in E(U)} x_e = 1 \land \sum_{e \in E(V)} x_e = 1 \) \},
all 0/1 perfect matchings.
\( O(m^{1.5} n^L) \)
$G = (V, E)$

$\sum_{e \in E} x_e \leq 1$

$G_1 \cong G$

$G_2 \cong G$

$\times e \in \text{0/1 matching (G')}$

$\times e \in \text{0/1 matching (G')}$

\[ x_e = 1 - \sum_{e \in E} x_e \]

\[ x_e \in \{0, 1\} \]

\[ \text{corr. comb. of 0/1 perfect matching in G'} \]

\[ \text{corr. (0/1 matchings) } \]