AMS 550.472/672: Graph Theory
Homework Problems - Week VI

1. Do Problems 3.2.3, 3.2.4, 3.2.11.

   Solution: Problem 3.2.3. Consider the stable matching with men m_1, m_2 and women w_1, w_2. Let the preferences be as follows:

$$m_1 : w_1 > w_2$$
$$m_2 : w_2 > w_1$$
$$w_1 : m_2 > m_1$$
$$w_2 : m_1 > m_2$$

   In this case, both perfect matchings are stable matchings.

   Problem 3.2.4. With left proposing, the algorithm ends with matchings uf, vc, wb, xa, yd, ze.

   Problem 3.2.11. We will show if man m is matched with woman w in SOME stable matching, then m is never rejected by w in the men proposing algorithm. Thus, in the men proposing algorithm, every man is matched with a woman who is highest in his list amongst all possible partners in stable matchings.

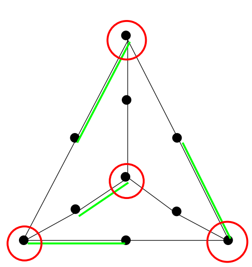
   As hinted in the textbook, during the execution of the men proposing algorithm, consider the FIRST instance of a rejection of a man m by woman w and there exists SOME stable matching M with mw as a matching edge. Since w is rejecting m, at this iteration, w must be saying "maybe" to some other man m' and so in w's preference list $m' \succ m$. In the other stable matching M, say m' is matched to w'. Now in m''s preference list, we must have $w' \succ w$, otherwise, $m'w$ is a conflict edge in the matching M: m' and w both prefer each other to their partners in M. Since m' has moved onto w in the men proposing algorithm (at the iteration when m is rejected by w), m' must have been rejected by w' in a previous iteration. But this contradicts the fact that mw was FIRST choice of such a rejection - the edge $m'w'$ is matching edge in M was is an example of a rejection BEFORE m was rejected by w.

2. Check the Tutte-Berge formula for an odd cycle, i.e., show that for an odd cycle C,

$$\nu(C) = \min_{\emptyset \subseteq A \subseteq V(C)} \frac{|V| - oc(V(C) \setminus A) + |A|}{2}.$$

   Solution: Set $A = \emptyset$. Then $|oc(V(C) \setminus A)| = 1$ and we obtain $\frac{|V|-1}{2} = \frac{|V|-oc(V(C)\setminus A)+|A|}{2}$. Also, there is a matching of size $(|V| - 1)/2$ thus illustrating the Tutte-Berge formula.

3. Determine whether the graph below has a perfect matching. Justify your answer. Find a maximum matching in the graph and give a short argument to show that it is indeed maximum.



Solution: The green edges above shows a matching of size 4, and the red vertices shows a vertex cover of size 4, so the maximum matching is of size 4. Hence, there cannot be a perfect matching.

4. Let $G = (V(G), E(G))$ be a general graph and $k \leq |V(G)|/2$ be a given positive integer. Construct a graph G' such that G' has a perfect matching if and only if G has a matching of size k. [Hint: Add some vertices and edges to G]

   Solution: Add $|V(G)| - 2k$ new vertices to G and connect them to all other vertices. Let this graph be G' and let the new vertices be X. If there is a matching of size k in G, we can match the remaining vertices in $V(G)$ to X and we have a perfect matching in G'. If we have a perfect matching in G', then all the vertices in X are matching to $|V(G)| - 2k$ original vertices from $V(G)$. Thus, the remaining $2k$ vertices are matched by edges in G, yielding a matching of size k in G.

5. Show that Tutte's perfect matching condition implies the Tutte-Berge formula. (Thus, they are equivalent) [Hint: Use the previous exercise]

   Solution: First, a little claim:

   Claim: For any subset $A \subseteq V$, $\frac{|V(G)| - oc(G \setminus A) + |A|}{2}$ is an integer.

   Proof of Claim. Let $V_1 \subseteq$ be the union of all vertices in the even components of $oc(G \setminus A)$. Let V_2 be the union of all vertices in the odd components of $G \setminus A$. Thus $|V(G)| = |V_1| + |V_2| + |A|$. Thus, we obtain $|V(G)| - oc(G \setminus A) + |A| = |V_1| + |V_2| + |A| - oc(G \setminus A) + |A| = |V_1| + 2|A| + |V_1| - oc(G \setminus A)$. Next observe that $|V_1|$ is an even number, because it is the union of vertices in even components. Also, $|V_2| - oc(G \setminus A)$ is an even number, because $oc(G \setminus A)$ and the total number of vertices in the odd components have the same parity. Thus, the numerator in $\frac{|V(G)| - oc(G \setminus A) + |A|}{2}$ is even, so the number is an integer. □

   Let G be a graph. Let $k = \nu(G) + 1$ and construct the graph G' (with X as the set of new vertices) as in the previous exercise for this value of k. Since there is no matching of size k in G, there is no perfect matching in G'. Thus, by Tutte's theorem, there exists a set $S \subseteq V(G')$ with $|S| < oc(G' \setminus S)$. If $S = \emptyset$, then $oc(G')$ is 1, since G' is connected. This means we have an odd number of vertices in G'. But G' has $|V(G)| + |V(G)| - 2k$ vertices, which is an even number. Thus, S cannot be the empty set. Thus, $|S| \geq 1$ and so $oc(G' \setminus S) \geq 2$. This means, that S must contain all of X, otherwise, the graph $G' \setminus S$ remains connected. Let $A^* = S \cap V(G)$. $|A| = |S| - |X| = |S| - |V(G)| + 2k$. And also, $oc(G' \setminus S) = oc(G \setminus A^*)$, since $G' \setminus S = G \setminus A^*$ because $X \subseteq S$. Therefore, $\frac{|V(G)| - oc(G \setminus A^*) + |A^*|}{2} = \frac{|V(G)| - oc(G' \setminus S) + |S| - |V(G)| + 2k}{2} = \frac{|S| - oc(G' \setminus S) + 2k}{2} < k = \nu(G) + 1$. Therefore, $\frac{|V(G)| - oc(G \setminus A^*) + |A^*|}{2} - 1 < \nu(G)$.

   Since $\frac{|V(G)| - oc(G \setminus A^*) + |A^*|}{2}$ is an integer by the **Claim** above, this means $\frac{|V(G)| - oc(G \setminus A^*) + |A^*|}{2} \leq \nu(G)$. But we know that $\frac{|V(G)| - oc(G \setminus A) + |A|}{2} \geq \nu(G)$ for any subset of vertices $A \subseteq V(G)$. Thus, we obtain

$$\min_{\emptyset \subseteq A \subseteq V(G)} \frac{|V| - oc(V(G) \setminus A) + |A|}{2} \leq \frac{|V(G)| - oc(G \setminus A^*) + |A^*|}{2} \leq \nu(G) \leq \min_{\emptyset \subseteq A \subseteq V(G)} \frac{|V| - oc(V(G) \setminus A) + |A|}{2}$$

   Thus, we must have equality throughout, yielding the Tutte-Berge formula.

6. Let $G = (V(G), E(G))$ be a simple graph and let $T \subseteq V(G)$ be a subset of vertices. Show that there is a matching M in G such that all vertices in T are covered by some edge in M

if and only if for every $\emptyset \subseteq W \subseteq V(G)$, the number of odd sized components in $G \setminus W$ that are fully contained in T is at most $|W|$. [Hint: Add some vertices and/or edges to G]

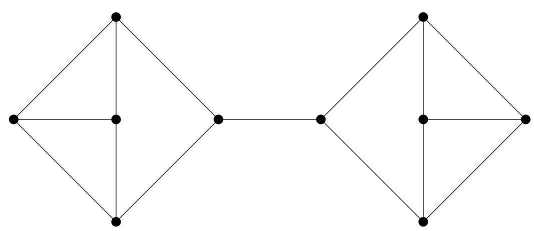
Solution: Add edges between all pairs of nonadjacent vertices in $V(G) \setminus T$. If G has an odd number of vertices, add an extra vertex which connects to all vertices in $V(G) \setminus T$. Call this new graph G'. If there is a matching in G that covers all vertices in T, then we can extend this to a perfect matching in G' by pairing up all remaining vertices. Similarly, a perfect matching in G' can matching the vertices in T using only edges from G, and thus yields a matching in G that covers all of T. Therefore we have:

G has a matching that covers T iff G' has a perfect matching.

We now show that that the stated condition for G is equivalent to Tutte's perfect matching condition in G'. Notice that since we all pairs of vertices in $V(G) \setminus T$ are connected in G', $oc(G' \setminus W')$ is simply equal to the number of odd components of $G \setminus (W \cap V(G))$ contained fully within T. Thus, the stated condition becomes equivalent to Tutte's perfect matching condition in G'.

7. Construct a 3-regular simple graph with a cut edge.

Solution:



8. Show that a tree T has a perfect matching if and only if for every vertex v, $T \setminus \{v\}$ has exactly one odd component.

Solution: ($\rightarrow$) follows from the Tutte perfect matching condition and the fact that if we have a perfect matching, we must have an even number of vertices.

($\leftarrow$) Proof by induction on the number of vertices in the tree. $n = 1$ does not satisfy the condition, so we start with $n = 2$, and the result is trivial.

For the induction step, we consider a leaf ℓ and its unique neighbor ℓ'. Since, removing ℓ' should leave exactly one odd component, this component must be the one formed by ℓ. Thus, removing ℓ' from the tree must leave some even components which are even-sized tree of smaller size. For each of these components we verify below that removing a single vertex from the component leaves exactly one odd component. By the induction hypothesis we will be done, because we can put together all the perfect matchings in these smaller components along with the edge $\ell\ell'$ to get a perfect matching in the original tree.

So we now consider an even component C in $T \setminus \{\ell'\}$. Since it is of even size, removing a vertex from C must leave at least one odd component from C. Suppose removing v^* from C leaves at least two odd components components in C. Notice that in $T \setminus \{v^*\}$, the components are those of $C \setminus \{v^*\}$, except for one which is connected to ℓ' and this component's size does not change parity because $V(T) \setminus V(C)$ has even size. Thus, $T \setminus \{v^*\}$ also has at least two odd components, which is a contradiction to the assumption on T. Therefore, removing a vertex from C leaves exactly one odd component.