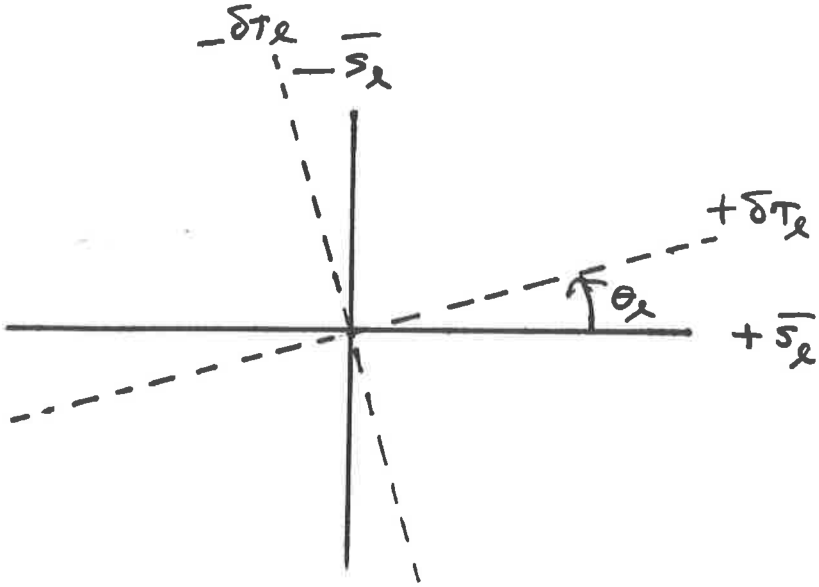
Now let us discuss the mechanisms of the inverse energy cascade and the approximations to stress τ_ℓ based on UV-scale locality. First, note in general that

$$\Pi_\ell = -\overline{S}_\ell : \tau_\ell < 0$$

requires that the large-scales do negative work on the small scales. That is, the response of the small-scale stress τ_ℓ is not to resist the large-scale strain but instead to assist the strain. Note we can write

$$\Pi_\ell = -\overline{S}_\ell : \mathring{\tau}_\ell$$

where $\mathring{\tau}_\ell = \tau_\ell - \frac{1}{d}\mathrm{Tr}(\tau_\ell)\mathbf{I}$ is the deviatoric or traceless part of the stress. In this way, Π_ℓ is written as the matrix dot-product of two symmetric, traceless matrices. The eigenvalues of each matrix are equal but opposite in magnitude, say $\pm\overline{s}_\ell$, $\pm\delta\tau_\ell$, resp. and the eigenvectors of each matrix define an orthogonal system. Let θ_ℓ be the angle between the eigenframes of $\overline{S}_\ell$ and $\mathring{\tau}_\ell$, as shown here:



It is not hard to check that

$$\Pi_\ell = -2\,\bar{s}_\ell\,\delta\tau_\ell \cos(2\theta_\ell)$$

so that Π_ℓ is most negative $\theta_\ell \approx 0$ and the two frames are nearly aligned.

How does this come about in 2D? Let us try to use the approximation from UV locality (nonlinear model):

$$\tau^{NL}_{\ell\, ij} = \frac{1}{2} C \ell^2 \frac{\partial \bar{u}_{\ell i}}{\partial x_k} \frac{\partial \bar{u}_{\ell j}}{\partial x_k}$$

which can be further calculated in 2D using

$$\frac{\partial u_i}{\partial x_j} = S_{ij} - \frac{1}{2}\epsilon_{ij}\,\omega$$

to be

$$\tau_\ell^{NL} = \frac{1}{2} C \ell^2 \left[\overline{S}_\ell^2 + \overline{\omega}_\ell \tilde{\overline{S}}_\ell + \frac{1}{4} |\overline{\omega}_\ell|^2 I \right]$$

where we introduce the <u>skew-strain matrix</u>

$$\tilde{S}_{ij} = S_{ik} \epsilon_{kj} = - \epsilon_{ik} S_{kj} .$$

One can check that it has the same eigenvalues as the strain, but its eigenframe is rotated by 45° so that

$$S : \tilde{S} = 0 .$$

But it is a consequence of this that

$$\Pi^{NL} = - \overline{S}_\ell : \tau_\ell^{NL} = 0 \quad !!!$$

In fact,

$$\overline{S}_\ell^2 = \overline{s}_\ell^2 I$$

is diagonal, as is the term $\frac{1}{4} |\overline{\omega}_\ell|^2 I$. Because the strain is traceless, $tr(\overline{S}_\ell) = 0$, any diagonal term also gives zero contribution to the energy flux. Hence, the entire contribution of Π^{NL} is null.

In fact, this is not surprising. Recall from Turbulence I, Coursenotes, Section IV (A) that

$$\Pi_\ell^{NL} = \frac{1}{2} C \ell^2 \left[-\mathrm{tr}(\overline{S}_\ell^3) + \frac{1}{4} \omega_\ell^T \overline{S}_\ell \omega_\ell \right]$$

with $\omega_\ell = \overline{\omega}_\ell \hat{z}$. These two terms, the strain skewness and the vortex-stretching rate, both vanish in 2D! In fact, recall from the Betchov relation that

$$\langle \Pi_\ell^{NL} \rangle_{space} = \frac{1}{2} C \ell^2 \langle \overline{\omega}_\ell^T \overline{S}_\ell \overline{\omega}_\ell \rangle$$

so the entire contribution in net is proportional to vortex-stretching, which is absent in 2D. This is just a physical-space version of the result of Kraichnan (1971) that the super-local contribution to energy flux vanishes identically in 2D as a consequence of the conservation of enstrophy (or lack of vortex-stretching). It follows that energy flux, unlike enstrophy flux, is an intrinsically multiscale phenomenon.

To deal with this situation, we can generalize the previous single-scale approximation by instead introducing a multiscale decomposition of the type of Paley-Littlewood for length-scales $\ell_n = 2^{-n}\ell$, as

$$u = \sum_{n=0}^{\infty} u^{[n]}$$

where

$$u^{[0]} = \bar{u}_{\ell_0} = u^{(0)}$$

and

$$u^{[n]} = \bar{u}_{\ell_n} - \bar{u}_{\ell_{n-1}} = u^{(n)} - u^{(n-1)}$$

are band-pass filtered fields. Likewise, introduce

$$\delta u(r;x) = \sum_{n=0}^{\infty} \delta u^{[n]}(r;x)$$

and substitue this expression into the formula for $T_\ell(u,u)$. The previous approximation corresponded to keeping just the first term $n=0$. Furthermore, since all of the band-pass fields are entire analytic, they have convergent Taylor polynomial approximations

$$\delta u^{(n,m)}(r;x) = \sum_{p=1}^{m} \frac{1}{p!}(r\cdot\nabla)^p u^{(n)}(x).$$

Substituting this one gets a sequence of <u>convergent approximations</u>

$$\tau^{(n,m)} = \int d^d r\, G_\ell(r)\, \delta u^{(n,m)}(r)\, \delta u^{(n,m)}(r)$$

$$- \left(\int d^d r\, G_\ell(r)\, \delta u^{(n,m)}(r) \right) \left(\int d^d r\, G_\ell(r)\, \delta u^{(n,m)}(r) \right)$$

such that

$$\tau = \lim_{n\to\infty} \lim_{m\to\infty} \tau^{(n,m)}.$$

For details, see

G. L. Eyink, "Multi-scale gradient expansion of the turbulent stress tensor," J. Fluid. Mech. <u>549</u> 159–190 (2006)

Unfortunately, the above expansion, while convergent, has a rate of convergence too slow to be practically useful. It also generates quite a lot of terms, such as those off-diagonal in scale, which are probably considerably smaller than others. Thus, a further <u>coherent subregions approximation</u> (CSA) was developed in Eyink (2006) which was argued to contain the most significant terms in the original expansion and to converge much more rapidly. To <u>first-order in gradients</u> it looks like

$$T_{\ell\, ij}^{CSA(1)} = \sum_{n=0}^{\infty} \frac{1}{d} C^{[n]} \ell_n^2 \frac{\partial u_i^{[n]}}{\partial x_k} \frac{\partial u_j^{[n]}}{\partial x_k}$$

where the constants $C^{[n]} \to C$ as $n \to \infty$. Thus, each term looks just like the old approximation, but now there is a term for each length-scale ℓ_n.

In 2D, this becomes

$$\mathbf{T}_\ell^{CSA(1)} = \sum_{n=0}^{\infty} \frac{1}{2} C^{[n]} \ell_n^2 \Big[(\mathbf{S}^{[n]})^2 + \omega^{[n]} \tilde{\mathbf{S}}^{[n]} + \frac{1}{4} |\omega^{[n]}|^2 \mathbf{I} \Big]$$

As before $(\mathbf{S}^{[n]})^2 = |s^{[n]}|^2 \mathbf{I}$ and the two diagonal terms contribute nothing to energy flux, but the middle term can now contribute for $n \geq 1$:

$$\Pi_\ell^{CSA(1)} = \sum_{n=1}^{\infty} \frac{1}{2} C^{[n]} \ell_n^2 \, \omega^{[n]} \left(\mathbf{S}^{(0)} : \tilde{\mathbf{S}}^{[n]} \right).$$

Here the term for given n represents the contribution of "eddies" or "vortices" of scale ℓ_n to the energy flux across scale ℓ. The quantity $\omega^{[n]}$ represents the vorticity of a small eddy of scale ℓ_n and $\mathbf{S}^{[n]}$ represents the strain matrix of that eddy. Because the skew-strain is rotated by $45°$, we can make the following interesting observation:

> The eddy of size ℓ_n makes a contribution to inverse energy cascade if its strain matrix $\mathbf{S}^{[n]}$ is rotated relative to $\mathbf{S}^{(0)} = \overline{\mathbf{S}}_\ell$ by an angle
>
> $$-45° \quad \text{if} \quad \omega^{[n]} > 0$$
>
> and
>
> $$+45° \quad \text{if} \quad \omega^{[n]} < 0$$

This is only a kinematic statement. Why should such a situation occur dynamically?

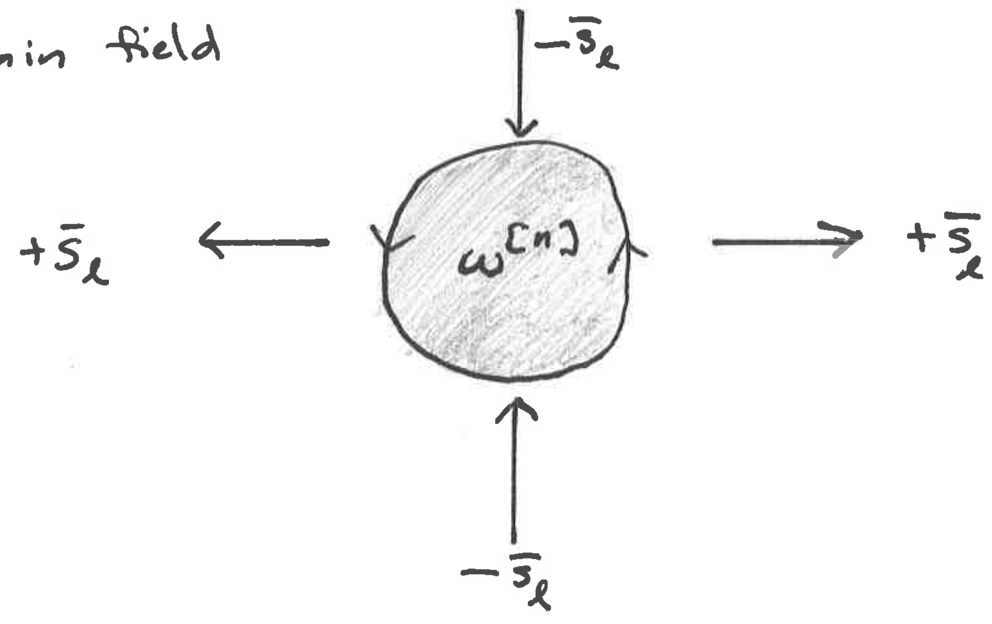
It turns out that this is exactly what happens in a mechanism of inverse cascade suggested by R. H. Kraichnan in 1976:

R. H. Kraichnan, "Eddy viscosity in two and three dimensions", J. Atmos. Sci. <u>33</u> 1521-1536 (1976)

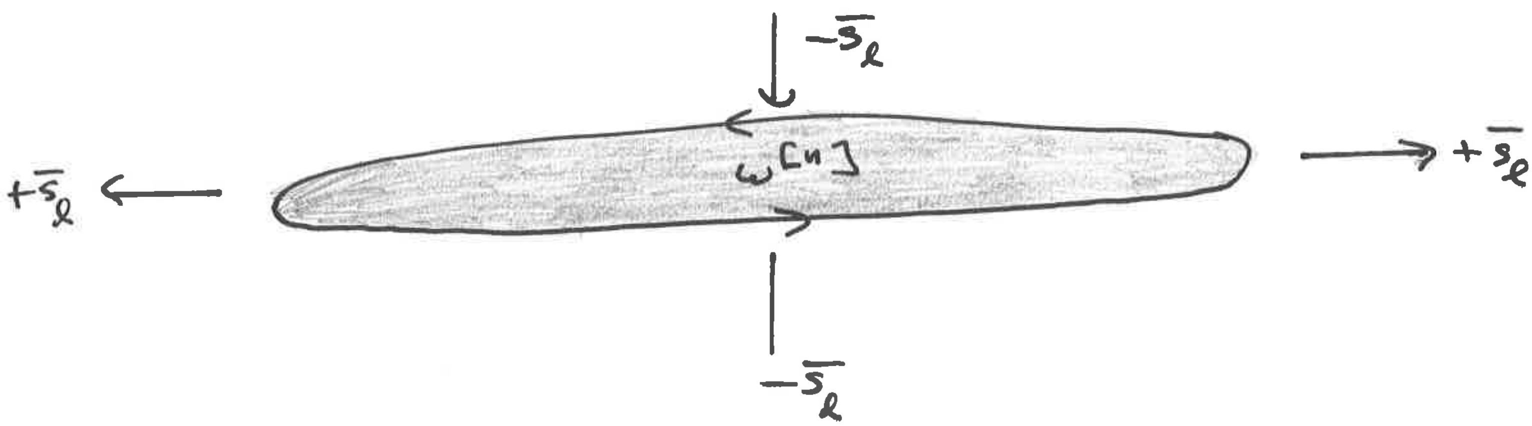
To quote from that paper (p. 1530):

"If a small-scale motion has the form of a compact blob of vorticity, or an assembly of uncorrelated blobs, a steady straining will eventually draw a typical blob out into an elongated shape, with corresponding thinning and increase of typical wavenumber. The typical result will be a decrease of the kinetic energy of the small-scale motion and a corresponding reinforcement of the straining field..."   — Kraichnan (1976)

It is easiest to explain this passage with a simple picture. Suppose that one begins with a nearly circular small-scale vortex in a larger-scale strain field

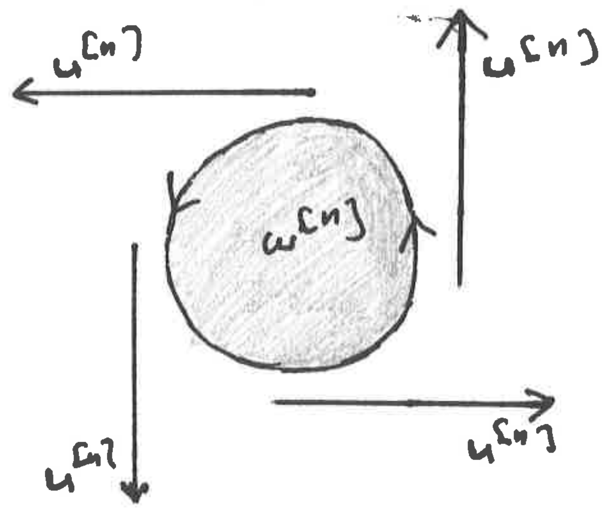


The vortex blob will become "thinned" and elongated, as follows, preserving its area,

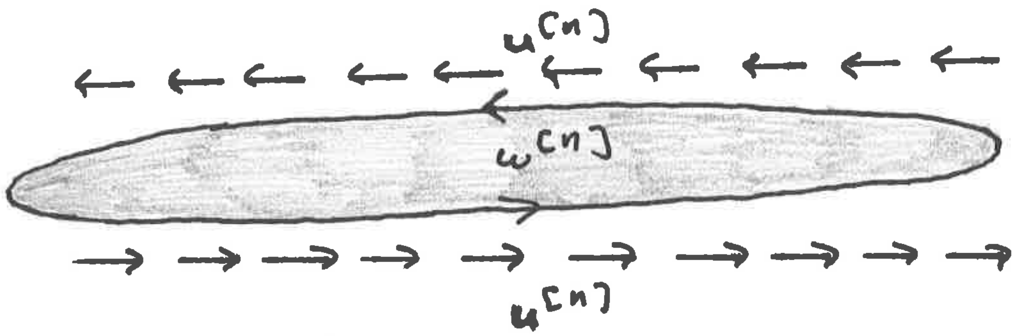


But notice that the <u>perimeter</u> of the vortex patch increases! Thus, by the Kelvin theorem, the velocity $u^{[n]}$ induced by the vortex $\omega^{[n]}$ must weaken, in order to preserve the circulation. The result is that the small-scale vortex blob is "spun down" and loses energy.

Where does the energy of the vortex blob go? It is not hard to see that it is transferred to the large scale. To see this, consider the subscale stress induced by the blob $\tau^{[n]} \propto u^{[n]}u^{[n]}$. The induced velocity changes during the process from



to



As we argued earlier, the velocity is weakened but it is also <u>rectified</u>. Now the velocity is mainly parallel or anti-parallel to the positive straining direction $\mathbf{e}_+$, so that

$$\tau^{[n]} \propto u^{[n]}u^{[n]} \propto \mathbf{e}_+\mathbf{e}_+$$

Thence, the deviatoric part becomes

$$\mathring{\tau}^{[n]} = \tau^{[n]} - \frac{1}{2}\mathrm{tr}(\tau^{[n]})\, I$$

$$\propto e_+ e_+ - \frac{1}{2}(e_+ e_+ + e_- e_-)$$

$$\propto \frac{1}{2}(e_+ e_+ - e_- e_-)$$

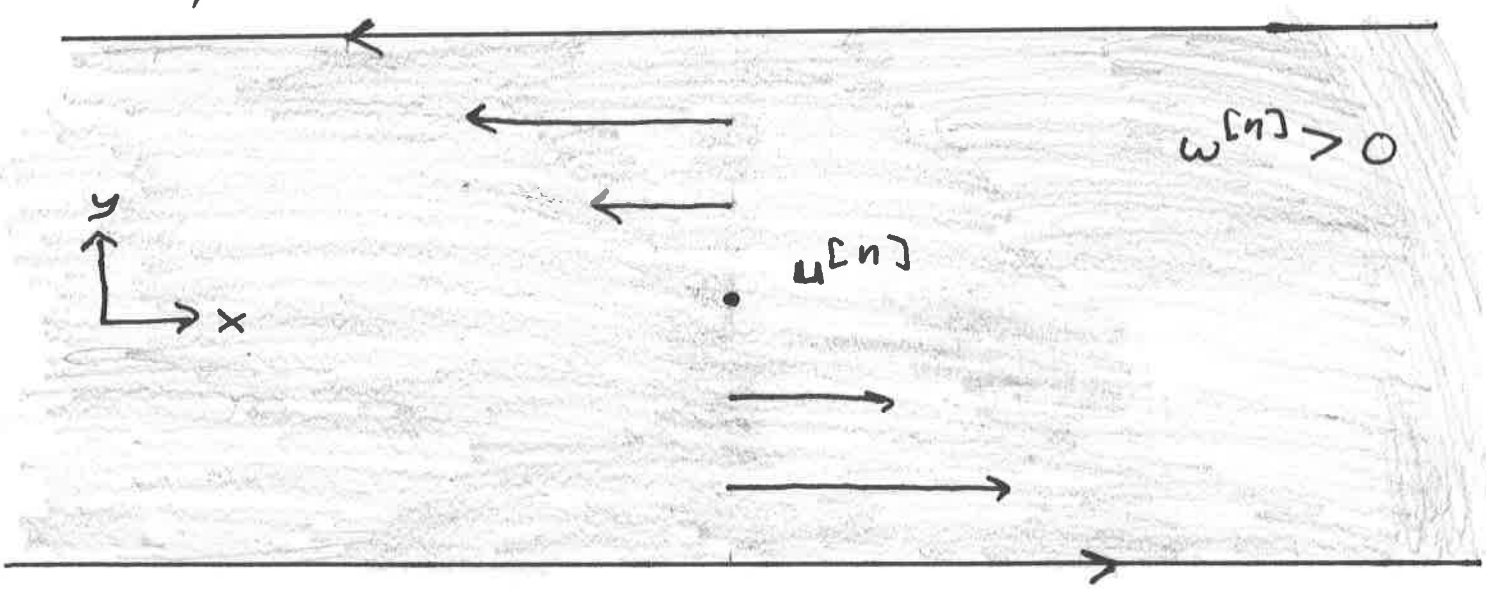
$$\propto \overline{S}_\ell$$

Hence, this "thinning process" creates a small-scale stress which is aligned with the large-scale strain, doing negative work and transferring energy from the small-scale blob to the large-scale field.

Finally, notice that this process also leads to

$$\omega^{[n]} \tilde{S}^{[n]} \propto \overline{S}_\ell$$

The reason is that the vortex blob is transformed into a small, thin <u>shear layer</u>. Zooming in, its velocity field appears as follows:



The corresponding strain matrix is

$$\mathbf{S}^{[n]} = \begin{pmatrix} 0 & -\omega^{[n]}(y) \\ -\omega^{[n]}(y) & 0 \end{pmatrix}$$

with $\omega^{[n]}(y) = -\dfrac{\partial u^{[n]}(y)}{\partial y}$. Its eigenvectors for $\omega^{[n]}(y) > 0$ are

$$\mathbf{e}_+^{[n]} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{e}_-^{[n]} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

which are rotated by $-45°$ relative to those of $\overline{\mathbf{S}}_\ell$. This leads to $\omega^{[n]}\widetilde{\mathbf{S}}^{[n]} \propto \widetilde{\mathbf{S}}^{[n]} \propto \overline{\mathbf{S}}_\ell$. One can easily see that for $\omega^{[n]} < 0$ the rotation of $\mathbf{S}^{[n]}$ is in the opposite sense $(+45°)$ relative to $\overline{\mathbf{S}}_\ell$, so that then $\omega^{[n]}\widetilde{\mathbf{S}}^{[n]} \propto -\widetilde{\mathbf{S}}^{[n]} \propto \overline{\mathbf{S}}_\ell$. It follows that this simple cartoon picture leads to the same prediction

$$\mathring{\boldsymbol{\tau}}^{[n]} \sim C\ell_n^2\, \omega^{[n]}\widetilde{\mathbf{S}}^{[n]}$$

as does the formal multiscale gradient expansion! It is interesting to compare these arguments with those for 3D forward energy cascade in Turbulence I, Course notes, Section IV(A).

We shall refer to this picture of the 2D inverse energy cascade as Kraichnan's <u>vortex-thinning mechanism</u>. This idea has become very popular in the geophysical fluid dynamics community and was promoted by

P. B. Rhines, "Geostrophic turbulence," Annu. Rev. Fluid Mech. <u>11</u> 401-441 (1979)

R. Salmon, "Geostrophic turbulence," in Topics in Ocean Physics, Proc. International School of Physics "Enrico Fermi" (ed. A. R. Osborne and P. M. Rizzoli), pp. 30-78 (North-Holland, 1982)

among others. Basic ideas of the picture go back to the atmospheric scientist Victor Starr

V. P. Starr, "Note concerning the nature of the large-scale eddies in the atmosphere," Tellus <u>5</u> 494-498 (1953)

and

V. P. Starr, <u>Physics of Negative Viscosity Phenomena</u> (McGraw Hill, 1968)

Starr proposed that in many large-scale, nearly two-dimensional motions in the atmosphere and the ocean, the smaller scale eddies should provide a negative eddy-viscosity, so that their stress would be of the form

$$\tau_\ell = -2\nu_\ell \mathbf{S}_\ell = +2|\nu_\ell|\mathbf{S}_\ell$$

and transfer energy into the large-scale motions. Notice that this indeed is essentially the effect described by the thinning mechanism, although it is not instantaneously true that $\tau_\ell \propto \overline{\mathbf{S}}_\ell$ and the phenomenon is essentially a multiscale one.

How do these ideas and predictions compare with laboratory experiments and numerical simulations? We discuss here results of

S. Chen et al., "Physical mechanism of the two-dimensional inverse energy cascade," Phys. Rev. Lett. 96 084502 (2006)

and

Z. Xiao et al., "Physical mechanism of the inverse energy cascade of two-dimensional turbulence: a numerical investigation," J. Fluid. Mech. 619 1-44 (2009)

Fig. 2 of the paper of Xiao et al. (2009) [next page] shows the energy spectra $E(k)$ and fluxes $\Pi(k)$ from four different simulations, at resolutions 512^3 to 2048^3 and with various (hyper) viscosities and inverse Laplacian damping at small and large scales. All of the spectra observed in these simulations are very close to $k^{-5/3}$ and the energy fluxes are constant and negative for up to two decades of wavenumbers below k_f. The spatial PDF of the flux is shown in their Fig. 3 [page after next], which is noticeably skewed to the left. It is interesting that this PDF is less skewed than the corresponding PDF of energy flux in 3D and more skewed than the PDF of enstrophy flux in the 2D forward cascade. The second curve in Fig. 3 is from an improvement of the CSA [referred to as "MSG" by Xiao et al. (2009)], which shall be discussed later.

Fig. 8 of the paper [page after next] shows the fractional contribution of the energy flux at scale ℓ which arises from smaller-scale eddies of size $\ell_n = 2^{-n}\ell$ (panel a) or $\ell_n = \left(\frac{2}{3}\right)^n \ell$ (panel b). As predicted by Kraichnan (1971), it is eddies about 2-4 times smaller which make the greatest contribution. Kraichnan's TFM closure gives a pretty good fit to the DNS curve for the cumulative flux fraction.

,

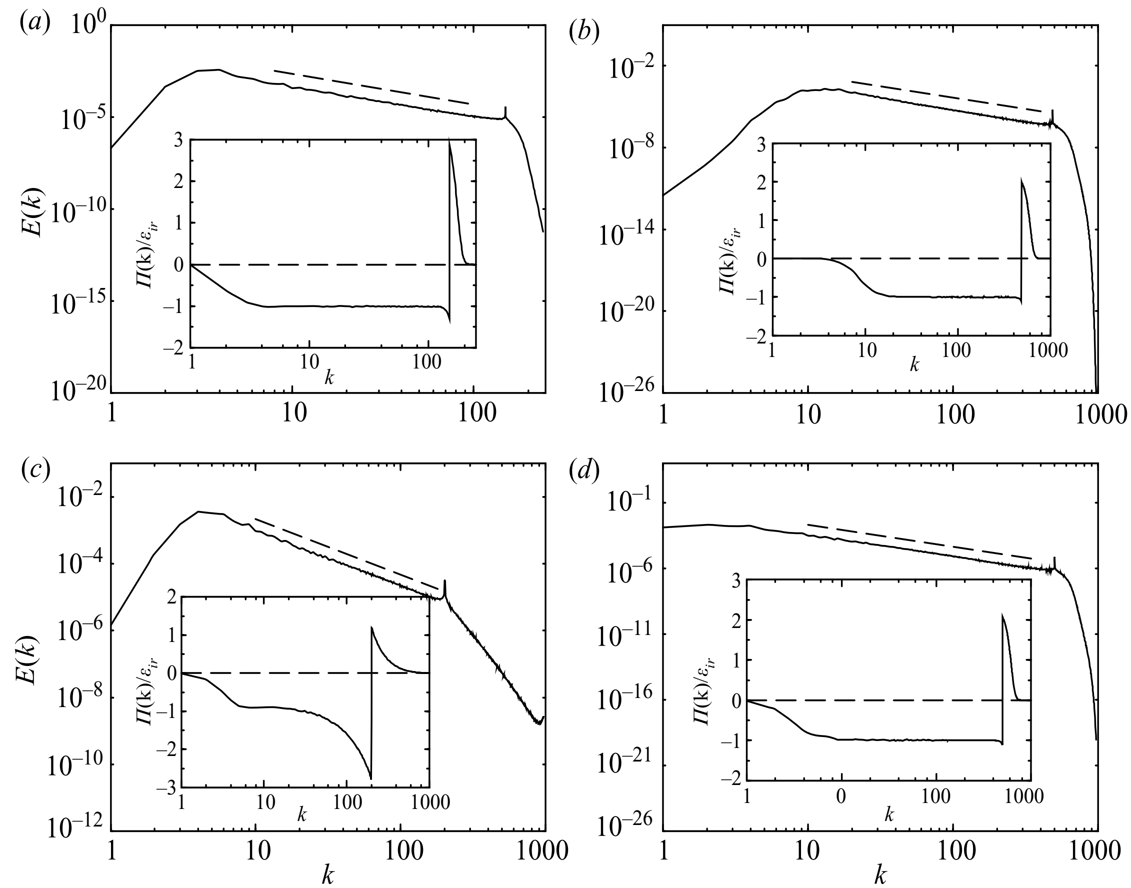


FIGURE 2. Energy spectrum functions $E(k)$ versus k at steady state. (*a*) *RUN* 2, (*b*) *RUN* 3, (*c*) *RUN* 4 and (*d*) *RUN* 5. Insets are the mean spectral energy fluxes normalized by large-scale (infrared) energy dissipation ϵ_{ir}.

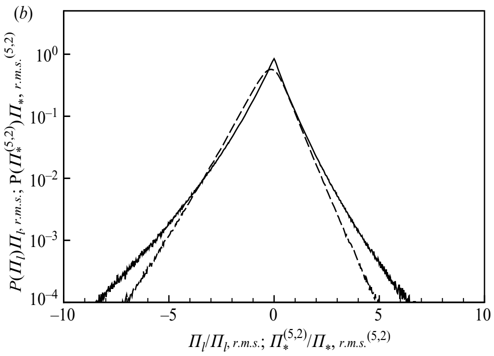


FIGURE 4. (*a*) The cumulative mean energy flux $\langle \Pi_\ell \rangle_{cum}(\rho)$, normalized by the mean flux $\langle \Pi_\ell \rangle$ versus ρ and (*b*) PDF of energy flux. Solid line: true flux Π_ℓ; dashed line: second-order MSG model flux $\overline{\Pi}_*^{2\text{nd}}$ (see § 5.2) with each normalized by its r.m.s. value.

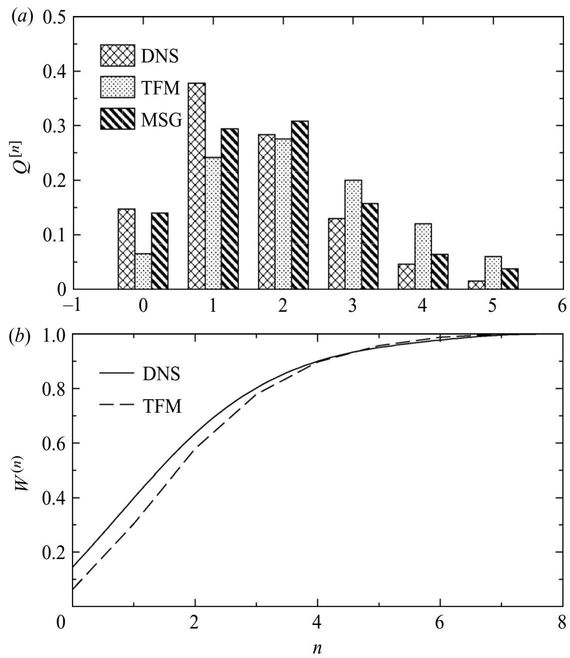


FIGURE 8. (*a*) Flux fraction $Q^{[n]}$ from length scale ℓ_n versus n, $\lambda = 2$, from DNS, TFM and second-order MSG; (*b*) cumulative flux fraction $W^{(n)}$ versus n, $\lambda = 1.5$, from DNS and TFM.

To get some insight where in the flow the inverse cascade is occurring, we reproduce on the next page [from Fig. 3 of Xiao et al. (2009)] a plot of the instantaneous vorticity field $\omega(x)$ and, from the same snapshot, a plot of the energy flux $\Pi_\ell(x)$ for ℓ in the inertial range. One can see that negative flux is not especially associated with the stronger vortices and there is even some general tendency for the green regions in both plots to coincide [low vorticity regions and low, negative flux]. It turns out that most of the mean flux arises from the "green" region of low, negative flux.

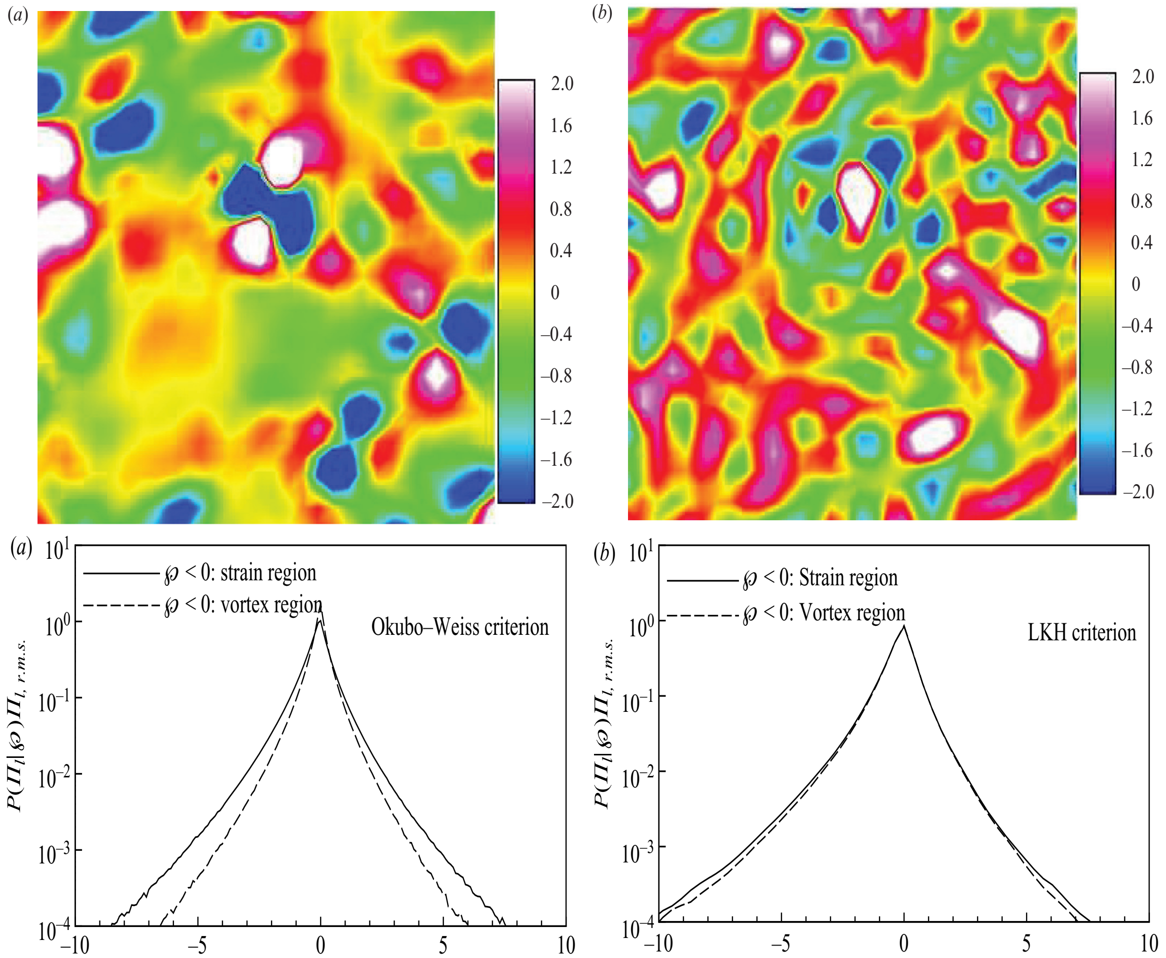
To further quantify this observation, Fig. 6 of their paper [next page, lower panels] plotted the PDFs of energy flux conditioned on values of

$$\wp = \Delta p = \frac{1}{2}\omega^2 - |S|^2,$$

with $\wp < 0$ giving "strain regions" and $\wp > 0$ "vortices". The second PDF uses a more sophisticated criterion of Lapeyre, Klein & Hua (1999) based on

$$r = \frac{\omega + 2D_t\alpha}{s}, \quad \alpha \text{ orientation angle of strain eigenframe}$$

with $|r| < 1$ giving "strain regions" and $|r| > 1$ "vortices." For either criterion, the inverse cascade is not associated with the vortices and even slightly stronger (more skewed) in strain regions.



Top left: Instantaneous snapshot of vorticity field ω. *Top right:* Instantaneous snapshot of energy flux Π_ℓ. *Bottom left:* Conditional PDFs of energy flux with Okubo-Weiss criterion. *Bottom right:* Conditional PDFs of energy flux with LKH criterion.

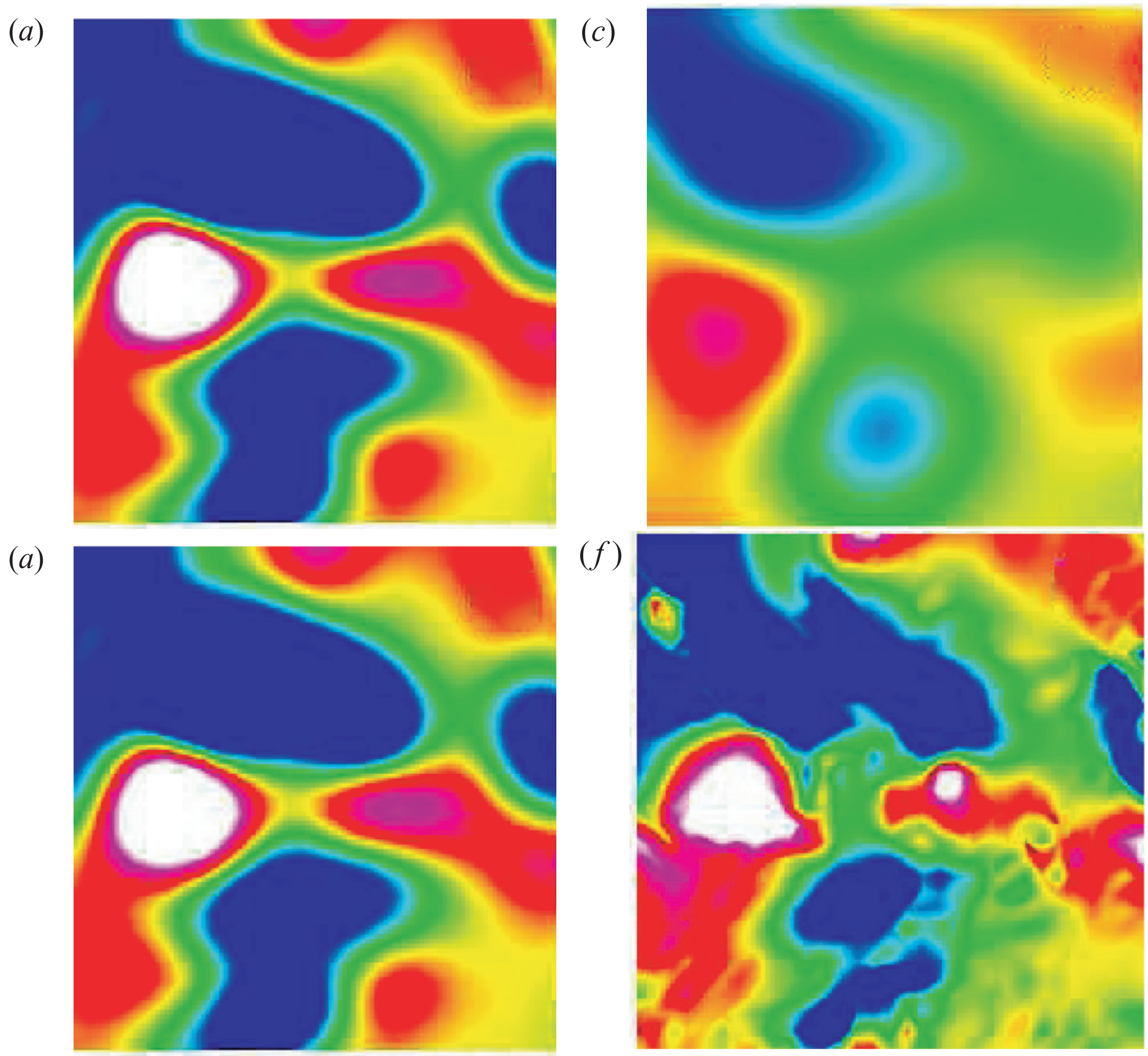
The paper of Xiao et al. (2009) also computed the quantity $\Pi_\ell^{CSA(1)}$ in their simulations and found that

$$\langle \Pi_\ell^{CSA(1)} \rangle \cong (0.6) \langle \Pi_\ell \rangle .$$

Hence, the CSA first-order in gradients does imply net inverse energy cascade, but gives only about 60% of the total mean flux. The spatial correlation of $\Pi_\ell(x)$ and $\Pi_\ell^{CSA(1)}(x)$ point-to-point is also quite poor, only about 0.65. This is illustrated by their Fig. 13, panels (a), (c) reproduced on the next page [top two panels]. There is only some general similarity between the two plots. Clearly, something is lacking in $\Pi_\ell^{CSA(1)}$!

As a matter of fact, there are important contributions that arise to <u>second-order in gradients</u>. Some of these are "super-local" and, while giving little net contribution to mean flux, are strongly correlated with $\Pi_\ell(x)$ point-to-point. Other 2nd-order contributions that are less local nevertheless contribute substantially to the mean value. These 2nd-order contributions were calculated in detail in

G. Eyink, "A turbulent constitutive law for the two-dimensional inverse energy cascade," J. Fluid. Mech. <u>549</u> 191–214 (2006)



Top left: Instantaneous snapshot of energy flux Π_ℓ. *Top right:* Instantaneous snapshot of energy flux $\Pi_\ell^{CSR,(1)}$. *Bottom left:* Same as top left. *Bottom right:* Instantaneous snapshot of energy flux $\Pi_\ell^{CSR,(2)}$.

The full 2nd-order result for the deviatoric stress contains terms that are diagonal in scale

$$\overset{\circ}{T}{}^{[n,2]}_{CSA,\ell} = \frac{1}{16} C_4^{[n]} \ell_n^4 (\nabla\omega^{[n]} \cdot \nabla) \tilde{S}^{[n]} \Big\} \text{ differential strain-rotation}$$

$$\text{vorticity-gradient stretching } \Big\{ + \frac{1}{64} C_4^{[n]} \ell_n^4 \left[\nabla^\perp\omega^{[n]} \nabla^\perp\omega^{[n]} - \nabla\omega^{[n]} \nabla\omega^{[n]} \right]$$

and another set off-diagonal in scale

$$\overset{\circ}{T}{}^{off,(N,2)}_{CSA,\ell} = \frac{1}{2} \left[\nabla\psi_*^{(N,2)} \nabla\psi_*^{(N,2)} - \nabla^\perp\psi_*^{(N,2)} \nabla^\perp\psi_*^{(N,2)} \right]$$

with

$$\nabla\psi_*^{(N,2)} = \frac{1}{4} \sum_{n=1}^{N} \frac{C^{[n]}}{\sqrt{N_n}} \ell_n^2 \nabla\omega^{[n]}, \quad N_n = \lambda^{2n}.$$

Of these, the most physically transparent is the term we have described as due to vorticity-gradient stretching. It is easily explained by Kraichnan's vortex-thinning picture. Note that thinning aligns $\nabla\omega^{[n]}$ with e_-, the strain compressing direction, so that

$$\nabla^\perp\omega^{[n]} \nabla^\perp\omega^{[n]} - \nabla\omega^{[n]} \nabla\omega^{[n]}$$

$$\propto e_+ e_+ - e_- e_- \propto \overline{S}_\ell .$$

It can be interpreted as a 2nd-order effect associated to vortex-thinning. Notice that its contribution to flux

$$\Pi^{[n],\,vgs}_{CSA,\,\ell} = -\,\overline{\mathbf{S}}_\ell : \mathring{T}^{[n],\,vgs}_{CSA,\,\ell}$$

$$= \frac{1}{32} C_4^{[n]} \ell_n^4 \left(\nabla\omega^{[n]}\right)^T \overline{\mathbf{S}}_\ell \left(\nabla\omega^{[n]}\right)$$

which is proportional to the rate of vorticity-gradient
stretching of blobs at scale ℓ_n by the strain $\overline{\mathbf{S}}_\ell$.
It will tend to be negative because of the
alignment of $\nabla\omega^{[n]}$ and $\mathbf{e}_-$. Notice also that
this term does not vanish for $n=0$! As we
can see from Fig. 8 of Xiao et al. (2009), that
term also does not vanish for the true flux:
Kraichnan's argument does not rule out such a term
for coarse-graining flux, but suggests that it should
be small.

What about the other diagonal contribution?
We say this term is associated to <u>differential</u>
<u>strain rotation</u> because its contribution to energy
flux for $n=0$ is

$$\Pi^{[0],\,dsr}_{CSA,\,\ell} = \frac{1}{8} C_4^{[n]} \ell_n^4 \left(\nabla\overline{\omega}_\ell \cdot \nabla\alpha_\ell\right) |\overline{\mathbf{S}}_\ell|^2$$

and α_ℓ is the orientation angle of the eigenframe of $\overline{\mathbf{S}}_\ell$
to a fixed (laboratory) frame. Thus,

$$\Pi^{[0],dsr}_{CSA,\ell} < 0 \iff$$

$\bar{S}_\ell$ rotates clockwise moving in the direction of increasing $\bar{\omega}_\ell$.

Interestingly, there is a 2D version of the Betchov relation for 2nd-order gradients which implies that

$$2\langle |\bar{S}_\ell|^2 \nabla\bar{\omega}_\ell \cdot \nabla\alpha_\ell \rangle = \langle (\nabla\bar{\omega}_\ell)^T \bar{S}_\ell (\nabla\bar{\omega}_\ell) \rangle$$

Hence, the differential strain rotation term gives a net negative contribution if and only if the vorticity-gradient stretching term does so. See Eyink (2006). For general n the contribution

$$\Pi^{[n],dsr}_{CSA,\ell} = \frac{1}{16} C_4^{[n]} \ell_n^4 \, \bar{S}_\ell : (\nabla\omega^{[n]} \cdot \nabla) \tilde{S}^{[n]}$$

is associated not only to differential strain rotation, but also to "differential strain magnification" (i.e. the change in the magnitude of the strain in space). As we shall now discuss, this term is one of the most significant 2nd-order terms in the simulation of Xiao et al. (2009).

Let us now compare in detail with those simulations. The contributions of the various CSA terms to the mean energy flux are given in the following table:

CSA term	fractional contribution to mean energy flux
skew-strain (1st-order)	60%
differential strain rotation & magnification (2nd-order)	40%
vorticity-gradient stretching (2nd-order)	21%
off-diagonal stretching (2nd-order)	−16%
TOTAL	105%

As can be seen the CSA flux to 2nd-order slightly overestimates the true mean flux. Also, the (diagonal) vorticity-gradient stretching and the off-diagonal stretching contributions nearly cancel each other. Finally, 2/3rds of the mean comes from the 1st-order term and 1/3rd of the mean comes from the 2nd-order terms.

However, the addition of the 2nd-order terms greatly improves the spatial correlation of the model with the true flux. Whereas $\rho(\Pi_\ell^{CSA(1)}, \Pi_\ell) \cong 0.65$, now

$$\rho\left(\Pi_\ell^{CSA,(2)}, \Pi_\ell\right) \cong 0.89$$

The improved agreement can be seen visually in Fig. 13, panels (a) & (f) of Xiao et al. (2009), where $\Pi_\ell^{CSA,(2)}$ is now a recognizable facsimile of Π_ℓ.

Finally, Fig. 8 of Xiao et al. (2009) shows the scale distribution of the energy flux, ie the fraction $\varphi^{[n]}$ of the flux that arises from interaction of the scales $> \ell$ with those at scale $\ell_n = \lambda^n \ell$. The predictions are in reasonable agreement with those of the true flux and also with the results of Kraichnan's TFM closure. It is interesting that the "super-local" $n=0$ terms (which are all 2nd-order), although they contribute only about 20% of the mean flux are highly correlated with the exact flux spatially [$\rho \cong 81\%$]. This is shown visually in Figs. 18 & 19 of Xiao et. al. (2009), not reproduced here. Thus, despite the weak locality of the cascade, a reliable indicator of inverse cascade is large-scale vorticity-gradient stretching and/or differential strain rotation!

To summarize: We have derived a set of analytical formulas for subscale stress and energy flux in the 2D inverse cascade regime by a multiscale-gradient expansion based on UV-locality. The terms in this expansion can be interpreted physically and be argued to produce inverse cascade based on a simple vortex-thinning picture of Kraichnan. Furthermore, the approximate expansion to 2nd-order in gradients is in quite reasonable agreement with the true flux when evaluated by a DNS of forced, steady-state 2D turbulence.

We cannot, of course, claim that the vortex-thinning mechanism is "proved." An interesting criticism of the thinning picture is presented in

G. Holloway, "Eddy stress and shear in 2D flow," J. Turbulence 11 14 (2010)

Among other points, Holloway correctly notes that energy is not additive over vortices, because of the long-range Coulomb (inverse Laplacian) potential $G(x,y)$. Hence, individual vortices could all be thinned and yet their total energy increase, because of long-range pair interaction energies. This means that inverse energy cascade is intrinsically a multi-vortex effect and one cannot claim to understand the phenomenon from a simple single-vortex cartoon. However, it is surprising how well the cartoon explains all of the terms in the systematic expansion. The good agreement with numerics argues that it contains an essential element of the truth!

Comparison of forced and decaying 2D turbulence. It is now a good time to compare some essential aspects of 2D steady-state, forced turbulence and 2D freely decaying turbulence.

We have seen that one of the prominent phenomena in decaying 2D turbulence is the appearance and merger of coherent vortices. It is very common in the literature to see this equated with inverse energy cascade. In fact, it is quite often that one encounters unqualified statements such as; "Vortex merger is the mechanism of 2D inverse energy cascade." The idea was already suggested by R. H. Kraichnan in his seminal 1967 paper. He wrote of his dual cascade picture of energy and enstrophy as follows:

> "This is consistent with a picture of the transfer process as a clumping-together and coalescence of similarly signed vortices, with the high-wavenumber excitation confined principally to thin and infrequent shear layers attached to the ever-larger eddies thus formed."
>
> — R. H. Kraichnan (1967)

If this suggestion is correct, then why did we make no mention of merger in our previous discursion on inverse energy cascade?

In fact, Xiao et al. (2009) carried out a detailed investigation of the role of mergers in their simulations. See their Fig. 11 [next page]. Mergers were defined in a conventional topological manner as a saddle-node bifurcation of one of a pair of vortex maxima/minima with an intervening saddle, leading to a single maximum/minimum. The first observation was that mergers were exceedingly rare events in the simulations of Xiao et. al. (2009). Taking as the "neighborhood" of a merger the set of points within a distance of 0.8 ℓ of the saddle-node bifurcation — which was larger than the typical radius of the merger region observed by eye — there was a probability of less than 0.08% to lie in such a neighborhood! Furthermore, vortex "splittings" — which appear as reverse saddle-node bifurcations with a new node and saddle appearing "out of the blue" — were about as common as mergers.

Finally, Xiao et al. (2009) found essentially no spatial correlation of negative energy flux with the (rare) merger events. This is illustrated in the bottom panel of Fig. 11, where the neighborhoods of mergers are indicated by the "rainbow rings" and the regions of negative flux are the black contours. There was found to be a slightly elevated level of negative flux in the neighborhood of mergers. This was of the order of 20% and might simply be due to the higher strains associated to mergers. However, because of their great rarity, mergers made less than 0.1% contribution to the total mean energy flux.

,

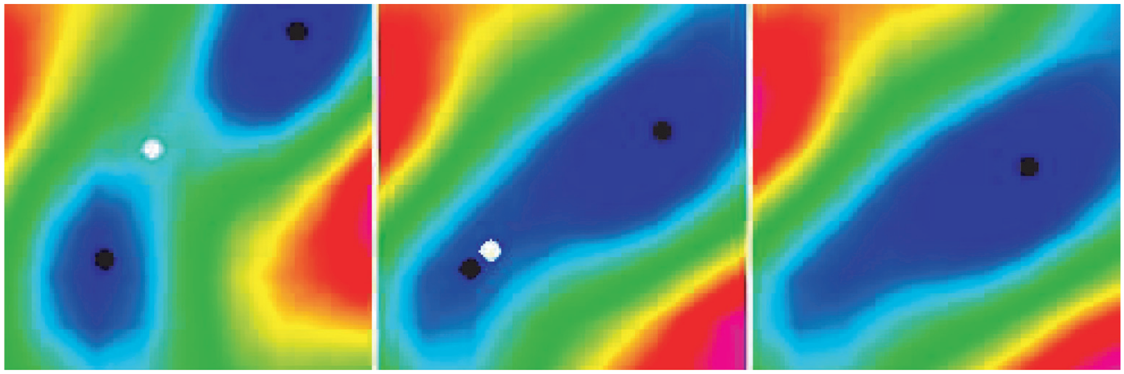


FIGURE 10. Snapshot of a merger event captured in our DNS, at times well before, immediately before and immediately after a saddle-node bifurcation; black circles represent nodes and white circles saddle points.

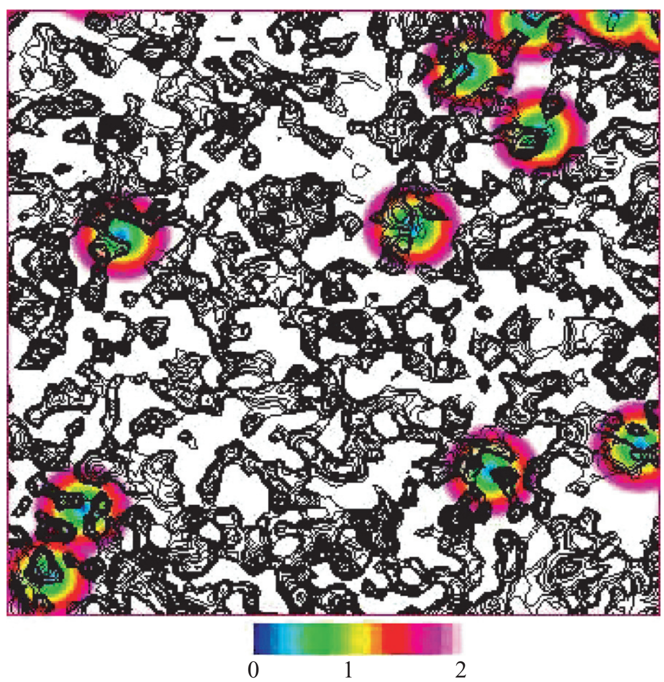


FIGURE 11. Vortex merger events, with colours indicating radial distance in units of filter length ℓ, overlaid by contour lines of energy flux, in black, showing regions of inverse cascade.

On the other hand, there really must be transfer of energy to large-scales associated with mergers in decaying 2D turbulence! Recall that in the simulations of A. Bracco et. al. (2000) there was a clear increase of $E(k)$ at low-wavenumbers. This implies $\Pi(k) < 0$ at some intermediate wavenumbers, and it is reasonable to associate this with the observed, frequent mergers in those decay simulations. As a matter of fact, a number of studies of individual merger events have shown that merger is associated to transfer of energy to larger scales/lower wavenumbers:

A. H. Nielsen et al., "Vortex merger and spectral cascade in two-dimensional flows," Phys. Fluids 8 2263–2265 (1996)

Ch. Josserand & M. Rossi, "The merger of two corotating vortices: a numerical study," European J. Mech. B/Fluids 26 779–794 (2007)

As far as we know, there has never been carried out a study of the local energy flux $\Pi_\ell(x)$ in decaying 2D turbulence. However, it seems plausible that the negative values of $\Pi_\ell(x)$ should here be associated mainly with vortex-mergers.

Recently, a detailed study of spectral energy flux $\Pi(k)$ for decaying 2D turbulence has been carried out in

P. D. Mininni and A. Pouquet, "Inverse cascade behavior in freely decaying two-dimensional fluid turbulence," Phys. Rev. E **87** 033002 (2013)

These authors carried out 50 simulations of decaying 2D turbulence at resolutions 2048^2, with an initial energy spectrum sharply peaked around $k_0 = 20$. Their results are shown in Figs. 1 and 5 reproduced on the next two pages.

Shown in Fig. 1 is the time development of a single such simulation of 2D decay, for energy spectrum $E(k,t)$, energy flux $\Pi(k)$, and enstrophy flux $Z(k)$ [which is denoted $\Sigma(k)$]. There is clearly a range with $\Pi(k) < 0$ for $k < k_0$ and $Z(k) > 0$ for $k > k_0$. Because of the shortness of the ranges, there is also considerable "leakage" of flux in the "wrong" directions. To get better statistics, averages over 50 runs and times $t = 0.5 - 6$ are considered in Fig. 5. Now the ranges with $\Pi(k) < 0$ and $Z(k) > 0$ are quite clear, although the fluxes are not nearly constant over these ranges. Nevertheless, Fig. 5(a) shows a narrow range of $k^{-5/3}$ energy spectrum where $\Pi(k) < 0$ and also a spectrum of about k^{-4} (distinctly steeper than k^{-3}) where $Z(k) > 0$.

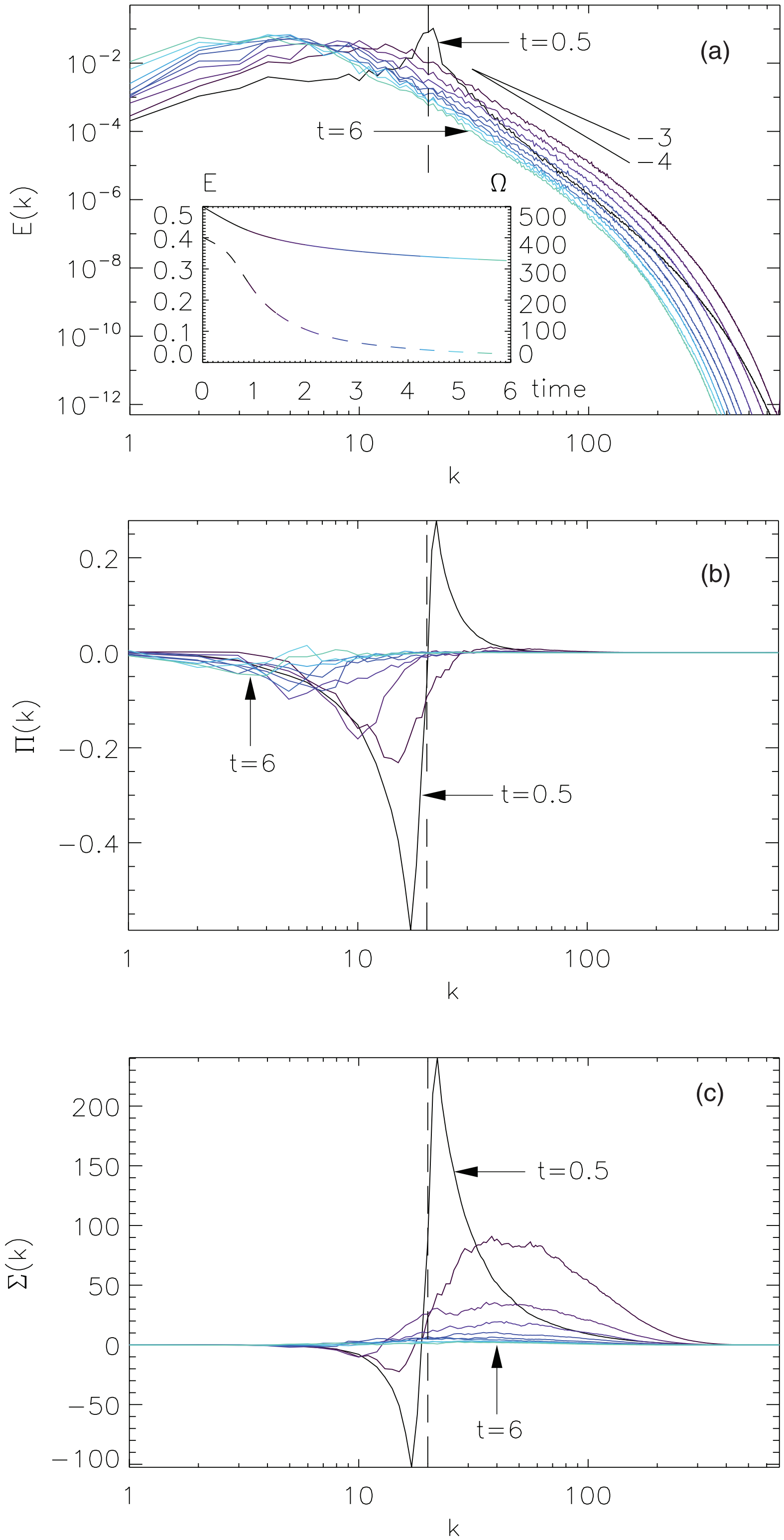


FIG. 1. (Color online) Time evolution of the (a) energy spectrum, (b) energy flux, and (c) enstrophy flux in a single 2048^3 simulation from $t = 0.5$ (black line) to $t = 6$ (light gray or light blue line). Slopes in the energy spectrum are indicated as references. The curves corresponding to $t = 0.5$ and 6 are indicated in all panels by arrows and the vertical dashed lines indicate the initial energy containing wave number k_0. In (b) and (c), note the displacement to smaller wave numbers of the minimum of energy flux and to larger wave numbers of the maximum of enstrophy flux. The inset in (a) shows the time evolution of the energy (solid line) and of the enstrophy (dashed line) in this run, with the color changing with time following the colors used for the different curves in the spectrum and fluxes.

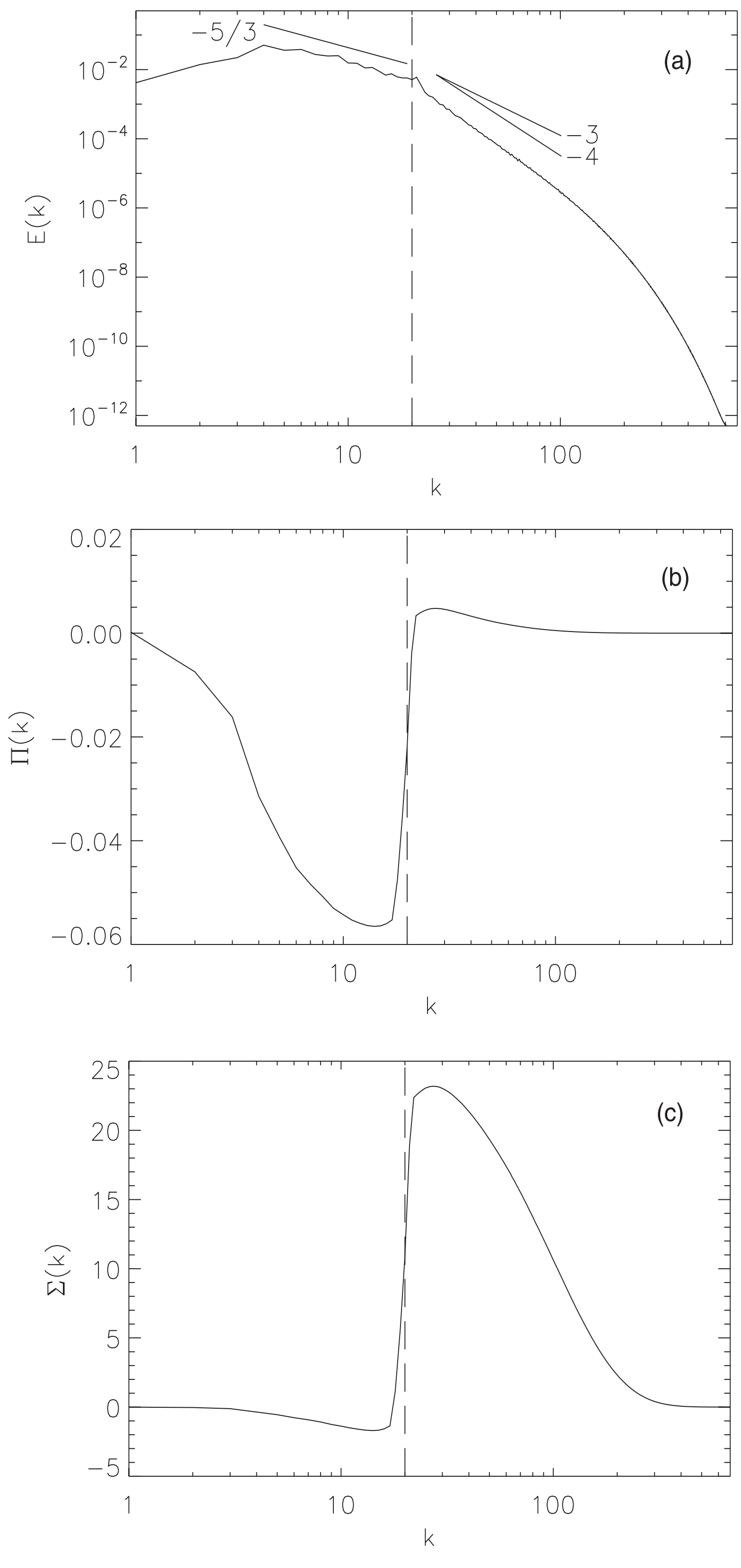


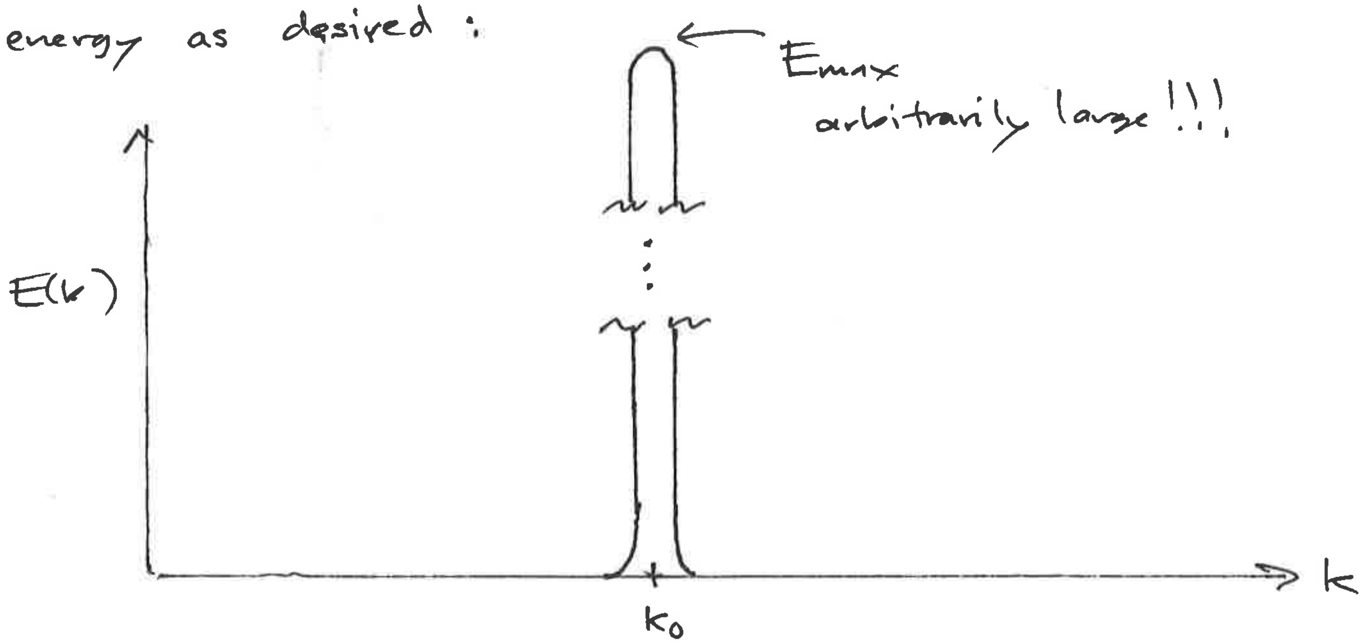
FIG. 5. (a) Time- and ensemble-averaged energy spectrum, (b) energy flux, and (c) enstrophy flux over the fifty 2048^2 simulations and from $t = 0.5$ to 6. Slopes in (a) are indicated as references. The vertical dashed lines correspond to the initial energy-containing wave number k_0.

These various results seem to suggest that inverse cascades in 2D forced and 2D decaying turbulence are rather different (but related) phenomena. All these matters are still controversial and subject to much debate. However, we would like to suggest the following general picture:

ANTITHESIS OF 2D INVERSE CASCADES

	FORCED	DECAYING
LOCALITY	Spectral cascade (local in scale)	Physical-space cascade
PHYSICAL MECHANISM	Vortex thinning	Vortex merger
SPECTRAL ENERGY FLUX	Constant in scale	Non-constant in scale
POWER-LAW SPECTRAL RANGE ("INERTIAL RANGE")	Arbitrarily-long ranges possible	Finite ranges only possible (?)

A few comments are required to explain this proposal. Regarding locality, we have already seen that forced, steady-state inverse cascades with any power-law of the type currently proposed must be (asymptotically) local-in-scale. On the other hand, vortex-merger is a process (mainly pairwise) which is local in physical space. In fact, it is well-known that merger requires approach of two vortices to a critical ratio of the vortex separation distance to their radii. E.g. see Josserand & Rossi (2007). After the vortices approach to within this critical ratio, merger becomes inevitable. We have already discussed the physical mechanisms. We have also seen that it is easy to prove existence of long ranges of constant (negative) energy flux for the forced case, under reasonable assumptions. There is no evidence that we know of the possibility of constant energy-flux ranges in the case of 2D decaying turbulence. Perhaps the most questionable proposal in the above dichotomy is that the scale range with power-law energy spectrum ($k^{-5/3} - k^{-2}$) and negative flux $\Pi(k) < 0$ must necessarily be finite for 2D decaying turbulence. At issue here is whether an initial datum with large amounts of energy spectral localized around initial wavenumber k_0 can mimic a spectrally localized body-force at that wavenumber. There is certainly no difficulty in providing as large a reservoir of initial energy as desired:



The difficulty is that that peak will successively <u>spread</u> in wavenumber and no longer remain well-localized at k_0 and thus give spectrally broader & broader forcing, unlike a fixed force.

Finally, let us comment briefly on the finding of Mininni & Pouquet (2013) that the forward enstrophy cascade regime has a spectrum steeper than Batchelor's prediction of k^{-3} and more like Saffman's k^{-4} prediction. However, there are other simulations which show that a simulation which is at high enough Reynolds number and which is sufficiently resolved at high wavenumbers k shows after the range with rapidly decaying spectrum $k^{-4}-k^{-5}$ (and containing most of the energy) there is a high-wavenumber "tail" with a k^{-3} spectrum, as originally proposed by Batchelor. On the following page we show Figs. 1 & 4 from the previously mentioned paper of Fox & Davidson (2010). Their data show reasonable evidence of a range $[k_1(t), k_2(t)]$ of k^{-1} enstrophy spectrum $\Omega(k,t)$ [denoted $E_\omega(k,t)$] which is growing in time. They find, however, that these are not constant enstrophy-flux ranges and do not scale with the viscous dissipation rate $\eta_{dis}(t) = \nu \langle |\nabla\omega(t)|^2 \rangle$. They instead propose that $\Omega(k,t) \sim [Z(k_1(t),t)]^{2/3} \frac{1}{k}$. We shall now present rigorous results which justify that $\eta_{dis}(t)$ is a poor choice.

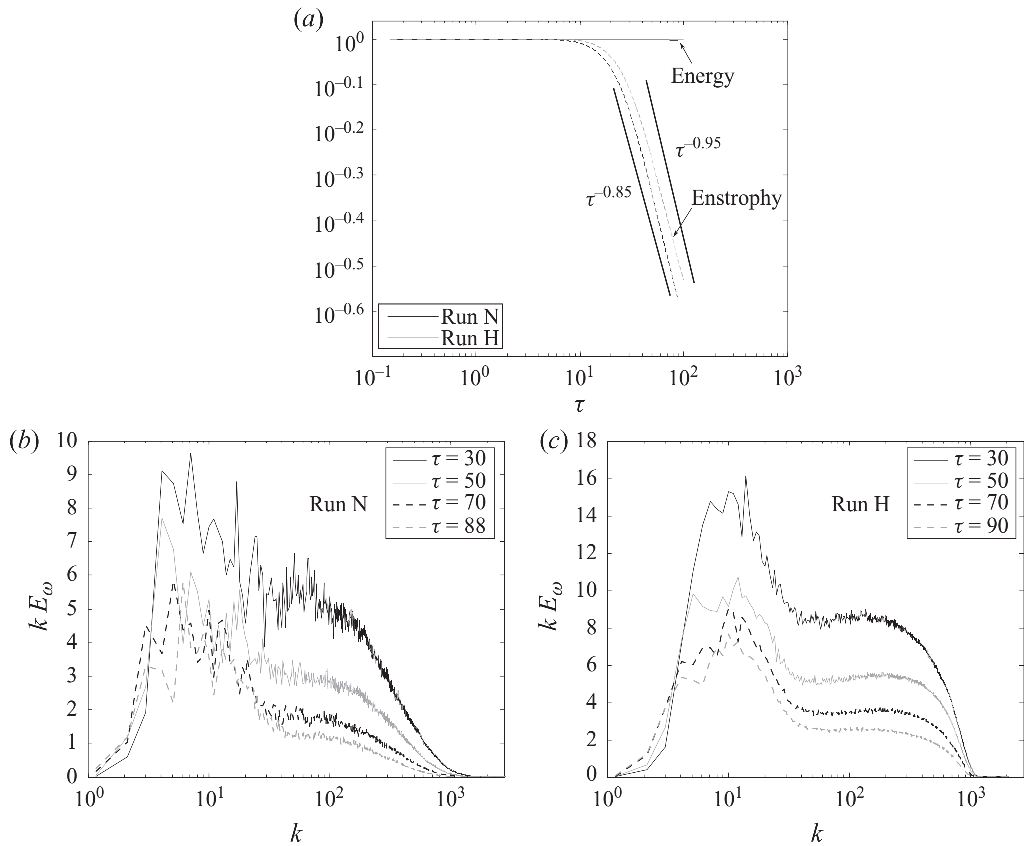


FIGURE 1. The evolution of (a) energy and enstrophy normalized by their initial values, and (b, c) the enstrophy spectra in runs N and H.

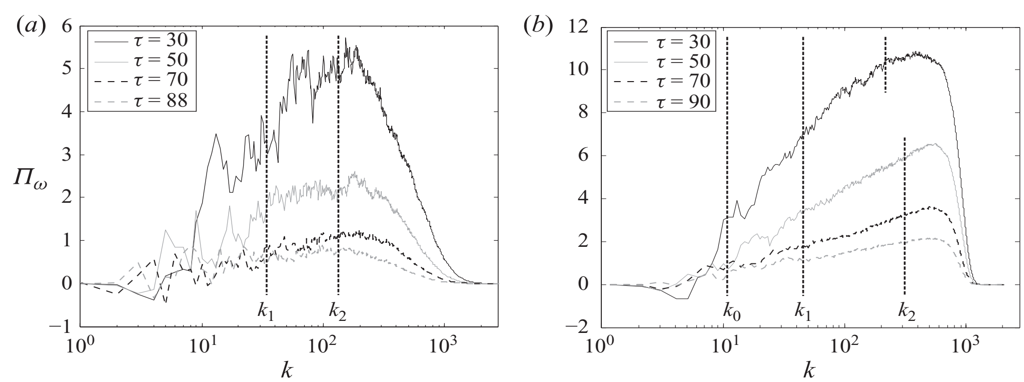


FIGURE 4. The enstrophy flux, Π_ω in (a) run N and (b) run H. k_1 and k_2 mark the limits of the k^{-1} region in the corresponding enstrophy spectrum E_ω.

Enstrophy cascade and the Di Perna-Lions $W^{1,1}$-theory

We shall now explain the implications for 2D enstrophy cascade of the celebrated "$W^{1,1}$-theory" of Di Perna & Lions:

R. J. Di Perna & P. L. Lions, "Ordinary differential equations, transport theory and Sobolev spaces," Invent. Math. 98 511-547 (1989)

This paper developed a theory of "renormalized solutions" of linear transport equations

$$\partial_t \theta + \mathbf{u} \cdot \nabla \theta = g$$

and the associated ODE's for the characteristics (flow maps)

$$\frac{d\mathbf{X}}{dt} = \mathbf{u}(\mathbf{X}, t)$$

in the case of rather rough velocity fields $\mathbf{u} \in W^{1,p}$ $p \geq 1$. (This will be explained in a moment). Their theory has the important implication for 2D turbulence that any weak/distributional/singular solution of 2D incompressible Euler equation

$$\partial_t \omega + \nabla\cdot(u\omega) = 0, \quad \nabla\cdot u = 0$$

for which the time-average enstrophy is finite

$$\frac{1}{T}\int_0^T dt\cdot \frac{1}{2}\int d^2x\, |\omega(x,t)|^2 < +\infty$$

must conserve enstrophy, or

$$\Omega(t) = \frac{1}{2}\int d^2x\, |\omega(x,t)|^2 = (\text{const.})$$

It turns out that the basic ideas of this theory are not hard to explain in physical terms. We shall do so now.

First, we should observe that the Euler solutions under consideration have actually been proved to exist as zero-viscosity limits of 2D NS solutions. It has been shown in

R. J. DiPerna & A. J. Majda, "Concentrations in regularizations for 2D incompressible flow," Commun. Pure Appl. Math. XL 301–345 (1987)

that the solutions u_ν of the incompressible 2D NS equation

$$\partial_t u + (u\cdot\nabla)u = -\nabla p + \nu\Delta u$$

in the periodic domain $\mathbb{T}^2$ (and also, with more assumptions, in $\mathbb{R}^2$)

for which the initial data have $\omega_0 \in L^p$ with $p > 1$ converge (strongly in $L^{p'}$ for some $p' > 2$) to a distributional solution of 2D incompressible Euler

$$\partial_t \mathbf{u} + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\nabla p$$

with $\omega \in L^\infty([0,T]; L^p)$, i.e.

$$\sup_{t \in [0,T]} \int d^2x \, |\omega(x,t)|^p < +\infty .$$

This is equivalent to saying that $\mathbf{u} \in L^\infty([0,T]; W^{1,p})$ where $W^{1,p}$ is the pth-order Sobolev space of functions $\mathbf{u} \in L^p$ whose distribution derivatives are also in L^p:

$$\|\mathbf{u}\|_{W^{1,p}} = \|\mathbf{u}\|_{L^p} + \|\nabla \mathbf{u}\|_{L^p} .$$

As just a remark, we note that $W^{s,p}$ can be defined for any real $s \in \mathbb{R}$ and is a kind of "Besov space" closely related to those which we have discussed. In fact,

$$B_p^{s-\epsilon,\infty} \subset W^{s,p} = B_p^{s,p} \subset B_p^{s,\infty}, \quad \text{any } \epsilon > 0$$

Thus, $W^{s,p}$ is "nearly" the same as $B_p^{s,\infty}$. In any case, returning to the Euler solutions obtained by Di Perna – Majda (1987), it was shown that they conserve kinetic energy and furthermore are distributional solutions of the vorticity equation

$$\partial_t \omega + \nabla\cdot(\mathbf{u}\,\omega) = 0$$

when $p > \frac{4}{3}$. Unfortunately, it is not known whether there is a unique solution for given $\omega_0 \in L^p$ and, in principle, one might obtain different limits along different subsequences of viscosity $\nu_k \to 0$ as $k \to \infty$.

We now consider the enstrophy conservation properties of the solutions (or any solutions satisfying the basic bound $\omega \in L^\infty([0,T]; L^p)$). One of the key ideas of Di Perna-Lions was to consider the higher-order enstrophy invariants of Euler. Recall that

$$I_h(t) = \int d^2x\; h(\omega(\mathbf{x},t))$$

for any local function h, is a formal invariant of 2D Euler. In fact, if h is a convex function with $h'' > 0$, then for a 2D NS solution ω_ν

$$\partial_t h(\omega_\nu) + \nabla\cdot\left[\mathbf{u}_\nu h(\omega_\nu) - \nu \nabla h(\omega_\nu)\right]$$

$$= -\nu h''(\omega_\nu)|\nabla\omega_\nu|^2 \leq 0$$

and these higher-order invariants are expected to be dissipated at high wavenumbers together with the enstrophy, which corresponds to $h(\omega) = \frac{1}{2}|\omega|^2$.

These higher-order invariants have been a focus of attention since the work of Kraichnan in 1967, who remarked:

> "One important difference between two and three dimensions is the existence of an infinite number of local inviscid constants of motion in the former: the vorticity of each fluid element. This implies that inertial forces alone cannot produce universal statistical distributions in the similarity ranges, independent of the statistical distributions of the driving forces." — R. H. Kraichnan (1967)

We shall return to the above argument a little later, when we talk about intermittency in 2D turbulence. Returning to the Di Perna-Majda solutions with $\omega \in L^p$ we note that not all of these invariants need be well-defined, but only those for which h has pth-order growth. More precisely, we take $h \in \mathcal{H}_p$ with

$$\mathcal{H}_p = \{ h \in C^1(\mathbb{R}) : |h'(\omega)| \leq C|\omega|^{p-1} \text{ for } |\omega| \geq R, \text{ with some } C, R > 0 \}.$$

In that case the invariants $I_h(t) = \int d^2x \; h(\omega(x,t))$ are finite numbers.

What can be said about the conservation of these invariants for the DiPerna-Majda solutions? It appears *a priori* that they may suffer anomalies. For example, it is not hard to study the higher-order fluxes by arguments analogous to those of Duchon-Robert (2000), which we discussed in Turbulence I, Coursenotes, Section III (C). If we use the coarse-graining approach, then it is elementary to derive the higher-order balance relations for the DiPerna-Majda solutions:

$$\partial_t h(\bar{\omega}_\ell) + \nabla\cdot\left[\bar{u}_\ell h(\bar{\omega}_\ell) + \sigma_\ell h'(\bar{\omega}_\ell)\right] = + h''(\bar{\omega}_\ell)\, \sigma_\ell \cdot \nabla\bar{\omega}_\ell$$

for $h \in \mathcal{H}_p$ and we see that

$$Z_\ell^{(h)} = - h''(\bar{\omega}_\ell)\, \sigma_\ell \cdot \nabla\bar{\omega}_\ell$$

is the subscale flux of the invariant associated to h. It is interesting that

$$Z_\ell^{(h)} = h''(\bar{\omega}_\ell)\, Z_\ell$$

where Z_ℓ is the enstrophy flux. Now taking the limit

$\ell \to 0$ of both sides it can be shown using the arguments of Duchon-Robert-type that

$$\partial_t h(\omega) + \nabla\cdot\big(\mathbf{u}\, h(\omega)\big) = -Z^{(h)}(\omega)$$

with the anomaly term

$$Z^{(h)}(\omega) = \mathcal{D}\text{-}\lim_{\ell\to 0} Z^{(h)}_{\ell}(\omega).$$

Thus, the anomaly term represents flux to infinitesimally small length-scales.

The above results hold for any $h\in\mathcal{H}_p$ when $p>2$ and for any $h\in\mathcal{H}_{p'}$, $p'<2$ when $p=2$. The difference for $p=2$ is an annoyance that we need to discuss. Recall that the Duchon-Robert argument is based on showing that

$$\mathcal{D}\text{-}\lim_{\ell\to 0} h(\bar{\omega}_\ell) = h(\omega)$$

$$\mathcal{D}\text{-}\lim_{\ell\to 0} \bar{\mathbf{u}}_\ell\, h(\bar{\omega}_\ell) = \mathbf{u}\, h(\omega)$$

$$\mathcal{D}\text{-}\lim_{\ell\to 0} \boldsymbol{\sigma}_\ell\, h'(\bar{\omega}_\ell) = \mathbf{0}$$

so that $$\mathcal{D}\text{-}\lim\Big\{\partial_t h(\bar{\omega}_\ell) + \nabla\cdot\big[\bar{\mathbf{u}}_\ell h(\bar{\omega}_\ell) + \boldsymbol{\sigma}_\ell h'(\bar{\omega}_\ell)\big]\Big\} = \partial_t h(\omega) + \nabla\cdot\big[\mathbf{u}\, h(\omega)\big].$$

This turns out to be easy for $p>2$, because it is known that

$$W^{1,p} \subset L^\infty \qquad p>2,\ d=2$$

and thus u is <u>bounded</u> for $p>2$, so that

$$\| u\,(h(\bar{\omega}_\ell) - h(\omega))\|_{L^1} \leq \|u\|_{L^\infty} \|\omega\|_{L^p}^{p-1} \|\bar{\omega}_\ell - \omega\|_{L^p}$$

using $h \in \mathcal{H}_p$ and this vanishes as $\ell \to 0$. However, u can be <u>unbounded</u> for $p=2$! This can be seen for very simple examples, such as

$$f(r) = |\ln r|^a \quad , \quad 0<a<\tfrac{1}{2}$$

which is unbounded for $r\to 0$, but such that

$$\int_0^1 2\pi r\, dr\, |f'(r)|^2 = 2\pi a^2 \int_0^1 \frac{dr}{r\,|\ln r|^{2(a-1)}}$$

$$= 2\pi a^2 \int_0^\infty \frac{du}{u^{2(1-a)}}$$

$$(u = \ln r)$$

$$< \infty$$

It turns out that for $p=2$, the condition $u\,h(\omega) \in L^1$ is only guaranteed when $h \in \mathcal{H}_{p'}$ with $p'<2$. This explains the (annoying) restriction in the statement of the Duchon-Robert-type result above.

This annoying restriction means that one only gets the enstrophy balance

$$\partial_t \left(\tfrac{1}{2}|\omega|^2\right) + \nabla\cdot\left(\tfrac{1}{2}|\omega|^2 \mathbf{u}\right) = -Z(\omega)$$

with

$$Z(\omega) = \mathcal{D}\text{-}\lim_{\ell\to 0} Z_\ell(\omega)$$

in the case $p>2$. However, in that case one can prove by the same Duchon-Robert-type arguments that a form of the "$\frac{4}{5}$th-law in 2D" holds, i.e.

$$Z(\omega) = \mathcal{D}\text{-}\lim_{\ell\to 0} \frac{1}{4}\int d^2r\, (\nabla G)_\ell(\mathbf{r})\cdot \delta\mathbf{u}(\mathbf{r})\,|\delta\omega(\mathbf{r})|^2$$

for any smooth, compactly supported filter kernel G.

The detailed derivation of all these Duchon-Robert-type results can be found in

G. L. Eyink, "Dissipation in turbulent solutions of 2D Euler equations," Nonlinearity 14 787-802 (2001)

which also derives corresponding results for any $p>\frac{4}{3}$ (but whose statement is just slightly more complicated).

It turns out that all of the above anomalies actually vanish, as can be shown from the DiPerna-Lions theory. Their idea was to define the coarse-grained enstrophy flux in a slightly different way, via

$$\partial_t h(\bar{\omega}_\ell) + \nabla\cdot\left[\bar{u}_\ell h(\bar{\omega}_\ell)\right] = -h'(\bar{\omega}_\ell)\nabla\cdot\sigma_\ell$$

so that
$$\tilde{Z}_\ell^{(h)} = + h'(\bar{\omega}_\ell)\nabla\cdot\sigma_\ell$$

They also confined themselves, initially, to

$$h\in C^1(\mathbb{R}),\quad h'\in L^\infty(\mathbb{R})$$

For this class of h's they show directly that

$$(\star)\qquad \lim_{\ell\to 0}\|\tilde{Z}_\ell^{(h)}\|_{p/2} = 0,\quad p\geq 2,$$

i.e. the flux of all these invariants vanishes in L^1 (and thus the average over space vanishes). The estimate $(\star)$ is what DiPerna & Lions describe as the "fundamental technical tool" of their entire theory of renormalized solutions!

Let's show how to get the estimate (★). Since h' is (for now) assumed bounded

$$\|\tilde{Z}_\ell^{(n)}\|_{p/2} \leq \|h'\|_\infty \|\nabla\cdot\sigma_\ell\|_{p/2}$$

and we must show that

$$\lim_{\ell\to 0} \|\nabla\cdot\sigma_\ell\|_{p/2} = 0.$$

Let us first symmetrize the definition of σ_ℓ as

$$\sigma_\ell(\mathbf{u},\mathbf{v}) = \frac{1}{2}[\tau_\ell(\mathbf{u},\omega_\mathbf{v}) + \tau_\ell(\mathbf{v},\omega_\mathbf{u})]$$

for $\omega_\mathbf{u} = \nabla\times\mathbf{u}$, $\omega_\mathbf{v} = \nabla\times\mathbf{v}$, so that $\sigma_\ell = \sigma_\ell(\mathbf{u},\mathbf{u})$.

As in our earlier proof of UV-locality, we make use of the vector calculus identity

$$\frac{1}{2}[\mathbf{u}\times(\nabla\times\mathbf{v}) + \mathbf{v}\times(\nabla\times\mathbf{u})]$$
$$= -\frac{1}{2}\nabla\cdot(\mathbf{uv}+\mathbf{vu}) + \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{v})$$

to show that

$$\sigma_\ell^\perp(\mathbf{u},\mathbf{v}) = \frac{1}{2}\nabla\cdot\tau_\ell(\mathbf{u},\mathbf{v}) + \frac{1}{2}\nabla\tau_\ell(\mathbf{v},\mathbf{u}) + \frac{1}{2}\nabla\tau_\ell(u_i,v_i)$$

and thus

$$\nabla\cdot\sigma_\ell(\mathbf{u},\mathbf{v}) = \frac{1}{2}\nabla\nabla\overset{\perp}{:}\left(\tau_\ell(\mathbf{u},\mathbf{v}) + \tau_\ell(\mathbf{v},\mathbf{u})\right).$$

Again as in our proof of UV-locality we use the "shift trick" for generalized central moments to write $\nabla\cdot\sigma_\ell(\mathbf{u},\mathbf{v})$ entirely in terms of increments

$$
\begin{aligned}
\nabla\nabla^\perp : \ & T_\ell(\mathbf{u},\mathbf{v}) \\
= \ & \frac{1}{\ell^2}\Big[\int d^2r\,(\nabla\nabla^\perp G)_\ell(\mathbf{r}):\delta\mathbf{u}(\mathbf{r})\,\delta\mathbf{v}(\mathbf{r}) \\
& -\left(\int d^2r\,\cancelto{0}{(\nabla_i\nabla_j^\perp G)_\ell(\mathbf{r})\,\delta u_i(\mathbf{r})}\right)
\left(\int d^2r\,G_\ell(\mathbf{r})\,\delta v_j(\mathbf{r})\right) \\
& -\left(\int d^2r\,G_\ell(\mathbf{r})\,\delta u_i(\mathbf{r})\right)
\left(\int d^2r\,(\nabla_i\nabla_j^\perp G)_\ell(\mathbf{r})\,\delta v_j(\mathbf{r})\right) \\
& -\left(\int d^2r\,\cancelto{0}{(\nabla_i G_\ell)(\mathbf{r})}\,\delta u_i(\mathbf{r})\right)
\left(\int d^2r\,(\nabla_j^\perp G_\ell)(\mathbf{r})\,\delta v_j(\mathbf{r})\right) \\
& -\left(\int d^2r\,(\nabla_j^\perp G)_\ell(\mathbf{r})\,\delta u_i(\mathbf{r})\right)
\left(\int d^2r\,(\nabla_i G)_\ell(\mathbf{r})\,\delta v_j(\mathbf{r})\right)\Big]
\end{aligned}
\tag{$\star\star$}
$$

and likewise for $\mathbf{u} \leftrightarrow \mathbf{v}$. From this formula it follows by Hölder inequality that

$$\|\nabla\cdot\sigma_\ell(u,v)\|_{p/2} \leq \frac{(\text{const.})}{\ell^2}\int d^2r\ |(\nabla\nabla G)_\ell(r)| \times \|\delta u(r)\|_p\, \|\delta v(r)\|_p .$$

$$+ \text{ similar terms}$$

But recall

$$\|\delta u(r)\|_p \leq |r|\cdot\|\nabla u\|_p \leq |r|\cdot\|u\|_{W^{1,p}} .$$

It therefore holds that

$$\|\nabla\cdot\sigma_\ell(u,v)\|_{p/2} \leq C\|u\|_{W^{1,p}}\,\|v\|_{W^{1,p}}$$

with a constant C that depends only on the filter kernel G.

You will notice that this bound does not vanish as $\ell\to 0$! So how could DiPerna-Lions show that its limit is zero? They used a density argument.

Notice because of the identity (☆☆) it is possible to check that, for smooth u, the limit

$$\lim_{\ell\to 0}\nabla\cdot\sigma_\ell(u,u) = 0 \quad \text{pointwise} \quad (u \text{ smooth}).$$

This must be true because there can be no non-vanishing flux as $\ell \to 0$ if u is smooth. Thus, DiPerna–Lions approximate u by a smooth function u_ε so that

$$\|u_\varepsilon\|_{W^{1,p}} \leq \|u\|_{W^{1,p}}, \quad \|u - u_\varepsilon\|_{W^{1,p}} \xrightarrow[\varepsilon \to 0]{} 0.$$

For example, one may take $u_\varepsilon = \bar{u}_\varepsilon = G_\varepsilon * u$. Now, write

$$(\nabla\cdot\sigma_\ell)(u,u) = (\nabla\cdot\sigma_\ell)(u_\varepsilon + (u-u_\varepsilon), u_\varepsilon + (u-u_\varepsilon))$$

$$= (\nabla\cdot\sigma_\ell)(u_\varepsilon, u_\varepsilon)$$

$$+ \; 2(\nabla\cdot\sigma_\ell)(u_\varepsilon, u-u_\varepsilon)$$

$$+ \; (\nabla\cdot\sigma_\ell)(u-u_\varepsilon, u-u_\varepsilon)$$

so that

$$\|\nabla\cdot\sigma_\ell(u,u)\|_{p/2} \leq \|\nabla\cdot\sigma_\ell(u_\varepsilon,u_\varepsilon)\|_{p/2}$$

$$+ \; 2C\, \|u\|_{W^{1,p}} \|u-u_\varepsilon\|_{W^{1,p}}$$

$$+ \; C\, \|u-u_\varepsilon\|_{W^{1,p}}^2$$

For any $\delta > 0$ there is an $\varepsilon_\delta > 0$ so that

$$\|\nabla\cdot\sigma_\ell(u,u)\|_{p/2} \leq \|\nabla\cdot\sigma_\ell(u_\varepsilon,u_\varepsilon)\|_{p/2} + \delta$$

for $\varepsilon < \varepsilon_\delta$.

Hence, because u_ε is smooth

$$\limsup_{\ell \to 0} \| \nabla \cdot \sigma_\ell(u,u) \|_{p/2}$$

$$\leq \overset{\nearrow 0}{\lim_{\ell \to 0} \| \nabla \cdot \sigma_\ell(u_\varepsilon, u_\varepsilon) \|_{p/2}} + \delta$$

$$= \delta$$

Because $\delta > 0$ is arbitrary, it follows that

$$\lim_{\ell \to 0} \| \nabla \cdot \sigma_\ell(u,u) \|_{p/2} = 0.$$

QED!

This completes the proof of (☆), that $\| \tilde{\Sigma}_\ell^{(h)} \|_{p/2} \to 0$ as $\ell \to 0$ for $h \in C^1$ with h' bounded. For such types of h it follows that

$$\partial_t h(\omega) + \nabla \cdot (u\, h(\omega)) = 0$$

and there is no anomaly! But what about more general $h \in \mathcal{H}_p$ (including enstrophy for $p > 2$)? Di Perna & Lions show that this follows from the above result by approximating general $h \in \mathcal{H}_p$ with the above type of h. This is a bit technical but here is the argument:

For any $h \in \mathcal{H}_p$ and $M > 0$ define

$$h_M(\omega) = h(0) + \int_0^\omega d\bar{\omega}\, \mathrm{sign}(h'(\bar{\omega})) \min\{|h'(\omega)|, M\}$$

so that $h_M \in C^1$ with

$$h_M'(\omega) = \mathrm{sign}(h'(\omega)) \min\{|h'(\omega)|, M\}$$

and $|h_M'(\omega)| \leq M$. Also,

$$\lim_{M \to \infty} h_M(\omega) = h(\omega) \qquad \text{pointwise in } \omega$$

by dominated convergence, using the properties of h.
But furthermore $h(\omega) = h(0) + \int_0^\omega d\bar{\omega}\, \mathrm{sign}(h'(\bar{\omega}))\, |h'(\bar{\omega})|$,
so that

$$h(\omega) - h_M(\omega) = \int_0^\omega d\bar{\omega}\, \mathrm{sign}(h'(\bar{\omega})) \max\{|h'(\bar{\omega})| - M, 0\}$$

and

$$\begin{aligned} |h(\omega) - h_M(\omega)| &\leq \int_0^\omega d\bar{\omega}\, \max\{|h'(\bar{\omega})| - M, 0\} \\ &\leq \int_0^\omega d\bar{\omega}\, |h'(\bar{\omega})| \\ &\leq C_1 + C_2 |\omega|^p \end{aligned}$$

by the properties of $h \in \mathcal{H}_p$.

Now, since $h_M \in C^1$ and h_M' is bounded, it follows from (✫) that

$$\partial_t h_M(\omega) + \nabla \cdot (u h_M(\omega)) = 0$$

But for $p > 2$ we know that $u \in L^\infty$ and thus $u|\omega|^p \in L^1$. Hence, one can use dominated convergence to show that

$$\|u h(\omega) - u h_M(\omega)\|_{L^1}$$
$$\leq \int d^2x \, |u(x)| \, |h(\omega(x)) - h_M(\omega(x))|$$
$$\longrightarrow 0 \quad \text{as} \quad M \longrightarrow 0$$

It follows that we can take the limit as $M \to \infty$ and obtain

$$\partial_t h(\omega) + \nabla \cdot (u h(\omega)) = 0$$

for all $h \in \mathcal{H}_p$. The same argument works for $p = 2$ except one is then restricted to $h \in \mathcal{H}_{p'}$ for some $p' < 2$, so that $u|\omega|^{p'} \in L^1$.

Putting this result together with the Duchon-Robert-type result, we see that for $p > 2$

$$\mathcal{D}\text{-}\lim_{\ell \to 0} \tilde{Z}_\ell^{(h)} = 0$$

for any $h \in \mathcal{H}_p$ and also

$$\mathcal{D}\text{-}\lim_{\ell \to 0} Z_\ell^{(h)} = 0$$

if $h \in \mathcal{H}_p \cap C^2$. This holds in particular if $h(\omega) = \frac{1}{2}|\omega|^2$ and then

$$\mathcal{D}\text{-}\lim_{\ell \to 0} Z_\ell = 0$$

Likewise,

$$\mathcal{D}\text{-}\lim_{\ell \to 0} \frac{1}{4} \int d^2r \, (\nabla G)_\ell(r) \cdot \delta u(r) \, |\delta\omega(r)|^2 = 0 .$$

If $p = 2$, then the first two results still hold if one takes instead $h \in \mathcal{H}_{p'}$ and $h \in \mathcal{H}_{p'} \cap C^2$, respectively, for $p' < 2$. However, for $p = 2$ one cannot obtain a local conservation law for the enstrophy itself!

On the other hand, for $p=2$ it is still possible to obtain global conservation of enstrophy. Note that for $h(\omega) = \frac{1}{2}|\omega|^2$

$$h_M(\omega) = \begin{cases} \frac{1}{2}|\omega|^2 & |\omega| < M \\ M|\omega| - \frac{1}{2}M^2 & |\omega| > M \end{cases}$$

and $h_M(\omega) \uparrow h(\omega)$ as $M \to \infty$. Integrating

$$\partial_t h_M(\omega) + \nabla \cdot (u\, h_M(\omega)) = 0$$

over space gets rid of the dangerous $u h_M(\omega)$ term

$$\frac{d}{dt} \int d^2x \, h_M(\omega(\mathbf{x},t)) = 0.$$

If one also integrates in time from 0 to t, one gets

$$\int d^2x \, h_M(\omega(\mathbf{x},t)) = \int d^2x \, h_M(\omega_0(\mathbf{x})).$$

Taking the limit $M \to \infty$ and using monotone convergence yields

$$\int d^2x \, h(\omega(\mathbf{x},t)) = \int d^2x \, h(\omega_0(\mathbf{x}))$$

for $h = \frac{1}{2}|\omega|^2$ and $p=2$. Thus, finiteness of enstrophy implies that enstrophy is (globally) conserved.

We have now completed our technical discussion of the DiPerna-Lions results as applied to 2D Euler, with just few more remarks.

Remark #1. All of the above results still hold if one adds a smooth body-force $\mathbf{f}$ to the momentum equation

$$\partial_t \mathbf{u} + \nabla\cdot(\mathbf{u}\mathbf{u}) = -\nabla p + \mathbf{f}$$

or, equivalently, a source $g = \nabla\times\mathbf{f}$ to the vorticity equation

$$\partial_t \omega + \nabla\cdot(\mathbf{u}\omega) = g .$$

The coarse-grain balance term now contains a contribution from the force

$$\partial_t h(\bar{\omega}_\ell) + \nabla\cdot(\bar{\mathbf{u}}_\ell h(\bar{\omega}_\ell)) = -h'(\bar{\omega}_\ell)\nabla\cdot\sigma_\ell + h'(\bar{\omega}_\ell)\bar{g}_\ell$$

but for smooth g as usually considered in turbulence theory (and even rougher things) it is easy to show for $h\in\mathcal{H}_p$ that

$$\mathcal{D}\text{-}\lim_{\ell\to 0} h'(\bar{\omega}_\ell)\bar{g}_\ell = h'(\omega)g .$$

Remark #2. The DiPerna–Lions theory gives far more detailed information about the 2D Euler solutions and their Lagrangian representation. For a general linear transport equation

$$\partial_t \theta + (\mathbf{u}\cdot\nabla)\theta = g, \quad \nabla\cdot\mathbf{u}=0 \tag{1}$$

with $\mathbf{u}\in W^{1,1}$ and θ only measurable, DiPerna & Lions showed that there always exist unique "renormalized solutions" (for given $\mathbf{u}$ field), defined by the condition that

$$\partial_t h(\theta) + \nabla\cdot(\mathbf{u}h(\theta)) = h'(\theta)g$$

for all $h\in\mathcal{H}_2$ that are also bounded and vanish at the origin ($h(0)=0$). Notice that our previous discussion has shown that the DiPerna–Majda solutions of 2D Euler for $p>2$ are, in fact, "renormalized solutions" of the 2D vorticity equation! DiPerna & Lions furthermore showed that the general ODE

$$\frac{d\mathbf{X}}{dt} = \mathbf{u}(\mathbf{X},t), \quad \mathbf{X}(\mathbf{a},0)=\mathbf{a} \tag{2}$$

with $\mathbf{u}\in W^{1,1}$ has a unique "renormalized solution" defined by the condition that

$$\frac{d}{dt}\beta(\mathbf{X}) = u(\mathbf{X},t)\cdot\nabla\beta(\mathbf{X}), \quad \beta(\mathbf{X}(a,0)) = \beta(a)$$

for all $\beta \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ such that $|\beta(x)|$ and $\dfrac{|\nabla\beta(x)|}{1+|x|}$ are bounded for $x \in \mathbb{R}^d$. These flows are volume-preserving when the velocity field is incompressible, $\nabla\cdot u = 0$. Finally, the two notions of "renormalized solutions" are related by the fact that

$$(3) \qquad \theta(x,t) = \theta_0(\mathbf{X}(x,-t)),$$

which means that the standard method of characteristics holds for this class of "renormalized solutions."

These results give a Lagrangian interpretation of the conservation properties of the DiPerna-Majda solutions for $p>2$. Because of the property (3), for any $h \in \mathcal{H}_p$

$$\begin{aligned} I_h(t) &= \int d^2x \, h(\omega(x,t)) \\ &= \int d^2x \, h(\omega_0(\mathbf{X}(x,-t))) \\ &= \int d^2x \, h(\omega_0(x)) = I_h(0) \end{aligned}$$

using the volume-preserving property of $\mathbf{X}$. It is very tempting to conjecture that hypothetical 2D Euler solutions of the "Kraichnan - Batchelor type" with $\omega \in B_2^{0,\infty}$ will dissipate enstrophy because of the non-uniqueness of the flows. This is suggested partly by the analogy with the Kraichnan model of turbulent passive scalar, where the mechanism of anomalous dissipation of such integrals as $I_h(t)$ is the non-uniqueness of Lagrangian flows, or "spontaneous stochasticity." Also, DiPerna & Lions have shown, by example, that uniqueness of flows can fail if the velocity field is just a bit less smooth than $W^{1,1}$, even $\mathbf{u} \in W^{s,1}$ for any $s<1$. Because of the standard embedding results for $d=2$

$$B_2^{1,\infty} \subset W^{s,1} \qquad \text{any } s<1$$

one can see the velocities $\mathbf{u} \in B_2^{1,\infty}$ (or vorticities $\omega \in B_2^{0,\infty}$) appropriate to the Kraichnan - Batchelor theory lie just outside the class where DiPerna - Lions work guarantees the existence of unique flows.

Remark #3. The argument we have presented above is based on estimates of enstrophy flux rather than viscous enstrophy dissipation. One can obtain results on viscous dissipation á la Duchon – Robert, if one assumes that the 2D NS solution ω_ν converges to the 2D Euler solution ω in an L^p norm for any $p > 2$ as $\nu \to 0$:

$$\lim_{\nu \to 0} \| \omega - \omega_\nu \|_p = 0 \qquad \text{some } p > 2$$

then it is not hard to show that

$$\mathcal{D}\text{-}\lim_{\nu \to 0} \tfrac{1}{2} \omega_\nu^2 = \tfrac{1}{2} \omega^2$$

$$\mathcal{D}\text{-}\lim_{\nu \to 0} \tfrac{1}{2} \omega_\nu^2 \mathbf{u}_\nu = \tfrac{1}{2} \omega^2 \mathbf{u}$$

and so

$$\mathcal{D}\text{-}\lim_{\nu \to 0} \left\{ \partial_t \left(\tfrac{1}{2} \omega_\nu^2 \right) + \nabla \cdot \left[\tfrac{1}{2} \omega_\nu^2 \mathbf{u}_\nu - \nu \nabla \left(\tfrac{1}{2} \omega_\nu^2 \right) \right] \right\}$$

$$= \partial_t \left(\tfrac{1}{2} \omega^2 \right) + \nabla \cdot \left(\tfrac{1}{2} \omega^2 \mathbf{u} \right).$$

It follows immediately that

$$\partial_t \left(\tfrac{1}{2} \omega^2 \right) + \nabla \cdot \left(\tfrac{1}{2} \omega^2 \mathbf{u} \right) = -D(\omega)$$

with

$$D(\omega) = \mathcal{D}\text{-}\lim_{\nu \to 0} \nu |\nabla \omega_\nu|^2 \geq 0 .$$

However, because the limiting solution $\omega \in L_p$ with $p > 2$, the Di Perna-Lions theory applies and one can infer in fact that the above inequality is strict equality

$$\mathcal{D}\text{-}\lim_{\nu \to 0} \nu |\nabla \omega_\nu|^2 = 0 \ !$$

Thus, viscous dissipation also vanishes on the above assumptions. For related results on vanishing enstrophy dissipation in decaying 2D turbulence, by an entirely different approach, see

C. V. Tran & D. G. Dritschel, "Vanishing enstrophy dissipation in two-dimensional Navier-Stokes turbulence in the inviscid limit," J. Fluid. Mech. 559 107-116 (2006)

and

D. G. Dritschel et al., "Revisiting Batchelor's theory of two-dimensional turbulence," J. Fluid Mech. 591 379-391 (2007)

Remark #4. The above results from Di Perna–Lions theory show that no 2D Euler solutions with an enstrophy spectrum of the form

$$\Omega(k) \sim k^{-(1+2s)}, \quad 0<s<1 \tag{4}$$

can have a non-vanishing enstrophy flux to infinitely high wavenumbers. In fact, such a spectrum implies

$$\omega \in B_2^{s,\infty} \subset L_p \quad \text{for} \quad p = \frac{2}{1-s} > 2$$

by Besov embedding theorem. This rules out a wide class of alternatives to the Batchelor–Kraichnan k^{-3} spectrum for forced steady-states, including those of Saffman, Moffatt and Polyakov.

The contradiction is particularly striking for Polyakov's conformal theory, because he claimed to construct exact inviscid solutions of the stationary Friedman–Keller hierarchy for vorticity correlations of 2D Euler equations, having spectrum (4) and also having a non-vanishing enstrophy flux to arbitrarily high wavenumbers! Where was the error? There are, in fact, many incomplete "details" in Polyakov's construction. E.g. he does not solve the problem of "matching" his inertial-range conformal models to the forcing scale (infrared problem) and to the viscous scale (ultraviolet problem). However, we would like to focus here on the realizability problem. It is known that exact solutions of hierarchy equations can fail to correspond to statistical ensembles of solutions of the underlying PDE (here, 2D Euler). E.g. see

G.L. Eyink & J. Xin, "Self-similar decay in the Kraichnan model of a passive scalar," J. Stat. Phys. 100 679-741 (2000)

which gives an example of "parasitic solutions" of the hierarchy equations — in an example where they are even closed! — which are not realizable by the underlying statistical model. This issue of "realizability" was long emphasized by R. H. Kraichnan as fundamental in statistical approaches to turbulence, e.g.

R. H. Kraichnan, "Variational method in turbulence theory," Phys. Rev. Lett. 42 1263-1266 (1979)

R. H. Kraichnan, "Realizability inequalities and closed moment equations," Ann. N.Y. Acad. Sci. 357 37-46 (1980)

The first paper, in particular, contain some very interesting ideas on how to construct statistically realizable approximations by a variational method.

The only obvious way to account for the failure of Polyakov's approach is a realizability problem. In fact, even without solving the matching problem, he claimed to construct a statistical ensemble of 2D Euler solutions whose properties contradict those shown to hold for <u>any</u> solution of the 2D Euler equations by the DiPerna & Lions theory. This seems to imply that his conformal solutions of the Friedman-Keller hierarchy for vorticity <u>are not realized by any Euler solutions</u>! The same problem is possible for any approach which only yields solutions of the stationary Friedman-Keller hierarchy without establishing their realizability by 2D Euler solutions. E.g.

M. Flohr, "2-dimensional conformal turbulence: Yet another conformal field theory solution," arXiv: hep-th/9606130

shows by an approach very similar to Polyakov's that one may obtain conformal "solutions" of the 2D Euler hierarchy consistent with the Kraichnan-Batchelor k^{-3} energy spectrum. But realizability is not established.

On the other hand, Polyakov's proposal that 2D turbulent cascades may possess conformal symmetry of some type may still be correct. Recently, numerical evidence for this has been obtained in the work of

D. Bernard et al., "Conformal invariance in two-dimensional turbulence," Nat. Phys. 2 124-128 (2006)

Rather than the forward enstrophy cascade, this work studied the inverse energy cascade and also followed a very different approach than that of Polyakov. Instead of focusing on the conformal properties of the random fields themselves, they focussed on the conformal properties of their level curves (e.g. the vorticity isolines). These were found to have the same properties as those of a well-known conformal model, 2D critical percolation. Similar results have so far been obtained in numerical studies of several other 2D turbulence models. Unfortunately, there has so far been little progess analytically in understanding these observations or in exploiting them for a deeper understanding of 2D turbulence.