Preface

Let's begin with a provocative claim: *Real numbers are useless.*

No real-world problems require real numbers for their solution. Whether determining how much mortgage a home buyer can afford, prescribing a medication dosage for a patient, or measuring the trajectory of a space probe, the rational numbers do perfectly well. One may require just a few digits of accuracy or perhaps a dozen digits to the right of the decimal point. Not enough? It is not likely there's a real problem to be solved that requires precision to 50 or more digits. Real numbers are irrelevant to the real world. Even if we reject this heretical assertion, it's possible to use real numbers without even knowing what they really are.

Do people *need* to know what a real number really is? Perhaps not, but I expect some *want* to know.

This is an interesting endeavor. Real numbers are not easy to define and it's an important accomplishment of generations of mathematicians that we can give a precise definition.

The focus of this book is on *definitions* and much less on theorems and their proofs. An A-to-Z completely rigorous development of the real number system is for upper-level mathematics majors in an analysis course. However, the core ideas can be enjoyed by a broader audience and those are the readers I hope to serve.

Audience

The objective of this book is to present a basic introduction to defining the real numbers, $\mathbb{R}$. The reader will come away understanding that: there are complete ordered fields and that all complete ordered fields are isomorphic; the real numbers are any one of these; and everything we need to know about the real numbers can be derived from the fact that $\mathbb{R}$ is a complete ordered field.

This book is aimed at self-learners, armchair mathematicians, mathematics students who want to understand what numbers actually are, and people who want a user-friendly path into mathematics. It would be a perfect book to read before, during, or after taking a course in calculus or real analysis, or for use in a seminar course.

Counting in the stone age?

The first math class.

From *Saturday Morning Breakfast Cereal* [12].

No calculus is required for this book, but readers need to be comfortable with pre-calculus level mathematics.

It is worth noting that this book is not a book on the foundations of mathematics. We don't start with the Zermelo-Fraenkel axioms and we make no mention of the Axiom of Choice. We don't prove everything we assert. We don't distinguish between *sets* and *proper classes* when we partition all finite sets into equivalence classes. We don't begin by setting $0 = \varnothing$, $1 = \{\varnothing\}$,

$2 = \{\emptyset, \{\emptyset\}\}$, and so forth.

To do everything "right" would mean never reaching readers who can still profit.

To the reader

This is a book about numbers. We begin with the most familiar numbers: 0, 1, 2, and so forth. These are exactly the numbers you need to answer questions of the form "How many?" We use these basic building blocks to carefully define the integers, then the rational numbers, then the real numbers, and—as a bonus—the complex numbers.

You have already learned something about the real numbers through decimal notation. That is, a real number consists of a (typically infinite) stream of symbols chosen from this collection:

$$- \quad . \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9$$

Certain sequences are valid such as -3.501 or 22, but others are nonsensical such as 3.3..2 or 53−.

Some real numbers have exactly one representation in this system, such as

$$\pi = 3.14159265358979323846264\ldots.$$

but others have more than one representation:

$$1 = 1.00000\ldots = 0.9999\ldots.$$

You've likely been using this notation long enough that it feels comfortable. Let me try to make you feel unsettled. When we add or multiply decimal numbers, we begin at the right. To multiply 37×18 we first calculate $7 \times 8 = 56$. We write down the 6 and carry the 5. Here's the first step:

$$\begin{array}{rr} & 37 \\ \times & 18 \\ \hline & 6 \end{array}$$

Likewise, when we add $37 + 18$ the first step is to calculate $7 + 8$; we always begin at the right.

So how do we calculate (say)

$$\pi \times e = 3.14159265358979323846264\ldots \times 2.71828182845904523536 0\ldots$$

when we can't go all the way to the right?

You know what a number is. Can you explain it?

Augustine of Hippo wrote:

> *Quid est ergo tempus? Si nemo ex me quaerat, scio; si quaerenti explicare velim, nescio.*
>
> *What then is time? If no one asks me, I know. If I want to explain it to someone who asks, I do not know.*

If no one asks you what a *number* is, it feels like you know. What happens, then, if you are asked to explain what a numbers is to someone else? Does an explanation evade you?

One of the questions we address in this book is: What is the square root of 2?

Here's a worthless answer: The square root of 2 is a number that when multiplied by itself gives the result 2.

Why is that a bad answer? Suppose you asked me: What is the secret to a happy life? Imagine I replied: It's hidden knowledge that makes your life happy. Are you happy with that answer?

A better answer to the $\sqrt{2}$ question is this: The square root of two is $7/5$. This is an incorrect answer, but it's a pretty good wrong answer because

$$\frac{7}{5} \times \frac{7}{5} = \frac{49}{25}$$

which is nearly 2.

A terrific answer to "What is the square root of 2?" would be a fraction a/b where a and b are whole numbers (integers). We could check if the answer is correct by calculating

$$\frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2} = 2.$$

Alas, as you may be aware, there is no such fraction.

To then say "Ah yes, but $\sqrt{2}$ is an irrational number" is just a fancy way to say $\sqrt{2}$ is not a rational number (is not a fraction). That says what it isn't. What is it? And how do we know there is any such number? I could begin to write down an infinite decimal number (complete with three dots) and say *that* is the square root of two, but how would we know it's correct? Can you really multiply this number

$$1.4142135623730950488016887242096980785696718753769480 73\ldots$$

by itself and see that you get exactly 2? Besides the fact that I haven't shown you most of the number, how could you ever do the calculation on an infinite string of digits?

There is a square root of 9: $\sqrt{9} = 3$ because $3 \times 3 = 9$. Is it possible there is no $\sqrt{2}$?

The good news is that there is a square root of two. The difficult, but really interesting fact is that $\sqrt{2}$ is a *real number*, but saying precisely what a real number is isn't easy. It's a long journey from 1, 2, 3 to $\sqrt{2}$, but it's one I hope you enjoy.

The content of this book

We start with the notion of *counting*. We present the natural numbers as the answers to counting questions. In Chapter 2 we present the concept of *finite set* and define two finite sets to be equivalent if there is a one-to-one correspondence between them. Natural numbers are then defined as equivalence classes of finite sets.

This is a paradigm that is repeated throughout the book. We begin with a basic structure (e.g., finite sets). We present an equivalence relation on those structures. We then create a new family of numbers as equivalence classes of those structures. We reinforce this with a typographical convention. In the chapter where they are defined, the newly created numbers (equivalence classes of simpler objects) are presented in **boldface**. See the boxed comment on page 28.

Thus in Chapter 3 we extend the natural numbers by creating an equivalence relation on pairs of natural numbers; the equivalence classes are the *integers*.

We take a brief detour in Chapter 4 to create modular integers as equivalence classes of integers.

In Chapter 5 we extend the integers by creating a new equivalence relation on pairs of integers; those equivalence classes are *rational numbers*.

Chapters 6 and 7 give two different definitions of the real numbers. In Chapter 6 we define real numbers as equivalence classes of left rays of ration numbers (Dedekind cuts). Chapter 7 constructs real numbers as equivalence classes of Cauchy sequences of rational numbers.

The tension of having two different definitions of real number is resolved in Chapter 8 in which we give the most important definition in this book: the real numbers are a complete ordered field. We explain the notion of isomorphism and assert that all complete ordered fields are isomorphic.

Each step in this journey is motivated by a "failure" of a given set of numbers. Natural numbers fail to provide subtraction. Integers fail to provide division. Rational numbers fail to provide a square root of 2. It is reasonable to note that the real numbers also have a "failure": there is no square root of -1.

Chapter 9 rectifies this last failure by extending the real numbers to the complex numbers. We show that not only do all complex numbers have square roots within the complex numbers, we present the Fundamental Theorem of

Numbers as equivalence classes

Numbers (Chapter)	Raw Materials	Equivalence Relation
$\mathbb{N}$ (2)	Finite Sets	Bijection
$\mathbb{Z}$ (3)	$\mathbb{N} \times \mathbb{N}$	$(a, b) \equiv (c, d)$ iff $a + d = b + c$
$\mathbb{Z}_m$ (4)	$\mathbb{Z}$	$a \equiv b \pmod{m}$
$\mathbb{Q}$ (5)	$\mathbb{Z} \times (\mathbb{Z} - \{0\})$	$(a, b) \equiv (c, d)$ iff $ad = bc$
$\mathbb{R}$ (6)	Left Rays	$\lvert L \bigtriangleup L' \rvert \le 1$
$\mathbb{R}$ (7)	$\mathbb{Q}$-Cauchy Sequences	See §7.2
$\mathbb{C}$ (9)	$\mathbb{R}[x]$	$p(x) \equiv q(x) \pmod{x^2 + 1}$

A recurring motif in our development of $\mathbb{R}$ is to create each new type of number as equivalence classes of more basic objects. This table summarizes our approach.

Algebra and its proof (but omitting some topological rigor) to make the case that $\mathbb{C}$ is a reasonable end to the journey.

That said, as a bonus we have Chapter 10 in which we give a gentle introduction to a variety of more exotic concepts of number such as the extended reals, tropical arithmetic, hyperreals, quaternions, and p-adic numbers.

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