

Intermittency in 2D forced cascades

It is now rather generally accepted that there is inertial-range intermittency in the 3D energy cascade, and corresponding anomalous scaling. In earlier days, the experiments and simulations were not as clear and there was considerable debate whether inertial-range intermittency existed or not and, if so, whether anomalous scaling depended upon the Kolmogorov dissipation scale $\eta = \nu^{3/4} \epsilon^{-1/4}$ or upon the integral scale L , i.e., whether velocity structure functions for $Re = \left(\frac{L}{\eta}\right)^{4/3} \gg 1$ would scale as

$$\langle |\delta u(r)|^p \rangle \sim (\epsilon r)^{p/3} \left(\frac{r}{\eta}\right)^{\delta \zeta_p}$$
$$w/ \delta \zeta_p = \frac{p}{3} + \delta \zeta_p$$

or as

$$\langle |\delta u(r)|^p \rangle \sim (\epsilon r)^{p/3} \left(\frac{r}{L}\right)^{\delta \zeta_p}.$$

This alternative can also be stated, somewhat more mathematically, as the question whether

$$\lim_{L \rightarrow \infty} \langle |\delta u(r)|^p \rangle \text{ exists}$$

or whether

$$\lim_{r \rightarrow 0} \langle |\delta u(r)|^p \rangle \text{ exists.}$$

For 3D energy cascade the consensus is that the second alternatives above hold, i.e. the anomalous scaling depends upon L and the limit as $\nu \rightarrow 0$ is innocuous for velocity increments. However, for the 2D cascades of energy and enstrophy in forced steady states there is presently no consensus on these issues. Let us discuss here some of the possibilities and concrete results.

We begin with the inverse energy cascade in 2D, which extends over $[k_{ir}, k_f]$, with k_{ir} the infrared cutoff wavenumber corresponding to the low-wavenumber damping α_p and k_f the wavenumber corresponding to the energy input by the forcing. There are then three broad possibilities:

Forcing-scale intermittency: The statistics of velocity increments have well-defined limits as $l_{ir} \rightarrow \infty$ (or $k_{ir} \rightarrow 0$) but depend anomalously on l_f (or k_f).

Damping-scale intermittency: The statistics of velocity increments have well-defined limits as $l_f \rightarrow 0$ (or $k_f \rightarrow \infty$) but depend anomalously on l_{ir} (or k_{ir}).

No intermittency: The statistics of velocity increments have well-defined limits as both $l_f \rightarrow 0$ and $l_{ir} \rightarrow \infty$ and depend anomalously on neither length-scale.

Let us first discuss the possibility of damping-scale intermittency, since it is the most like the 3D case. Notice first that the total energy E should remain finite in this limit, i.e., the velocity field u should remain in L^2 . Hence, dynamically it is plausible that this limit yields a weak solution of the 2D Euler equation:

$$\partial_t u + \nabla \cdot (u u) = -\nabla p, \quad \nabla \cdot u = 0.$$

The spectrally localized forcing function f vanishes in the sense of distributions

$$\mathcal{D}\text{-}\lim_{l_f \rightarrow 0} f = 0$$

because of the Riemann-Lebesgue lemma. This is quite analogous to what is expected to happen in the $\nu \rightarrow 0$ limit for 3D turbulence, except that now the energy conservation anomaly would be "negative dissipation" rather than positive dissipation as in 3D. One can then also define p -th-order scaling exponents as

$$\sigma_p = \liminf_{r \rightarrow 0} \frac{\ln \| \delta u(r) \|_p}{\ln (r/l_f)}$$

after having taken the limit $l_f \rightarrow 0$ first. This is

exactly the same as we did for 3D in Turbulence I, Course notes, Section III(B), where we also showed that this exponent σ_p corresponds to the maximal Besov index of order p for the limiting velocity field u . As we discussed at length there, several important properties follow immediately from this definition. Setting $\zeta_p = p \sigma_p$:

Proposition 1: ζ_p is a concave function of $p \in [0, \infty)$, i.e. for all $t \in [0, 1]$, $p, p' \geq 0$

$$\zeta_{tp + (1-t)p'} \geq t\zeta_p + (1-t)\zeta_{p'}.$$

Corollary 1: σ_p is non-increasing in p .

Proposition 2: If u is bounded, then ζ_p is non-decreasing in p .

Corollary 2: If u is bounded, then $\sigma_p \geq 0$ for all $p \geq 0$.

All of these results directly carry over to 2D energy cascade in the scenario of damping-scale intermittency, with just L replaced by dil . Furthermore, the bound that

we proved earlier for energy flux

$$\|\Pi_\ell\|_{p/3} \leq \frac{(\text{const.})}{\ell} \left[\sup_{|\tau| \leq \ell} \|\delta u(\tau)\|_p \right]^3$$

for $p \geq 3$, implies that constant ζ_p ^(negative) energy flux at length-scales $\ell \ll \ell_{ir}$ requires

$$\zeta_p \leq \frac{p}{3} \quad \text{for } p \geq 3$$

Thus, intermittency can only decrease ζ_p below the K41 prediction for $p \geq 3$. If we assume that $\zeta_3 = 1$ and use $\zeta_0 = 0$, then concavity implies that

$$\zeta_p \geq \frac{p}{3} \quad \text{for } 0 \leq p \leq 3$$

All of these statements are exactly as in 3D.

Things are quite different for the scenario of forcing-scale intermittency. In the first place, the energy is expected to diverge in this limit! This means one will no longer have limiting $u \in L^2$. However, velocity-increments will still remain well-defined. Dynamically, one may consider Lagrangian evolution of increments, according to the equation

Turbulence I,
Currents
Section II(C)

$$D_t \delta u(r) + \delta u(r) \cdot \nabla_x u + \delta u(r) \cdot \nabla_x \delta u(r) \\ = -\nabla(\delta p(r))$$

where $\delta u(r; x) = u(x+r) - u(x)$ as before. Notice that the space-gradients $\nabla_x u$, $\nabla_x p$ are expected to remain well-defined with l_f fixed, even if $V \rightarrow 0$ according to the Kraichnan-Batchelor theory of the enstrophy cascade (at least as distributions!)

The definition and properties of scaling exponents also change drastically in this scenario. Now one must take

$$\sigma_p = \liminf_{r \rightarrow \infty} \frac{\ln \|\delta u(r)\|_p}{\ln(r/l_f)}$$

after having taken the limit $l_f \rightarrow \infty$ first. This definition corresponds to a minimal exponent, in the sense that for all $\epsilon > 0$

$$\forall r > l_f, \quad \|\delta u(r)\|_p \leq (\text{const.}) \left(\frac{r}{l_f} \right)^{\sigma_p + \epsilon}$$

$$\exists r_k > l_f, \quad \|\delta u(r)\|_p \geq (\text{const.}) \left(\frac{r}{l_f} \right)^{\sigma_p - \epsilon}$$

$r_k \rightarrow \infty$

The mathematical properties of the exponents are a mirror image of those in the previous scenario:

Proposition 1*. ξ_p is a convex function of $p \in [0, \infty)$, i.e., for all $t \in [0, 1]$, $\xi_p(p) \geq 0$

$$\xi_{tp + (1-t)p'} \leq t \xi_p + (1-t)\xi_{p'}$$

Corollary 1*. σ_p is non-decreasing in p .

There is no obvious analogue of Proposition 2 and Corollary 2, however, since velocity u (and also velocity increment δu) will not remain bounded as $\ell \rightarrow \infty$. On the other hand, we can deduce from the bound on energy flux that

$$\xi_p \geq \frac{p}{3} \quad \text{for } p \geq 3,$$

in order to permit a non-vanishing mean energy flux at length-scales $\ell \gg \ell_f$. If we assume that $\xi_3 = 1$ and use $\xi_0 = 0$, then convexity implies that

$$\xi_p \leq \frac{p}{3} \quad \text{for } 0 \leq p \leq 3$$

Thus, all of the inequalities for the case of forcing-scale intermittency are exactly opposite to those for the case of damping-scale intermittency (and 3D cascade).

In the third case, that both limits $\ell_f \rightarrow 0$ and $\ell_{ir} \rightarrow \infty$ may be taken for velocity-increments, the scaling exponents s_p must be both concave and convex, i.e. linear. In that case, constant energy flux implies

$$s_p = \frac{P}{3},$$

which is indeed a case of no intermittency. The above result can in fact be deduced by K41-style dimensional analysis, if the limits $\ell_f \rightarrow 0$ and $\ell_{ir} \rightarrow \infty$ are both "safe", so that they can be ignored in the inertial range.

Let us now consider the arguments and evidence in support of each of these scenarios. There seem to be few proponents of the forcing-scale intermittency picture (with a caveat below). Kraichnan did briefly flit with this idea in

R.H. Kraichnan, "Statistical dynamics of two-dimensional flow," J. Fluid Mech. 67 155-175 (1975), Section 7

motivated by early numerical studies of Joyce & Montgomery (1973) and Edwards & Taylor (1974). However, no later simulations that I know support an energy spectrum $E(k) \sim k^{-n}$ with $n < \frac{5}{3}$ in the inverse cascade range.

On the other hand, there have been persistent claims for many years of inverse cascade spectra that are steeper than Kraichnan's prediction, i.e. $n > \frac{5}{3}$, including Borne (1993), Danila & Gurarie (2001), Scott (2007), Vallgren (2011) and, most recently,

J. Fontane, D.G. Dritschel & R.K. Scott, "Vortical control of forced two-dimensional turbulence," *Phys. Fluids* 25 015101 (2013),

whose spectra we have already exhibited. All of those simulations (which include both transient simulations without large-scale damping, but mostly steady-state simulations with large-scale damping) report the appearance of coherent vortices at the forcing scale l_f which then merge and grow to the largest scales of the simulation (l_{ir} , with damping). On the next page we reproduce the Fig. 1 of Fontane et. al. (2013), which shows the time-development of the vorticity field in their simulation of the transient growth problem with no large-scale damping.

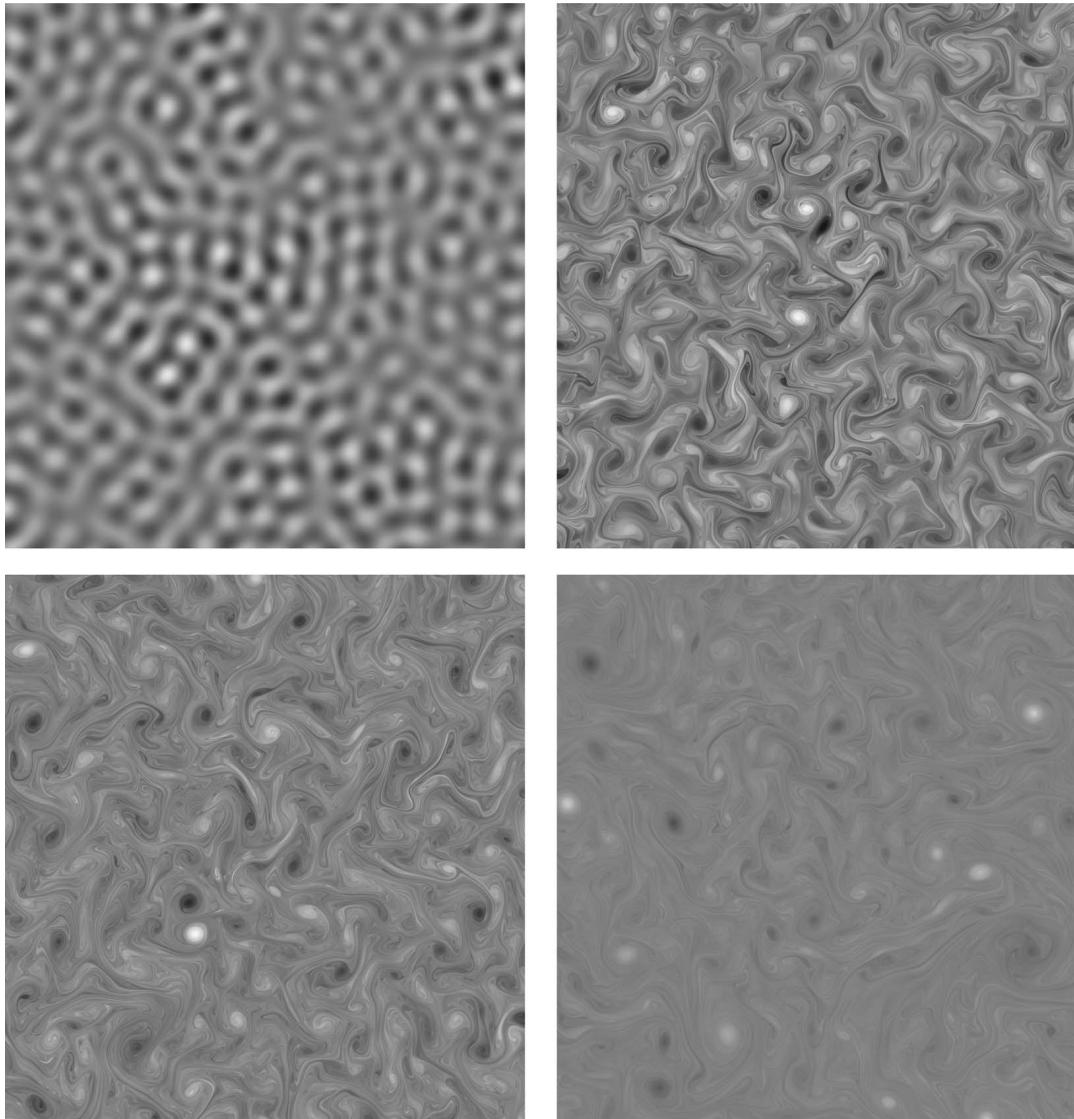


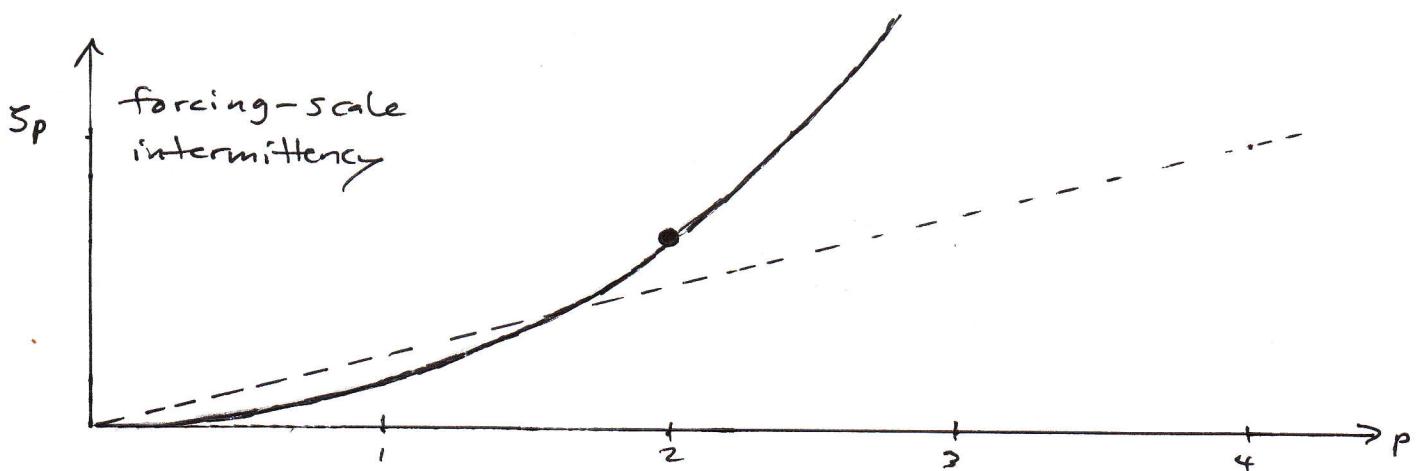
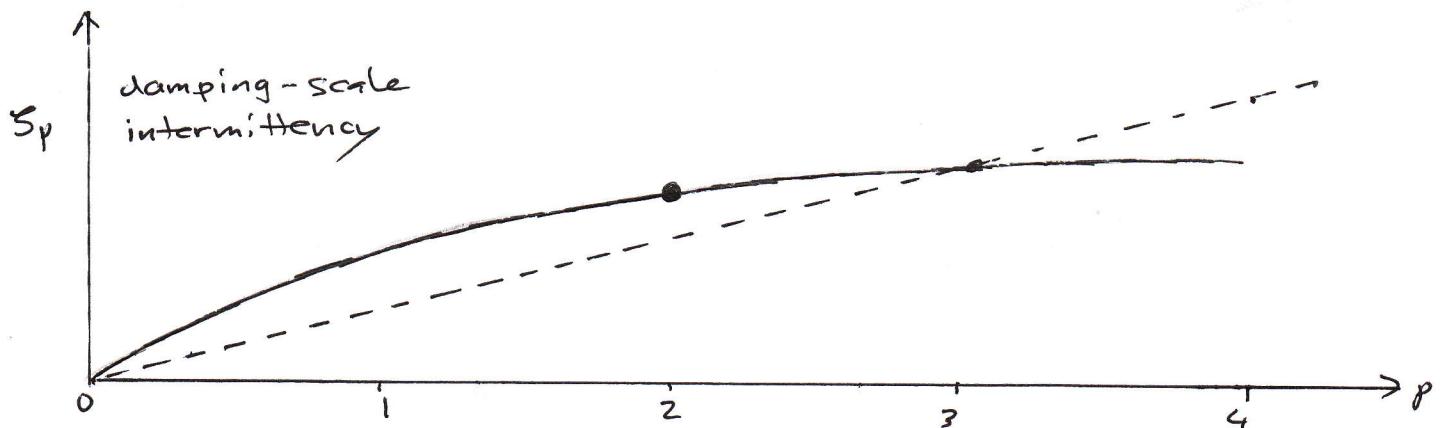
FIG. 1. The vorticity field ω at times $t = 1, 5, 10$, and 50 from left to right and top to bottom. A linear grey scale is used with white being the highest level of vorticity value and black being the lowest. The fields are taken from one simulation in set B and only a sixteenth of the domain is represented.

The most straightforward interpretation of these results is as a possible instance of damping-scale intermittency, which allows $n > \frac{5}{3}$. However, there is a chance that these claims could be consistent with forcing-scale intermittency, if

$$\langle \delta u_L^3(r) \rangle \ll \langle |\delta u_L(r)|^3 \rangle.$$

Since $\frac{\delta u_L^3(r)}{r}$ is a measure of the energy flux using the $\frac{3}{2}$ -law, but we have defined scaling exponents with absolute values, this would imply $\zeta_3 > 1$.

The two possibilities are represented schematically below:



However, while mathematically possible, most numerical simulations (as far as the data is available) seem consistent with

$$\langle \delta u_L^3(r) \rangle \simeq \langle |\delta u_L(r)|^3 \rangle$$

and $S_3 \doteq 1$. Thus, the damping-scale intermittency scenario seems more compatible with such claims.

As far as I know, however, none of the proponents of $n > \frac{5}{3}$ have ever attempted a scaling analysis to see whether anomalous power-laws scale with $\ell_{ir} \propto \ell_f$.

The claim that $n = \frac{5}{3}$ (within numerical & statistical errors) has also received support from a very large number of numerical simulations, including those of Smith & Yekhot (1993), Boffetta et al. (2000), Xiao et al. (2009), Boffetta & Musacchio (2010) and many others. We have already seen some of the results of the latter two simulations.

Let us here show instead some of the results obtained in the study of

G. Boffetta, A. Celani & M. Vergassola, "Inverse energy cascade in two-dimensional turbulence: Deviations from Gaussian behavior," Phys. Rev. E 61 R29-R32 (2000)

The study forced steady-states with linear damping, at resolution 2048^2 .

which systematically studied the issue of intermittency corrections (or their absence) in the inverse cascade,

In their Figs. 1 & 2 (reproduced on the next page) are shown their results for the energy spectrum and the 3rd-order longitudinal structure function.

Their energy spectrum is consistent with the $\frac{5}{3}$ -prediction of Kraichnan and they obtain good agreement with the (asymptotically) exact " $\frac{3}{2}$ -law". In accord with other simulations that observe the $\frac{5}{3}$ spectrum, they see essentially no vortices with size much larger than l_f in their runs. They also see that scaling of velocity increments is self-similar and, in fact, nearly Gaussian, except for the "slight" non-Gaussianity required by the " $\frac{3}{2}$ -law." In their Fig. 3 (page after next) is shown the 5th and 7th-order structure functions, which scale indistinguishably from $r^{5/3}$ and $r^{7/3}$, respectively.

In their Fig. 6 (page after next, bottom) they plot the symmetric part of the PDF of $\delta u_L(r)$ [to remove the asymmetry associated to cascade] and the PDF of $\bar{\delta u}_T(r)$ [not associated to flux]. Both are extremely close to Gaussian!

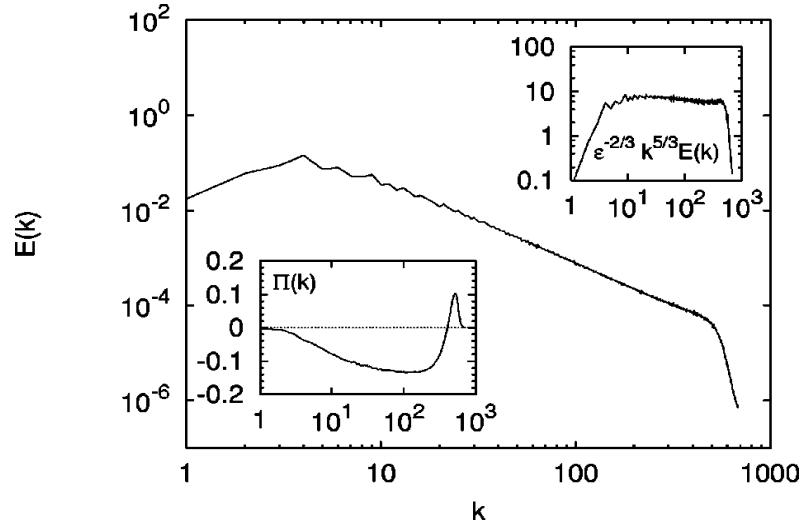


FIG. 2. Energy spectrum $E(k)$. In the lower inset the energy flux $\Pi(k)$ is shown. In the upper inset is the compensated spectrum $\epsilon^{-2/3}k^{5/3}E(k)$.

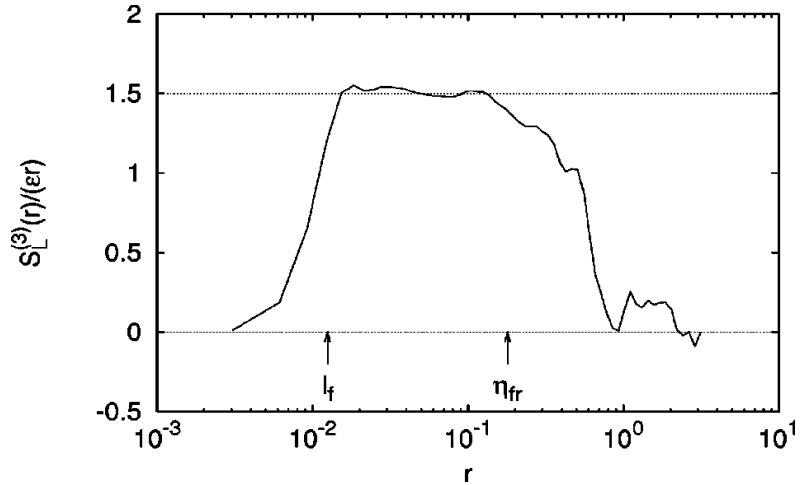


FIG. 1. Compensated third order longitudinal structure function $S_L^{(3)}(r)/(\epsilon r)$. The dotted line is the value $3/2$. Note the linear vertical scale. The labels l_f and η_{fr} indicate the forcing and the friction length scales, respectively.

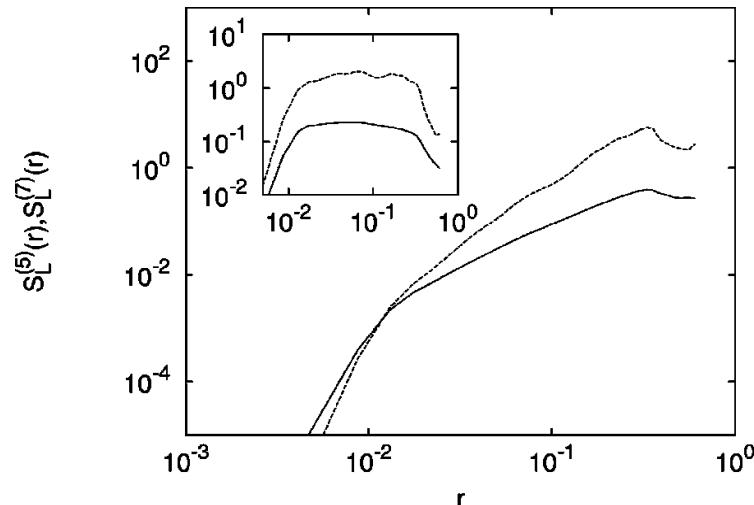


FIG. 3. Structure functions of order 5 (lower line) and 7 (upper line). The compensated curves $S_L^{(5)}(r)/(C_L^{(2)} \epsilon^{2/3} r^{2/3})^{5/2}$ (lower line) and $S_L^{(7)}(r)/(C_L^{(2)} \epsilon^{2/3} r^{2/3})^{7/2}$ (upper line) are shown in the inset.

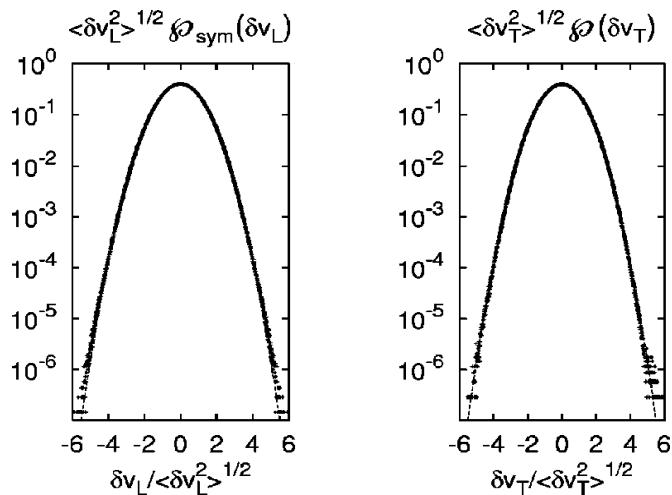


FIG. 6. Left: symmetric part of the longitudinal velocity difference PDF. Right: PDF of transverse velocity differences. The forcing is restricted to a band of wave numbers. Gaussian distributions are shown as solid lines.

Can anything more be said theoretically? It is an interesting fact that there are several toy models which are exactly soluble and that show inverse cascade, with all of them (to my knowledge) showing self-similarity (no intermittency). For example, it was pointed out by

M. Chertkov et. al., "Inverse versus direct cascades in turbulent advection," Phys. Rev. Lett. 80 512-515 (1998)

that passive scalars can show an inverse cascade, if the advecting velocity field is "sufficiently compressible" (mainly potential). This observation was studied at great length in the paper

K. Ganguly & M. Vergassola, "Phase transition in the passive scalar advection," Physica D 138 63-90 (2000)

who studied this problem in the Kraichnan model of random advection

$$\partial_t \theta + (\mathbf{u} \cdot \nabla) \theta = K \Delta \theta - \alpha \theta + f$$

with \mathbf{u} a Gaussian random field that is Hölder continuous in space with exponent $h \in (0, 1)$ and white-noise in time.

In the strongly compressible phase of this model there is an inverse cascade if the force injects scalar energy at the constant rate χ . In the absence of a damping term α , this corresponds to a transient regime with

$$e_{\theta}(t) = \frac{1}{2} \langle \theta^2(t) \rangle = \chi t \quad (\kappa \rightarrow 0)$$

and $e_{\theta}(t) \rightarrow \infty$ as $t \rightarrow +\infty$. However, the statistics of $\delta\theta(r)$ go to a stationary limit as $t \rightarrow +\infty$, which is perfect self-similar. E.g. the PDF of the scalar increment magnitude satisfies at $t \rightarrow \infty$

$$P(\delta\theta, r) = \frac{1}{r^{1-h}} F\left(\frac{\delta\theta}{r^{1-h}}\right)$$

and

$$F(x) = \frac{\sqrt{z'} x^b \Gamma(2b)}{2^{2b-1} [\Gamma(b)]^2} \frac{1}{[x + z' x^2]^{b+\frac{1}{2}}}$$

with z' and b some calculable positive constants that depend upon the space-dimensionality and the degree of compressibility of the velocity. It is interesting that the statistics of $\delta\theta(r)$, although self-similar, are very non-Gaussian. In fact, moments of $\delta\theta(r)$

[i.e. structure functions of θ] of order $p \geq 2b$ do not exist in this problem. If a positive friction is added to add to prevent the blow-up of the scalar energy for $t \rightarrow \infty$ ($\alpha > 0$), then all of the structure functions of θ exist and have the behavior

$$S_p^\theta(r) \sim \begin{cases} c_1 r^{(1-h)p} & p < 2b \\ c_2 r^{(1-h)p} \ln\left(\frac{lir}{r}\right) & p = 2b \\ c_3 r^{(1-h)p} \left(\frac{lir}{r}\right)^{(1-h)(p-2b)} & p > 2b \end{cases}$$

where $lir = \left(\frac{2Z'}{\alpha}\right)^{\frac{1}{2(1-h)}}$ is the damping scale associated to the friction. Thus, the scaling of the moments which were already finite for $\alpha = 0$ remain self-similar, but the moments which diverged for $\alpha = 0$ now exhibit damping-range intermittency.

If this behavior carried over to 2D inverse energy cascade, then all moments would be self-similar, since the statistic appear to be close to Gaussian for $\alpha = 0$ (transient regime) and all structure functions remain finite (apparently) as $t \rightarrow \infty$.

Because all known exactly soluble models which exhibit inverse cascade show also such properties of self-similarity, it has often been suggested that every inverse cascade must be self-similar. An intuitive argument in favor of this is that the number of modes at a given wavenumber k scales as $N_k \sim k^{d-1}$ in dimension d . Thus, a forward cascade spreads a conserved quantity over increasingly many modes as k increases. In this case, intermittency builds up as fluctuations are amplified in the branching process (the standard multiplicative cascade model). However, an inverse cascade compounds a conserved quantity from many modes into fewer modes as k decreases. In this case, fluctuations are smoothed out in the process of merging and intermittency can be expected to decrease. If the higher wavenumber modes being summed together are sufficiently weakly correlated, one can even imagine that the statistics will become close to Gaussian, by a central limit theorem argument. These arguments are far from convincing, of course, and the issue of intermittency in 2D inverse cascade (and inverse cascades generally) is quite open.

Now let us consider the possibility of intermittency & anomalous scaling in the 2D enstrophy cascade. The definition of structure functions for this case is a bit delicate, since in the KB theory the vorticity field ω is too rough ($h_\omega = 0$) to allow the use of $\langle |\delta\omega(r)|^p \rangle$ and the velocity field u is too smooth ($h_u = 1$) to allow the use of $\langle |\delta u(r)|^p \rangle$. One may instead define a "Paley-Littlewood structure-function" for the vorticity, as

$$S_p^\omega(l_N) \equiv \| \omega_N \|^p_p, \quad \text{w/} \quad l_N = 2^{-N} l_f$$

and a "second-difference structure-function" for the velocity, as

$$S_p^u(r) \equiv \| \delta^2 u(r) \|^p_p.$$

The first choice for the vorticity works for any possible scaling exponents and the second choice for velocity works for any exponents between 0 and 2. Thus, these definitions of "structure functions" allow us to consider theories that include the KB predictions as one possibility.

It is generally expected that the limit $v \rightarrow 0$ is innocuous in the 2D enstrophy cascade range, and the only possible source of anomalous scaling is forcing-scale intermittency, so that

$$S_p^\omega(l_N) \sim \eta^{2/3} \left(\frac{l_N}{l_f} \right)^{\xi_p^\omega}, \quad l_N \ll l_f$$

or

$$S_p^u(r) \sim \eta^{2/3} r^p \left(\frac{r}{l_f} \right)^{\xi_p^u}, \quad r \ll l_f$$

so that $\xi_p^u = p + \delta \xi_p^\omega = \xi_p^\omega + p$. Notice that these relations are only required to hold up to additional logarithmic factors. E.g. Kraichnan's log-corrected prediction could give

$$S_2^\omega(l_N) \sim \eta^{2/3} \ln^{2/3}(l_f/l_N)$$

but this would still correspond to $\xi_2^\omega = 0$ (as a minimal exponent for $l_N \rightarrow 0$). Note with the assumption that any intermittency in the 2D enstrophy cascade is from the forcing-scale, the resulting scaling exponents ξ_p^ω and ξ_p^u have all of the properties as did those for damping-scale intermittency in the 2D inverse cascade (Propositions 1 & 2, Corollaries 1 & 2). In particular, ξ_p^ω and ξ_p^u are both concave in p and

$$\sigma_p^\omega = \frac{\xi_p^\omega}{p}, \quad \sigma_p^u = \frac{\xi_p^u}{p} = \sigma_p^\omega + 1$$

are both non-increasing in p .

The DiPerna-Lions theory tells us that

$$\zeta_p^\omega \leq 0 \quad \text{for } p \geq 2$$

and, as a consequence,

$$h_{\min}^\omega = \sigma_\infty^\omega = \lim_{p \rightarrow \infty} \sigma_p^\omega \leq 0$$

We recall that h_{\min} so-defined is the "minimum Hölder singularity" of the vorticity field, as in the Parisi-Fractal multifractal model. Assuming the steady-state exists for $V \rightarrow 0$, only two possibilities exist:

intermittent scenario: $h_{\min} < 0$

non-intermittent scenario: $h_{\min} = 0$

Let us discuss these two in turn.

In the first scenario, $\zeta_p = 0$ at $p=0$ (by definition) and then possibly $\zeta_p = 0$ for some interval $[0, p_*]$ but $\zeta_p < 0$ for $p > p_*$. Presumably $p_* \geq 2$, since enstrophy spectra shallower than the K-B $\frac{1}{k}$ spectrum have never been reported in any simulation or experiment, to my knowledge. On the other hand, there have been reports of negative Hölder exponents in simulations of decaying 2D hyperviscous Navier-Stokes equation (square-Laplacian dissipation) with initial k^{-3} energy spectrum, associated with "cusps" in the centers of coherent vortex structures;

R. Benzi and M. Vergassola, "Optimal wavelet analysis and its application to two-dimensional turbulence," *Fluid Dyn. Res.* 8 117-126 (1991)

M. Farge and M. Holschneider, "Interpretation of two-dimensional turbulence energy spectrum in terms of quasi-singularity in some vortex cores," *Europhys. Lett.* 15 739-743 (1991)

Because the initial data u_0 are sampled from a Gaussian ensemble with energy spectrum k^{-3} they are monofractal fields with everywhere $h^u=1$ or $h^\omega=0$. More mathematically, $w_0 \in B_\infty^{0,\infty}$ not merely $w_0 \in B_2^{0,\infty}$ (but spectrally truncated to the finite set of wavenumbers of the simulation). The above papers claim that the solutions of the 2D hyperviscous NS equations develop solutions with negative Hölder (near) singularities. On the following page see Fig. 6 of the paper of Benzi & Vergassola (1991), which shows the distribution of Hölder exponents obtained by a wavelet analysis. There seem to be exponents for the vorticity as small as $h^\omega=-1$! The interpretation of these results is not entirely clear. It should be noted that the hyperviscous NS equations in 2D obey no maximum principle for the vorticity, which could exceed greatly its initial maximum.

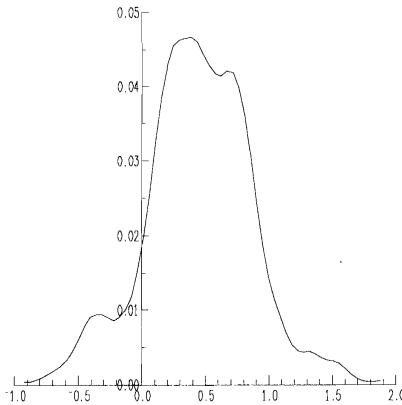


Fig. 6. Probability density distribution $P(h)$ of the local scaling exponents obtained by using the optimal wavelet transform on the vorticity signal of fig. 3. The vertical axis represents $P(h)$ in arbitrary units and the horizontal axis represents h .

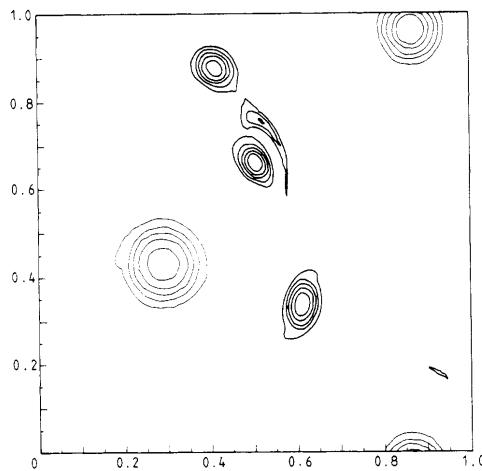


Figure 7. Vorticity contours for the spectral 128×128 run with superviscosity dissipation scheme with $\nu = 5 \times 10^{-7}$ at $t = 30$. The contours are the same as in figure 2. The comparison with figure 4 shows less detail on small scales.

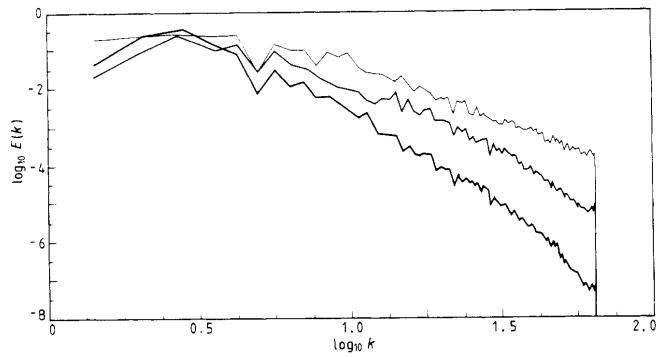


Figure 8. Energy spectra for the superviscosity run at $t = 0, 20, 40$.

The preceding numerical results are especially puzzling when compared with the classical 1933 results of Wolibner and Hölder. If one instead sampled the initial conditions from a Gaussian ensemble with energy spectrum $k^{-(3+2s)}$ for any tiny $s > 0$, then $\omega_0 \in C^s = B_\infty^{s,\infty}$. In that case, the results of Wolibner-Hölder imply that $\omega(\cdot, t) \in C^{s(t)}$ with $s(t) \geq \exp(-C\|w\|_\infty t) s$ for 2D Euler, and one can never obtain any negative exponents from the Euler evolution! Furthermore, it is known that these Euler solutions with Hölder-continuous vorticity fields are obtained as the inviscid limit of the corresponding solutions of 2D NS. These results suggest (but do not prove, of course) that instead the vorticity in the 2D enstrophy cascade has $h_{\min} = 0$ as $\nu \rightarrow 0$.

The scenario with $h_{\min} = 0$ is indeed non-intermittent, since, in that case

$$\zeta_p^\omega = 0 \text{ for all } p \geq 0.$$

This is obviously true for all $p \geq 2$, since

$$0 = \sigma_\infty^\omega \leq \sigma_p^\omega \leq 0 \text{ for } p \geq 2,$$

But also $\zeta_0^\omega = 0$ by definition. This implies $\zeta_p^\omega = 0$ for all $p \geq 0$, since a concave function such as ζ_p^ω can agree with a linear function $ap + b$ at three points $p_1 < p_2 < p_3$ only if $\zeta_p^\omega = ap + b$ on the entire interval $[p_1, p_3]$.

An illuminating heuristic derivation of the previous result can be obtained from the Parisi-Frisch formula

$$S_p^\omega = \inf_{h \in [h_{\min}^\omega, h_{\max}^\omega]} \left\{ ph + (2 - D^\omega(h)) \right\}.$$

If $h_{\min}^\omega = 0$, then for $p \geq 0$ both of the terms ph and $2 - D^\omega(h)$ are non-negative for all $h \in [h_{\min}^\omega, h_{\max}^\omega]$. Since we know from the DiPerna-Lions result that

$$S_p^\omega \leq 0 \text{ for } p \geq 2,$$

it follows that for any such p

$$ph_p = 0 \text{ and } 2 - D^\omega(h_p) = 0$$

where h_p is the Hölder exponent where the infimum is achieved. But this can only be true if $h_p = h_{\min}^\omega = 0$ for all $p \geq 2$ and if

$$D^\omega(0) = 2,$$

i.e. if the set $S(0)$ of points where $h^\omega(x) = 0$ is space-filling.

In that case also for any $0 \leq p < 2$, the infimum in the Parisi-Frisch formula is achieved with $h_p = h_{\min}^\omega = 0$, since both negative terms are made as small as possible (0!) by that choice.

This multifractal interpretation is plausible with reference to the numerical results of Chen et al. (2003). Recall their Fig. 3, which plotted the fields $\phi_\ell(x) = \frac{1}{2}\bar{\omega}_\ell^2(x) - \bar{s}_\ell^2(x)$ and the enstrophy flux $Z_\ell(x)$ [reproduced next page].

The set $S(0)$ with dimension 2 is presumably the strain-dominated region ($\phi_\ell(x) < 0$), with the positive enstrophy flux in the finely filamented lines. Of course, this region does not presumably correspond to actual discontinuities (like Burgers) in the limit $\nu \rightarrow 0$, but instead a space-filling set of logarithmic singularities. Note that a general log-corrected KB spectrum

$$E(k) \sim C \eta^{2/3} k^{-3} \ln^{-p}(k/k_f), \quad 0 < p < 1$$

corresponds to a vorticity correlation

$$\langle \omega(r) \omega(0) \rangle = \frac{C \eta^{2/3}}{1-p} \ln^{1-p}(l_f/r) \left[1 + O\left(\frac{1}{\ln(l_f/r)}\right) \right].$$

It has been argued in

G. Falkovich & V. Lebedev, "Nonlocal vorticity cascade in two dimensions," Phys. Rev. E 49 R1800-1803 (1994)

that general p th-order correlations of vorticity $\langle \omega(0) \omega(r_1) \dots \omega(r_{p-1}) \rangle$ are to leading order Gaussian with the above covariance, but with corrections only logarithmically smaller.

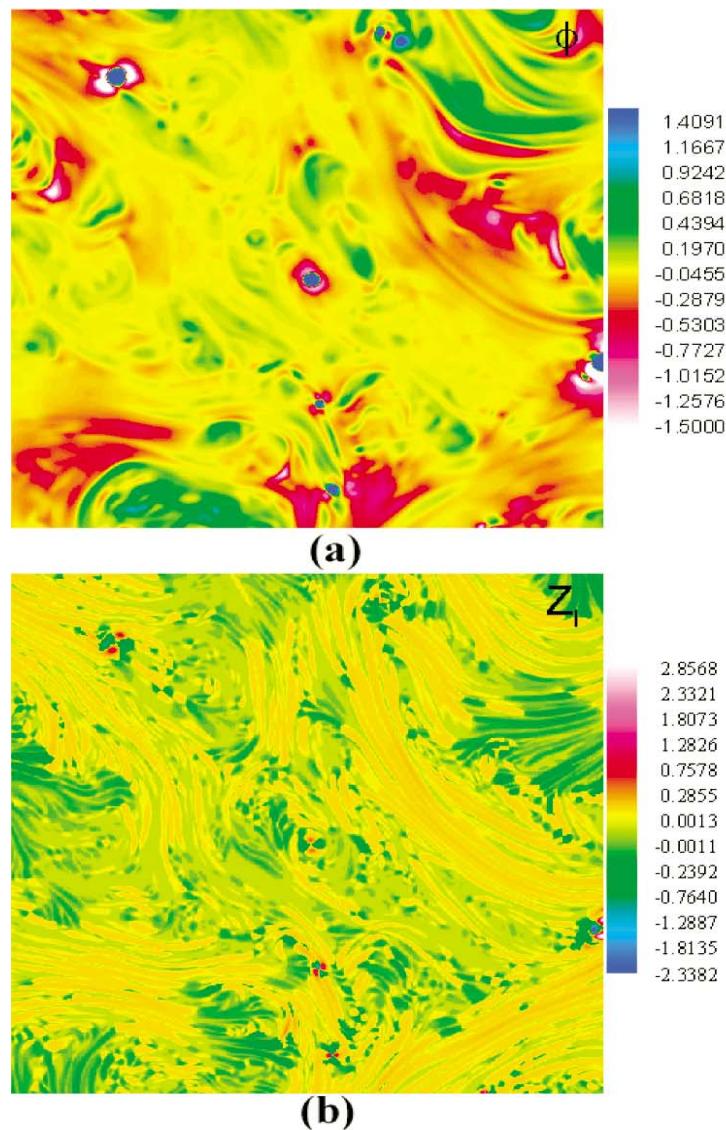


FIG. 4 (color). Instantaneous snapshot of (a) $\phi_\ell(\mathbf{r}, t)$ and (b) $Z_\ell(\mathbf{r}, t)$, for $\ell = \pi/130$ in a 512^2 subdomain. The red regions in (a) are dominated by strain and the green by vorticity.

Does $h_{\min}^\omega = 0$ imply that there is no intermittency at all in the 2D enstrophy cascade? This idea seems clearly rather strange with reference to Fig. 3 of Chen et al. (2003), which shows a great deal of space-variation! It is plausible from this figure that there should be Hölder exponents of the vorticity $h > 0$, corresponding to regions more smooth than $h_{\min}^\omega = 0$ (e.g. the vortices). As a matter of fact, the 2D enstrophy cascade corresponds to a peculiar situation where h_{\min}^ω is the most probable (space-filling) value $h_*^\omega = \left. \frac{dS_p}{dp} \right|_{p=0}^{\omega}$. For an extensive discussion of this concept and others in the Parisi-Frisch multifractal model, see Turbulence I, Causenotes, Section(E). As we discussed there, to detect exponents $h > h_*^\omega$ one must consider negative-index structure functions for $p < 0$. Better for data analysis are the inverse structure functions with $p > 0$, which are defined by "exit-times". For example, if we employ velocity 2nd-differences (since $h^\omega > 0 \Rightarrow h^4 > 1$), one defines

$$R(\delta u) = \inf \left\{ r \in \mathbb{R}^+ : |\delta u(r)| = \delta u \text{ for some } |r|=r \right\}$$

and then the (velocity 2nd-difference) inverse structure function

$$I_p(\delta u) = \langle |R(\delta u)|^p \rangle$$

with exponents

$$x_p^u = \liminf_{\delta u \rightarrow 0} \frac{\ln I_p(\delta u)}{\ln (\delta u / \eta^{1/3} l_f)}$$

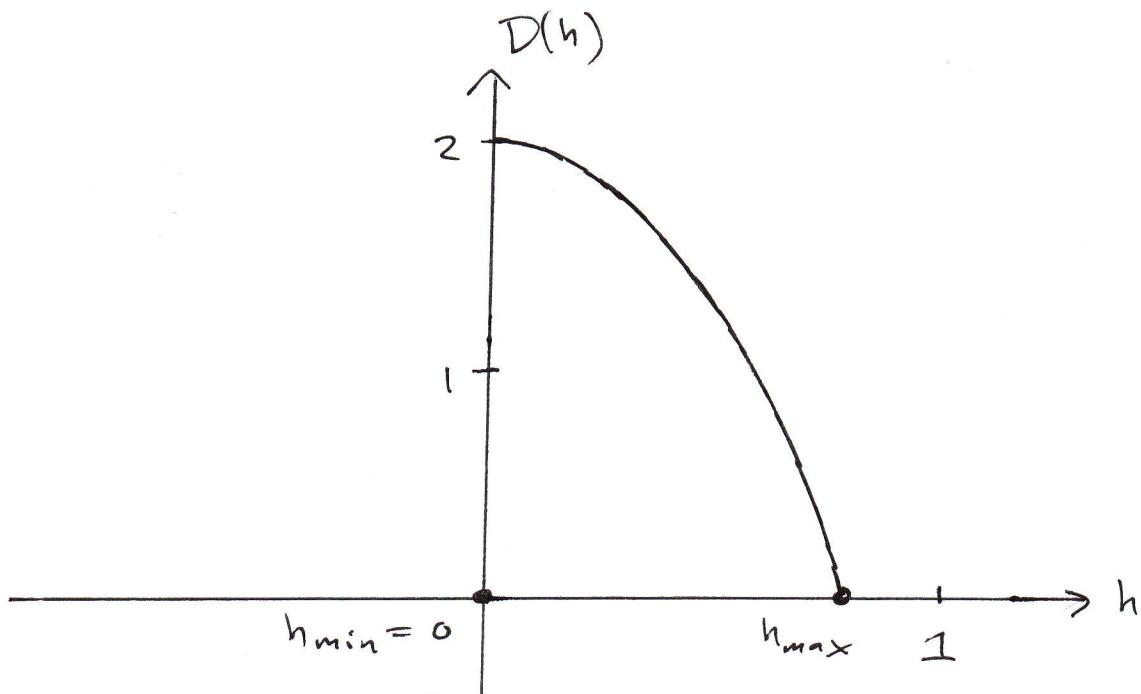
or, more heuristically,

$$I_p(\delta u) \sim (\text{const.}) \left(\frac{\delta u}{\eta^{1/3} l_f} \right)^{x_p^u}.$$

These exponents are related to the multifractal dimension spectrum by

$$x_p^u = \inf_{h \in [h_{\min}^u, h_{\max}^u]} \left\{ \frac{p + (2 - D(h))}{h} \right\}$$

and it is not hard to see the values of x_p^u for $p > 0$ are determined by $h > h_*^u$. These exponents may show non-trivial (nonlinear) scaling if $D^u(h)$ has the form



Of course, $D^{\omega}(h) = D^{\omega}(h-1)$. This simple observation seems to have been first made in

G. Eyink, "Exact results on scaling exponents in the 2D enstrophy cascade," Phys. Rev. Lett. 74 3800-3803 (1995)

and

G. Eyink, "Exact results on stationary turbulence in 2D: consequences of vorticity conservation," Physica D 91 91-142 (1996)

To test this idea, one must use inverse structure functions for velocity 2nd-differences (or inverse structure functions for Paley-Littlewood structure functions of vorticity).

As far as I know, this study has never been carried out. Amazingly, one paper

L. Biferale et al. "Inverse velocity statistics in two-dimensional turbulence," Phys. Fluids 15 1012-1020 (2003)

has studied inverse structure functions of 1st-order velocity differences and ordinary (direct) structure functions of 2nd-order velocity differences, but not inverse structure functions of 2nd-order velocity differences! So, the question is still open.

Effect of linear damping on the 2D enstrophy cascade. The enstrophy cascade is a very fragile cascade. It is nonlocal in scale, driven by shear rates $S_f \sim \eta^{1/3}$ at the forcing scale l_f . Thus, the cascade is non-accelerated, with each step from length-scale $l_n = \bar{2}^n l_f$ to $l_{n+1} = \bar{2}^{(n+1)} l_f$ taking the same amount of time $\eta^{-1/3}$. As we have seen, a non-vanishing flux to infinitesimally small lengths can only occur if the enstrophy itself is infinite. The enstrophy cascade barely manages to happen at all!

Such a fragile cascade is very easy to disrupt. Consider the 2D NS with a linear Ekman friction:

$$\partial_t w + (\mathbf{u} \cdot \nabla) w = \nu \Delta w - \alpha w + q.$$

As we have seen, this type of linear damping is a reasonable description of drag produced by Ekman boundary layers in geophysical (rotating) fluids. Similar drag also occurs in laboratory experiments on 2D turbulence, e.g. due to interaction with the air in flowing soap films. The effect of such drag on the 2D enstrophy cascade is profound.

Note that in one cascade step a fraction $\frac{\alpha}{\eta^{1/3}}$ of the enstrophy carried across a given scale will be dissipated by such friction. In that case, however, the enstrophy spectrum must fall steeper than the KB $\frac{1}{k}$ and, in consequence, ^{cascade} will shut off! One cannot expect enstrophy cascades to infinitesimally small length-scales in the presence of linear friction.

The above argument can be made more precise, taking into account that the shear rates in a smooth, chaotic flow are actually not a fixed value $S_f \sim \gamma^{1/3}$ but fluctuate from point-to-point in the flow. This is the notion of a finite-time Lyapunov exponent in dynamical systems theory. Using these ideas one may argue that indeed (1) the enstrophy spectrum will fall off as a power-law $k^{-(1+\xi)}$ with $\xi > 0$ an exponent which depends upon the damping rate α , (2) the enstrophy cascade will shut off and enstrophy flux tend to zero at small scales, and (3) the vorticity will become intermittent with a nontrivial multifractal spectrum.

These ideas were first discovered in the context of passive scalar advection by a smooth velocity field, in works of

M. Chertkov, "On how a joint interaction of two innocent partners (smooth advection and linear damping) produces a strong intermittency," Phys. Fluids 10 3017–3019 (1998)

K. Nam et al., "k Spectrum of finite lifetime passive scalars in Lagrangian chaotic fluid flows," Phys. Rev. Lett. 83 3426–3429 (1999)

Z. Neufeld et al., "Multifractal structure of chaotically advected chemical fields," Phys. Rev. E 61 3857-3866 (2000)

These ideas were then extended, in a self-consistent manner, to the 2D enstrophy cascade in the papers

D. Bernard, "Influence of friction on the direct cascade of the 2D forced turbulence," Europhys. Lett. 50 333-339 (2000)

K. Nam et al., "Lagrangian chaos and the effect of drag on the enstrophy cascade in two-dimensional turbulence," Phys. Rev. Lett. 84 5134-5137 (2000)

The basic ideas of these works are easily explained in an intuitive way, as we now discuss.

The Lyapunov exponent λ_* in dynamical systems measures the rate of divergence of infinitesimally nearby trajectories:

$$\lambda_* = \lim_{t \rightarrow \infty} \lim_{r \rightarrow 0} \frac{1}{t} \ln |X(x+r, t) - X(x, t)|.$$

It is a consequence of the multiplicative ergodic theorem that the above limit is the same for almost every initial point x , except for a set of zero-measure with respect to the stationary, invariant measure μ of the flow $X(\cdot, t)$. For an incompressible fluid, the flow preserves the

Lebesgue measure μ_L , so that $\mu = \mu_L$ and almost every point with respect to Lebesgue measure gives the same limit as above. The above statement can be refined by considering the finite-time Lyapunov exponents

$$\lambda_t(x) = \lim_{r \rightarrow 0} \frac{1}{t} \ln (x(x+r,t) - x(x,t))$$

which do depend on the specific point x . For sufficiently chaotic dynamics it is known that the finite-time Lyapunov exponents satisfy a large-deviations principle, so that the probability density of $\lambda_t(x)$ (over space) has the form

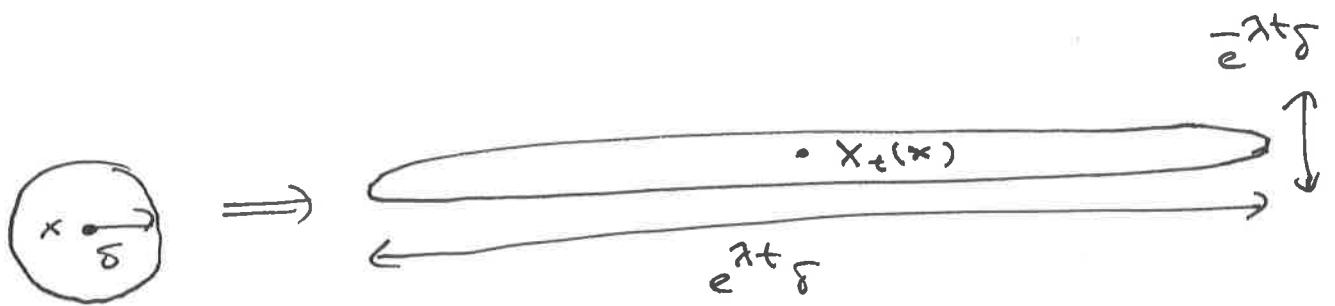
$$P(\lambda, t) \propto \exp[-tG(\lambda)]$$

with a rate function $G(\lambda)$ which is ≥ 0 , convex in λ , and

$$G(\lambda) = 0 \text{ iff } \lambda = \lambda_*$$

thus, the probability for $\lambda_t(x)$ to take on any other value than λ_* becomes exponentially small as $t \rightarrow \infty$. In particular, $\lim_{t \rightarrow \infty} \lambda_t(x) = \lambda_*$ almost surely for all x except a set of Lebesgue measure zero.

The above results can be restated in the language of multifractal theory. In a 2D incompressible flow, volumes are preserved in time. Thus, a little disk around a point where $\lambda_t(x) = \lambda$ is stretched in one direction by a factor $e^{\lambda t}$ in time t , but simultaneously compressed in the other by a factor $e^{-\lambda t}$:



Let us denote the thickness of this stretched region $\delta_\lambda(t) \sim e^{-\lambda t} \delta$. As $t \rightarrow \infty$, the area of the points where $\lambda_t(x) = \lambda$ goes to zero as $\delta A_\lambda(t) \sim e^{-t G(\lambda)} \delta^2$. If the space is covered with little disks at time t of radius $\delta_\lambda(t)$, then the number of disks $N_\lambda(t)$ required to cover the set where $\lambda_t(x) = \lambda$ is

$$N_\lambda(t) \sim \frac{\delta A_\lambda(t)}{[\delta_\lambda(t)]^2} \sim e^{[(2\lambda - G(\lambda))t]} \sim [\delta_\lambda(t)]^{\frac{G(\lambda)}{\lambda} - 2}$$

with $\delta_\lambda(t) \rightarrow 0$ as $t \rightarrow \infty$. Hence, we are on a safe

that the fractal dimension $D(\lambda)$ of the set where $\lambda_t(x) = \lambda$ as $t \rightarrow \infty$ is

$$D(\lambda) = 2 - \frac{G(\lambda)}{\lambda}.$$

Notice that $D(\lambda_x) = 0$, so that $\lambda_t(x) \xrightarrow[t \rightarrow \infty]{} \lambda_x$ corresponds to a set of points x which is space-filling.

How does this theory apply to the 2D enstrophy cascade in the presence of linear friction? If one ignores the viscous dissipation (which can be shown to play no essential role), then it is easy to solve the vorticity equation formally with the Lagrangian flow maps, as:

$$\omega(x, t) = \int_{-\infty}^t q(X(x, s), s) e^{-\alpha(t-s)} ds$$

where we have assumed a statistical steady-state for which $t_0 \rightarrow -\infty$. This implies that for neighboring points x and $x' = x + r$

$$\delta\omega(r; x) = \omega(x', t) - \omega(x, t)$$

$$= \int_{-\infty}^t [q(X(x', s), s) - q(X(x, s), s)] e^{-\alpha(t-s)} ds$$

The size of $\delta w(r)$ thus depends upon two competing factors. Because the forcing $q(x,s)$ is assumed to be at scale l_f

$$|q(x',s) - q(x,s)| \approx \begin{cases} \frac{r}{l_f} |\nabla q|, & r \ll l_f \\ q_{\text{rms}}, & r \gg l_f \end{cases}$$

Hence, as the separation $X(x',s) - X(x,s)$ grows (backward in time s), the factor $q(X(x',s),s) - q(X(x,s),s)$ also grows proportionally, until the scale l_f is reached and this quantity saturates. On the other hand, the factor $e^{-\alpha(t-s)}$ decays exponentially backward in time and penalizes trajectories which separate slowly. Pairs of trajectories whose separation reaches l_f most quickly are rewarded with the greatest increments.

The above argument can be formalized by the notion of the exit-time $T_f(r;x)$, which is the time for a pair of particles at $x, x+r$ to separate to distance l_f (here, backward in time). Heuristically, it is clear that

$$T_f(r) \sim \frac{1}{\gamma} \ln\left(\frac{l_f}{r}\right)$$

relates the exit-time to the finite-time Lyapunov exponent. Thus, one can estimate

$$\delta\omega(r) \sim \frac{q_{rms}}{\alpha} e^{-\alpha T_f(r)}$$

$$\sim \left(\frac{q_{rms}}{\alpha}\right) \left(\frac{r}{l_f}\right)^{\alpha/\lambda}, \quad \alpha < \lambda$$

This argument works for $\alpha < \lambda$. For $\alpha > \lambda$ it would also apply to 2nd-differences of the vorticity, $\delta^2\omega(r)$, but first differences will satisfy

$$\delta\omega(r) \sim \left(\frac{q_{rms}}{\alpha}\right) \cdot \left(\frac{r}{l_f}\right), \quad \alpha > \lambda.$$

The two estimates can be stated together as

$$\delta\omega(r) \sim \left(\frac{q_{rms}}{\alpha}\right) \cdot \left(\frac{r}{l_f}\right)^{\min\{1, \frac{\alpha}{\lambda}\}}.$$

If we use our earlier estimate of the fractal dimension $D(\lambda)$, we can evaluate the scaling exponents of the vorticity increment by a Parisi-Frisch multifractal argument. Note that

$$2 - D(\lambda) = \frac{G(\lambda)}{\lambda}.$$

Hence,

$$\langle (\delta w(r))^p \rangle \sim (\text{const.}) \left(\frac{r}{l_f} \right)^{\min_{\lambda} \left\{ \min \left\{ p, \frac{\alpha p}{\lambda} \right\} + \frac{G(\lambda)}{\lambda} \right\}}$$
$$\sim (\text{const.}) \left(\frac{r}{l_f} \right)^{\zeta_p^w}$$

with

$$\begin{aligned} \zeta_p^w &= \min \left\{ \min_{\lambda} \left\{ p + \frac{G(\lambda)}{\lambda} \right\}, \min_{\lambda} \left\{ \frac{\alpha p + G(\lambda)}{\lambda} \right\} \right\} \\ &= \min \left\{ p, \min_{\lambda} \left\{ \frac{\alpha p + G(\lambda)}{\lambda} \right\} \right\} \\ &= \min_{\lambda} \left\{ p, \frac{\alpha p + G(\lambda)}{\lambda} \right\}. \end{aligned}$$

Of course, if we had considered a Paley-Littlewood structure function of the vorticity, rather than a 1st-order difference structure function, then we would have obtained from this argument instead

$$\zeta_p^w = \min_{\lambda} \left\{ \frac{\alpha p + G(\lambda)}{\lambda} \right\}.$$

This result is "better", because it includes a non-trivial contribution from $\lambda < \alpha$.

Since $\min_{\lambda} \left\{ \frac{G(\lambda)}{\lambda} \right\} = 0$, we can see that

$$\xi_p^\omega = \min_{\lambda} \left\{ p, \frac{\alpha p + G(\lambda)}{\lambda} \right\} > 0$$

for $p > 0$ and $\alpha > 0$. However, taking $\alpha \rightarrow 0$

$$\xi_p^\omega = \min_{\lambda} \left\{ p, \frac{G(\lambda)}{\lambda} \right\} = 0$$

and one recovers the KB prediction $\xi_p^\omega = 0$ for all $p \geq 0$.

In particular, for $p=2$

$$\xi = \xi_2^\omega = \min_{\lambda} \left\{ 2, \frac{2\alpha + G(\lambda)}{\lambda} \right\} > 0$$

and one obtains an enstrophy spectrum

$$S(k) \sim k^{-(1+\xi)}$$

The above discussion can be made rigorous for the passive scalar advection problem, if there is sufficient knowledge about the chaotic properties of the smooth flow. For example, it can be demonstrated analytically for the Kraichnan white-noise advection model, with a smooth velocity field for which the velocity-gradient matrices are Gaussian with correlation

$$D_{ij} = \frac{\partial u_i}{\partial x_j}$$

$$\begin{aligned} & \langle D_{ij}(t) D_{kl}(t') \rangle \\ &= D_0 \left[(d+1) \delta_{ik} \delta_{jl} - \delta_{ij} \delta_{kl} - \delta_{il} \delta_{kj} \right] \\ & \quad \times \delta(t-t') \end{aligned}$$

in space dimension d . In that case, it was already shown in

R. H. Kraichnan, "Convection of a passive scalar by a quasi-uniform random straining field," J. Fluid. Mech. 64 737-762 (1974)

that

$$G(\lambda) = \frac{d}{4D_0(d-1)} (\lambda - D_0(d-1))^2.$$

In this case one can get an analytic expression for the exponents S_p^θ of a passive scalar θ advected by the white-noise velocity (Chertkov, 1998).

For 2D enstrophy cascade, the above argument is only self-consistent and by analogy with the passive scalar case. In particular, there is no analytical theory for $G(\lambda)$ in the 2D enstrophy cascade.

On the other hand, some parts of the above argument can be made mathematically rigorous. The papers

P. Constantin and F. Ramos, "Inviscid limit for damped and driven incompressible Navier-Stokes equation in \mathbb{R}^2 ," Commun. Math. Phys. 275 529-551 (2007)

H. Bessaih and B. Ferrario, "Inviscid limit of stochastic damped 2D Navier-Stokes equation," Nonlinearity 27 1-15 (2014)

These papers have used the methods of the DiPerna-Lions theory to study the invariant measures of the stochastically-forced 2D NS equation with linear friction. It has been proved that the inviscid ($\nu \rightarrow 0$) limit of the stationary measures exist, that viscous dissipation of enstrophy vanishes in the limit, and that the realizations of the random ensemble are "renormalized solutions" of the driven & linearly damped Euler equations, in the sense of DiPerna-Lions. In the limit $\nu \rightarrow 0$ there is a balance

$$\eta = \alpha \langle w^2 \rangle$$

between the enstrophy input from the force and the dissipation of enstrophy by the linear damping.

The more detailed predictions of the physical theories have been verified in numerical simulations and laboratory experiments. Numerical results were already presented in the work of Nam et al. (2000) and more detailed investigations were published in

G. Boffetta et al. "Intermittency in two-dimensional Ekman - Navier-Stokes turbulence," Phys. Rev. E 66 026304 (2002)

Laboratory experiments are discussed in

G. Boffetta et al., "Effects of friction on 2D turbulence: an experimental study of the direct cascade," Europhys. Lett. 71 590-596 (2005)

We shall briefly review here some of the results of the paper of Boffetta et al. (2002). That paper performed numerical simulations of the 2D NS equation with linear Ekman friction and also, for comparison, simulations of a passive scalar in the 2D NS flow. Figs. 1 and 5 of Boffetta et al. (2002) [reproduced on the following page] show snapshots of $\omega(x,t)$ at a given instant t and of $\Theta(x,t)$ at the same time. It is already apparent that (i) the two fields are rather similar and (ii) the vorticity field is smoother in appearance than for typical 2D enstrophy cascade fields without linear damping.

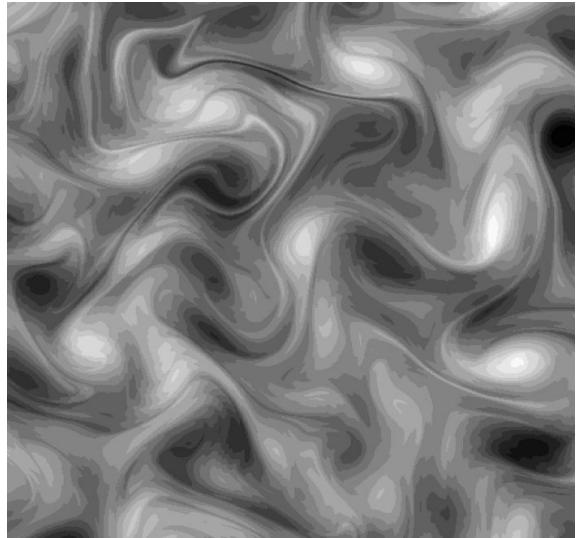


FIG. 1. Snapshot of the vorticity field resulting from the numerical integration of Eq. (1). Details are given in Ref. [2].

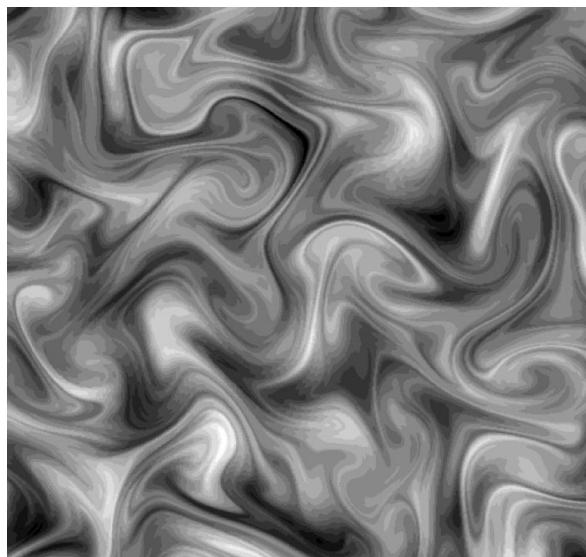


FIG. 5. Snapshot of the passive scalar field, simultaneous to the vorticity field shown in Fig. 1

A comparison of the spectra of ω and θ [not shown], shows that they are very similar at high-wavenumbers, but that the spectrum of ω is enhanced at low wavenumbers due to inverse energy cascade. In Fig. 2 of Boffetta et al. (2002) [next page] is shown the enstrophy spectrum $S_2(k)$ [denoted $Z(k)$] for three different choices of α . The spectra are all steeper than the BK k^{-1} , increasingly so as α increases.

In Fig. 4 of the paper [next page, bottom] are plotted the spectral enstrophy flux $Z(k)$ [denoted $\Pi_{\omega}(k)$] for $\alpha = 0, 15$ and two choices of viscosity ν . The enstrophy flux is now decaying as $Z(k) \sim \eta \left(\frac{k}{k_f}\right)^{-\xi}$ in the range $k_f \ll k \ll k_{uv} = \frac{\eta^{1/6}}{\nu^{1/2}}$. At the upper limit

$$\begin{aligned} Z(k_{uv}) &\cong \nu \langle |\nabla \omega|^2 \rangle \\ &\cong C\eta \left(\frac{k_f}{k_{uv}}\right)^{\xi} = C\eta \left(\frac{\nu k_f^2}{\eta^{1/3}}\right)^{\xi/2}, \end{aligned}$$

both vanishing $\propto \nu^{\xi/2}$ as $\nu \rightarrow 0$. Thus, as expected, the enstrophy cascade to very high-wavenumbers is cut off for $\xi > 0$ (i.e. $\alpha > 0$) and the viscous dissipation of enstrophy is negligible.

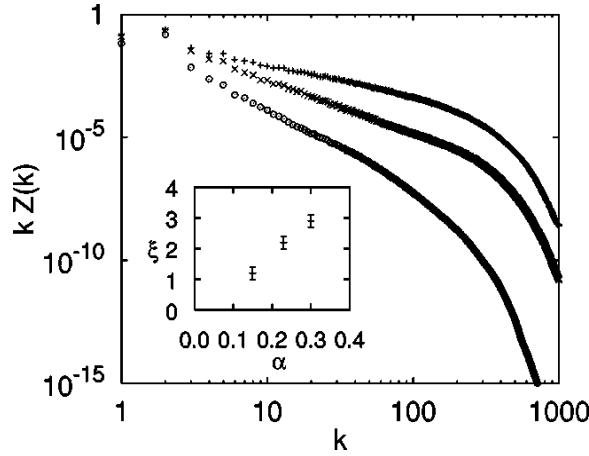


FIG. 2. The vorticity spectrum $Z(k) \sim k^{-1-\xi}$ becomes steeper by increasing the Ekman coefficient α . Here $\alpha = 0.15$ (+), $\alpha = 0.23$ (x), and $\alpha = 0.30$ (o). In the inset, the exponent ξ is shown as a function of α .

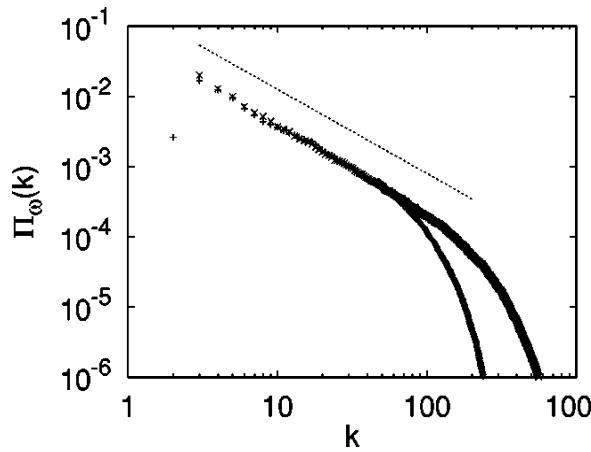


FIG. 4. Enstrophy flux $\Pi_\omega(k) \sim k^{-\xi}$ for $\nu = 5 \times 10^{-5}$ (+) and $\nu = 1.5 \times 10^{-5}$ (x). Here $\alpha = 0.15$. Reducing ν , the remnant enstrophy flux at small scales tends to zero as ν^ξ (see text), allowing to disregard viscous dissipation.

The paper of Boffetta et al. (2002) also made a detailed comparison of the statistics of vorticity-increments $\delta\omega(r)$ and passive scalar increments $\delta\theta(r)$ for their simulations. In their Figure 7 [reproduced on next page], they compare PDF's of these increments for four small values of r inside the power-law "inertial range". As can be seen the PDF's are very similar!

Because of the close agreement of the PDF's, it must be true that $\delta_p^\omega \approx \delta_p^\theta$. Boffetta et al. use this fact to make a further analysis of the theoretical predictions. First, the theory predicts that $\langle \exp[-\alpha_p T_f(r)] \rangle \sim r^{\frac{\delta_p^\theta}{\alpha_p}}$ and in the simulations both sides can be measured independently. Fig. 8 of the paper [next page, bottom] shows satisfactory agreement. Furthermore, the distributions of both the random variables $T_f(r)$ and λ_t can be obtained and the quantity

$$G_t(\lambda) \equiv -\frac{1}{t} \ln P(\lambda, t)$$

be plotted for large t . The good collapse is a test both of the existence of the above limit for $t \rightarrow \infty$ and of the relation $T_f(r) \approx \frac{1}{\lambda} \ln \left(\frac{\lambda_f}{r} \right)$.

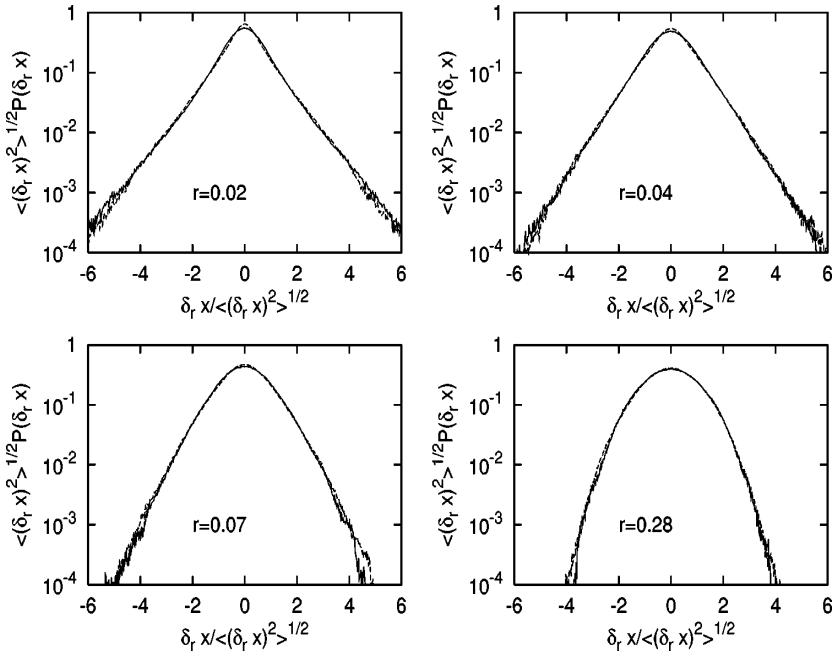


FIG. 7. Probability density functions of vorticity differences (solid line) and of passive scalar ones (dashed line), normalized by their respective standard deviation at different scales r within the scaling range.

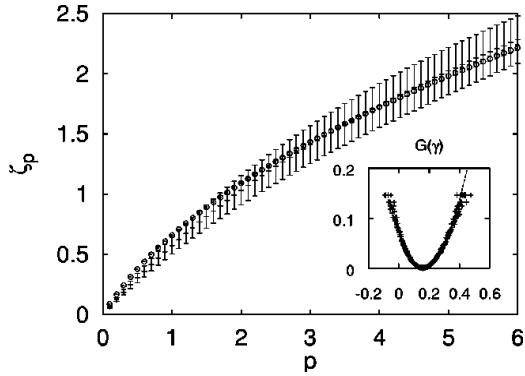


FIG. 8. The scaling exponents of the passive scalar ζ_p^θ (+). We also show the exponents obtained from the exit-times statistics (○) according to $\langle \exp[-\alpha p T_L(r)] \rangle \sim r^{\zeta_p^\theta}$, with an average over about 2×10^5 couples of Lagrangian particles. The error bars are estimated by the rms fluctuation of the local slope. In the inset we plot the Cramer function $G(\gamma)$ computed from finite-time Lyapunov exponents (symbols) and exit-time statistics (line).

CONCLUSION

The previous discussion has hopefully made clear some of the established facts of 2D turbulence and also some of the open questions still being debated in the literature. In addition to the problems already discussed, there are many others. Let us just mention here a few, including the role of higher-order vorticity invariants:

J. C. Bowman, "Casimir cascades in two-dimensional turbulence," J. Fluid. Mech. 729 364–376 (2013);

2D turbulence with forcing in multiple spectral bands, supporting "nonlinear superpositions" of energy & enstrophy cascades:

M. Cencini et al., "Nonlinear superposition of direct and inverse cascades in two-dimensional turbulence forced at large and small scales," Phys. Rev. Lett. 107 174502 (2011);

on-going work to understand the conformal properties of 2D inverse cascade and effects of spatial geometry:

G. Falkovich & K. Gawedzki, "Turbulence on hyperbolic plane: the fate of inverse cascade," arXiv:1402.1797 [nlin.CD];

and the role of spatial boundaries :

H. J. H. Clercx & G.J. F. van Heijst, "Two-dimensional Navier-Stokes turbulence in bounded domains," Applied Mechanics Reviews 62 020802 (2009).

The latter effects are certainly important in geophysical flows, due to the influence of orography (mountains and hills) on the atmosphere and bottom topography and continental boundaries on ocean dynamics.

2D turbulence continues to develop in two important and related directions. On the one hand, basic unresolved issues internal to 2D turbulence continue to be investigated, both for intellectual interest and for the perspective that they give on related turbulence problems (e.g. 3D cascades). A second important direction is comparison and contrast with geophysical fluid turbulence, which shares some features of 2D but also brings in many new aspects, including the existence of linear waves of various types!

We now turn to this subject.