(c) Forced Steady-State 2D Turbulence

We are now going to consider the standard problem (but not for us!) of statistically stationary measures for forced 2D fluids, described, for example, by the 2D NS dynamics with an added body force

$$(*) \qquad \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\Delta\mathbf{u} + \mathbf{f}$$

It is mathematically very attractive to take the force to be a Gaussian random vector field that is white in time, with covariance

$$\langle f_i(\mathbf{x},t) f_j(\mathbf{x}',t')\rangle = 2F_{ij}(\mathbf{x}-\mathbf{x}')\,\delta(t-t').$$

The dependence only on $\mathbf{x}-\mathbf{x}'$ means that the dynamical model is space-translation invariant and thus an invariant measure, if it is unique, must also be statistically homogeneous. The most attractive feature of the white-noise in time property is that <u>injection rates of energy and enstrophy are independent of the flow</u>.

For those who know Ito calculus, it is a simple exercise to show that the mean rate of injection of energy, averaging over the white-noise force, is

$$\boxed{\varepsilon = F_{ii}(0)}$$

Likewise the mean rate of injection of enstrophy can be inferred from the curl of (*)

$$\partial_t \omega + (u\cdot\nabla)\omega = \nu\Delta\omega + g$$

with $g = \nabla^{\perp}\cdot f$ so that

$$\langle g(x,t)\, g(x',t')\rangle = 2\,G(x-x')\,\delta(t-t')$$

with $G(r) = -\nabla_i^{\perp}\nabla_j^{\perp} F_{ij}(r)$. One finds by Ito rule that the injection rate of enstrophy is

$$\boxed{\eta = G(0)}$$

We give here a derivation of these results using not Ito calculus but a different method which applies to a larger range of Gaussian random forces, based on the so-called <u>Gaussian integration-by parts identity</u> (or Furutsu-Donsker-Novikov identity; see the monograph of U. Frisch (1995), section 4.1)

This identity is very easy to derive for a finite-dimensional normal random vector $\mathbf{x} \in \mathbb{R}^d$ with the probability density

$$N(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^d \operatorname{Det} \mathbf{C}}} \exp\left[-\frac{1}{2}\mathbf{x}^T \mathbf{C}^{-1} \mathbf{x}\right]$$

where, for simplicity, we assume $\mathbf{x}$ has zero mean and $C_{ij} = \langle x_i x_j \rangle$ is the <u>covariance matrix</u>.

The basic property of the Gaussian is that

$$\frac{\partial}{\partial x_i} N(\mathbf{x}) = -(\mathbf{C}^{-1}\mathbf{x})_i N(\mathbf{x}).$$

Thus, for any differentiable function f on $\mathbb{R}^d$

$$\begin{aligned}
\langle x_i f(\mathbf{x}) \rangle &= \int x_i f(\mathbf{x}) N(\mathbf{x})\, d^d x \\
&= \sum_j C_{ij} \int (\mathbf{C}^{-1}\mathbf{x})_j f(\mathbf{x}) N(\mathbf{x})\, d^d x \\
&= \sum_j C_{ij} \int f(\mathbf{x}) \cdot -\frac{\partial}{\partial x_j} N(\mathbf{x})\, d^d x \\
&= \sum_j C_{ij} \int \frac{\partial f}{\partial x_j}(\mathbf{x}) N(\mathbf{x})\, d^d x
\end{aligned}$$

by integration by parts. This gives

$$\boxed{\langle x_i f\rangle = \sum_j C_{ij}\left\langle \frac{\partial f}{\partial x_j}\right\rangle}$$

the Gaussian integration by parts identity. It turns out that this result is true not only for finite dimensional random vectors, but also for random fields. If we apply it to the forced 2D NS, we find that the energy input term

$$\varepsilon = \langle u(x,t)\cdot f(x,t)\rangle$$

can be evaluated for <u>any</u> Gaussian random force as

$$\varepsilon = 2\int_{-\infty}^{t} dt' \int_D d^2x'\, F_{ij}(x,t;x',t')\, G_{ij}(x,t;x',t')$$

where

$$2F_{ij}(x,t;x',t') = \langle f_i(x,t)\, f_j(x',t')\rangle$$

and

$$G_{ij}(x,t;x',t') = \left\langle \frac{\delta u_i(x,t)}{\delta f_j(x',t')}\right\rangle$$

is the <u>mean response function</u> of the velocity to the random force. By causality, it vanishes for $t < t'$, so we integrate only over $t' < t$ above.

One can see that in general the input depends on the statistics of the flow through the response function. However, if f is white-noise in time so that

$$F_{ij}(x,t;x',t') = F_{ij}(x,x';t)\,\delta(t-t')$$

then we can see from the relation

$$u_i(x,t) = u_i(x,t_0) + \int_{t_0}^{t} ds \left[-P^{\perp}(u(x,s)\cdot\nabla)u(x,s) + f(x,s) \right]$$

that

$$\frac{\delta u_i(x,t^+)}{\delta f_j(x',t)} = \delta_{ij}\,\delta^2(x-x')$$

and then we get

$$\varepsilon = 2\int_{-\infty}^{t} dt' \int_D d^2x'\, F_{ij}(x,x';t)\,\delta(t-t') \times \delta_{ij}\,\delta^2(x-x')$$

$$= F_{ii}(x,x;t)$$

where we used $\int_{-\infty}^{t} dt'\,\delta(t-t') = \frac{1}{2}$, since only half the delta function is included in the integration range. For a homogeneous, stationary force we recover the result $\varepsilon = F_{ii}(0)$. A similar calculation gives the enstrophy input.

For the white noise case we therefore get the steady-state energy balance

$$\varepsilon = F_{ii}(0) = \nu \langle |\nabla u|^2 \rangle$$

and steady-state enstrophy balance

$$\eta = G(0) = \nu \langle |\nabla \omega|^2 \rangle$$

with inputs balanced by viscous dissipation. If we work in the periodic domain $D = \mathbb{T}^2 = [0, 2\pi]^d$, these relations can be represented by Fourier coefficients

$$\varepsilon = \sum_{\mathbf{k}} \hat{F}_{ii}(\mathbf{k}) = \sum_{\mathbf{k}} \nu |\mathbf{k}|^2 \langle |\hat{u}(\mathbf{k})|^2 \rangle$$

and

$$\eta = \sum_{\mathbf{k}} |\mathbf{k}|^2 \hat{F}_{ii}(\mathbf{k}) = \sum_{\mathbf{k}} \nu |\mathbf{k}|^4 \langle |\hat{u}(\mathbf{k})|^2 \rangle$$

where. notice that $\hat{F}_{ii}(\mathbf{k}) \geq 0$ for all $\mathbf{k}$, as the Fourier transform of a positive-definite covariance. We have used here the assumption that $F_{ij}(\mathbf{r})$ is divergence-free on each index i, j, as would be true if the body force $\mathbf{f}$ is solenoidal.

The problem considered by Kraichnan (1967) was one of a spectrally localized force, non-vanishing only for $|k| \approx k_f$. It is only relatively recently that unique invariant measures have been mathematically proved to exist for white-noise forced 2D NS in $\mathbb{T}^2$ with force compactly supported in Fourier space. See

M. Hairer & J. C. Mattingley, "Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing," Ann. Math. <u>164</u> 993–1032 (2006)

and, for the generalization to a wider class of PDE's,

M. Hairer & J. C. Mattingley, "A theory of hypoellipticity and unique ergodicity for semilinear stochastic PDE's," Electronic J. Prob. <u>16</u> 658–738 (2011).

It turns out, however, that forced 2D Navier-Stokes equation is the <u>wrong problem</u> to study to investigate Kraichnan ideas as a steady-state law!, It is also the wrong system to study from the point of view of geophysical applications. Let us explain this point.

Necessity of large-scale damping. Kraichnan (1967) was interested in the limit of 2D turbulence at high Reynolds number and predicted a state with finite energy in the limit of infinite Re. However, we now show that the steady-state of forced 2D NS has energy diverging to infinity as $\nu \to 0$! The proof is based on the energy and enstrophy balances and the simple Cauchy-Schwartz inequality

$$\sum_{\mathbf{k}} |\mathbf{k}|^2 \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle \leq \sqrt{\sum_{\mathbf{k}} \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle \sum_{\mathbf{k}} |\mathbf{k}|^4 \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle}$$

so that

$$E = \frac{1}{2} \sum_{\mathbf{k}} \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle \geq \frac{1}{2\nu} \frac{\left(\nu \sum_{\mathbf{k}} |\mathbf{k}|^2 \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle \right)^2}{\nu \sum_{\mathbf{k}} |\mathbf{k}|^4 \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle} = \frac{1}{2\nu} \frac{\varepsilon^2}{\eta} \longrightarrow \infty \text{ as } \nu \to 0!$$

If we assume a spectrally localized force at wavenumbers near k_f, then $\eta \cong k_f^2 \varepsilon$ and we can write this as

$$E \geq \frac{(\text{const.})}{2\nu} \frac{\varepsilon}{k_f^2}$$

Let us define a "dissipation wavenumber" k_{dis} by

$$\varepsilon = 2\nu k_{dis}^2 E,$$

so that the previous inequality becomes

$$k_{dis} \leq k_f .$$

We see that the energy is piling up at a wavenumber $k_{dis} \leq k_f$ until the amplitude of the energy is sufficient to provide a dissipation which balances the input ε. This is consistent with Kraichnan's idea of an <u>inverse cascade</u> of energy to low wavenumbers. However, Kraichnan did <u>not</u> consider a statistical steady-state regime, but instead a transient regime in which energy first flows to low wavenumber and then begins to pile up there. Thus, 2D NS is not the right model to consider to model Kraichan's inverse cascade as a statistical steady state!

To realize Kraichnan's ideas in a statistical steady-state one must add a large-scale damping at wavenumbers $k_{dis} < k_f$, to dissipate (more effectively than viscosity) the energy flow to low wavenumbers. This may be done by adding to the NS equation an inverse Laplacian dissipation, such as

$$\partial_t u + (u\cdot\nabla)u = -\nabla p + \nu\Delta u - \alpha_p(-\Delta)^{-p} u .$$

This gives, in Fourier space, a damping term to the energy balance

$$-\alpha_p \sum_{k\neq 0} \frac{\langle|\hat{u}(k)|^2\rangle}{|k|^p}$$

which is most effective at low wavenumbers. This should "eat up" the energy that flows to low wavenumbers and prevent it from piling up there. It should now be possible to take the limit $\nu \to 0$ and obtain a statistical steady state with finite energy E, as expected in Kraichnan's picture. From now on we study only the NS equation with such large-scale damping terms added in addition to viscosity.

As a matter of fact, such large-scale damping is a feature of both laboratory experiments on 2D turbulence and in geophysical flows. In soap films it arises from interaction with the air and in fluid layers it arises from interaction with both the air and the solid bottom. In geophysics this is generally modeled by a <u>linear Ekman friction</u> of the previous form with $p=0$, or

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\Delta\mathbf{u} - \alpha\mathbf{u}$$

or

$$\partial_t \omega + (\mathbf{u}\cdot\nabla)\omega = \nu\Delta\omega - \alpha\omega$$

See

J. Pedlosky <u>Geophysical Fluid Dynamics</u> 2nd Ed. (Springer, 1987), Section 4.6

for a detailed discussion. Briefly, α represents a vortex stretching effect of Eckman "suction" induced by tiny vertical velocities at the top of the Ekman boundary layer of thickness $(\nu/\Omega)^{1/2}$.

Spectral balances of energy and enstrophy. We obtain more information by summing the local energy conservation equation in wavenumber space

$$(*) \quad \partial_t\left(\tfrac{1}{2}|\hat{\mathbf{u}}(\mathbf{k})|^2\right) = \sum_{\mathbf{pq}} T_{\mathbf{kpq}} + \sum_{\mathbf{k}} \mathrm{Re}(\hat{\mathbf{u}}_{\mathbf{k}}^* \cdot \hat{\mathbf{f}}_{\mathbf{k}})$$
$$- \nu|\mathbf{k}|^2|\hat{\mathbf{u}}(\mathbf{k})|^2 - \alpha_p|\mathbf{k}|^{-2p}|\hat{\mathbf{u}}(\mathbf{k})|^2$$

over different ranges of wavenumber and averaging. Summing over $|\mathbf{k}|<K$ with $K>k_f$ gives

$$\boxed{\Pi(K) = \varepsilon - \sum_{|\mathbf{k}|<K} D(\mathbf{k})\langle|\hat{\mathbf{u}}(\mathbf{k})|^2\rangle, \quad K>k_f}$$

with $D(\mathbf{k}) = \nu|\mathbf{k}|^2 + \alpha_p|\mathbf{k}|^{-2p}$ and

$$\Pi(K) = -\sum_{\substack{|\mathbf{k}|<K \\ \mathbf{pq}}} \langle T_{\mathbf{kpq}}\rangle$$

the <u>mean spectral energy flux</u>, which measures the rate of flux of energy out of the wavenumbers $|\mathbf{k}|<K$ due to nonlinear interactions. Because $\sum_{\mathbf{kpq}} T_{\mathbf{kpq}} = 0$ (at finite Reynolds) due to detailed conservation of energy, one can also write

$$\Pi(K) = + \sum_{\substack{|\mathbf{k}| \geq K \\ pq}} \langle T_{\mathbf{k}pq} \rangle ,$$

representing a flux of energy into the wavenumbers $|\mathbf{k}| \geq K$. Using the overall balance

$$\varepsilon = \sum_{\mathbf{k}} D(\mathbf{k}) \langle |\hat{u}(\mathbf{k})|^2 \rangle$$

we can also write

$$\boxed{\Pi(K) = \sum_{|\mathbf{k}| \geq K} D(\mathbf{k}) \langle |\hat{u}(\mathbf{k})|^2 \rangle , \quad K > k_f}$$

This shows that $\Pi(K) \geq 0$ for $K > k_f$. Instead summing over $|\mathbf{k}| < K$ with $K < k_f$, then the input term gives no contribution, so that

$$\boxed{\Pi(K) = -\sum_{|\mathbf{k}| < K} D(\mathbf{k}) \langle |\hat{u}(\mathbf{k})|^2 \rangle , \quad K < k_f}$$

and thus $\Pi(K)$ must be negative for $K < k_f$. The negative sign implies that energy must flow to low wavenumbers. In both cases, the flow of energy is away from the source at wavenumber k_f.

The above steps can be repeated for enstrophy by first multiplying $(*)$ by $|\mathbf{k}|^2$ and then performing the same operations as above. With the <u>mean spectral enstrophy flux</u> defined by

$$Z(K) = -\sum_{\substack{|\mathbf{k}|<K \\ \mathbf{pq}}} |\mathbf{k}|^2 \langle T_{\mathbf{kpq}} \rangle$$

$$= + \sum_{\substack{|\mathbf{k}|\geq K \\ \mathbf{pq}}} |\mathbf{k}|^2 \langle T_{\mathbf{kpq}} \rangle ,$$

one can readily show that

$$\boxed{\begin{aligned} Z(K) &= \eta - \sum_{|\mathbf{k}|<K} |\mathbf{k}|^2 D(\mathbf{k}) \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle \\ &= \sum_{|\mathbf{k}|\geq K} |\mathbf{k}|^2 D(\mathbf{k}) \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle , \quad K > k_f \end{aligned}}$$

and

$$\boxed{Z(K) = -\sum_{|\mathbf{k}|<K} |\mathbf{k}|^2 D(\mathbf{k}) \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle , \quad K < k_f}$$

Thus, it is again true for enstrophy flux that $Z(K) \geq 0$ for $K > k_f$ and $Z(K) \leq 0$ for $K < k_f$.

We now begin to address the important issue: to which scales are energy and enstrophy mainly transferred? It turns out that one can closely follow the reasoning of Fjørtoft also for the case where energy and enstrophy are injected by the force rather than initial data only. In particular, we can derive Fjørtoft-type bounds on fluxes, using the positivity of the dissipation function $D(k)$. For $K > k_f$

$$\Pi(K) = \sum_{|k| \geq K} D(k) \langle |\hat{u}(k)|^2 \rangle$$

$$\leq \frac{1}{K^2} \sum_{|k| \geq K} |k|^2 D(k) \langle |\hat{u}(k)|^2 \rangle$$

$$= \frac{1}{K^2} Z(K)$$

or

$$\boxed{\Pi(K) \leq \frac{1}{K^2} Z(K), \quad K > k_f}$$

Likewise, by an identical argument, one gets

$$\boxed{|Z(k)| \leq k^2 |\Pi(k)|, \quad k < k_f}$$

where $Z(k)$ and $\Pi(k)$ are both negative. We shall call these the Fjørtoft-type flux bounds, which were derived in

G.L. Eyink, "Exact results on stationary turbulence in 2D: consequences of vorticity conservation," Physica D 91 97–142 (1996), section 3.2.3

in a slightly weaker form (e.g. $\Pi(k) \leq \frac{\eta}{k^2}$) and in the above sharper form in

S. Danilov, "Non-universal features of forced 2D turbulence in the energy and enstrophy ranges," Discrete & Continuous Dynamical Systems B 5 67–78 (2005)

We can rewrite them slightly using

$$\eta \cong k_f^2 \varepsilon$$

for a spectrally localized forcing function, as

$$\frac{\Pi(k)}{\varepsilon} \leq (\text{const.}) \left(\frac{k_f}{k}\right)^2 \frac{Z(k)}{\eta}, \quad k > k_f$$

and

$$\frac{|Z(k)|}{\eta} \leq (\text{const.}) \left(\frac{k}{k_f}\right)^2 \frac{|\Pi(k)|}{\varepsilon}, \quad k < k_f$$

These results tell us that it is impossible for $\Pi(k) = f \cdot \varepsilon$ for a fixed fraction $f \in (0,1)$ at arbitrarily large $k > k_f$, because this would imply $Z(k) > \eta$ at large enough k! Likewise, it is impossible for $Z(k) = -f \cdot \eta$ for fixed $f \in (0,1)$ at arbitrarily small $k < k_f$, because this would imply $\Pi(k) > \varepsilon$ at small enough k. This is again the mutual effect of spectral blocking of the invariants of energy and enstrophy on the transfer of the partner invariant. The only consistent possibility seems to be that most of the enstrophy input η is carried by $Z(k)$ to high wavenumbers where it is disposed of by viscosity and that most of the energy input ε is carried by $\Pi(k)$ to low wavenumbers where it is disposed of by the large-scale damping.

There is a third possibility, however. It could be that the inputs ε and η "pile up" in the range of wavenumbers $k \approx k_f$, because nonlinear transfers are not sufficiently effective to carry them away.
Cannot reach wavenumbers
In that case, the enstrophy input η where viscosity is effective, nor the energy input ε reach low wavenumbers where the large-scale damping is effective. In such a situation the energy & enstrophy will pile-up in the forcing range $k \cong k_f$ until the dissipation at these wavenumbers is sufficient to absorb the inputs. This will require an energy

$$E > \text{const.} \min\left\{ \frac{\varepsilon}{\nu k_f^2}, \frac{\varepsilon k_f^{2p}}{\alpha_p} \right\}$$

depending upon whether viscosity ν or large-scale damping α_p is more effective at $k \approx k_f$, and likewise an enstrophy $\Omega \cong k_f^2 E$. For small ν and α_p there will be a large "spike" of energy and enstrophy at $k \cong k_f$. This possibility is unlikely, if there are reasonable transfer rates in k-space by the nonlinear dynamics. In any case, we shall see that numerical simulations do not support this "pile-up" scenario!

Kraichnan dual cascade picture. The above conclusions were reached using somewhat different but related arguments in the landmark paper

R. H. Kraichnan, "Inertial ranges in two-dimensional turbulence," Phys. Fluids 10 1417–1423 (1967)

Kraichnan argued that in the limit of very small viscosity most of the energy input ε would cascade to low wavenumbers because of the "blocking" effect of the enstrophy flux, with just a little bit of "leakage" flux $\varepsilon_{uv} \ll \varepsilon$ going to high-wavenumbers. He likewise argued that most of the enstrophy input η would cascade to high-wavenumbers, with just a bit of "leakage" flux $\eta_{ir} \ll \eta$ going to low-wavenumbers. Using dimensional reasoning and other plausible physical arguments, Kraichnan argued that there should be an inverse energy cascade range at scales greater than the forcing scale ℓ_f, with an energy spectrum

$$E(k) \sim C\varepsilon^{2/3}k^{-5/3}, \quad k_{ir} \ll k \ll k_f,$$

just like Kolmogorov's spectrum for 3D but with energy flux in the opposite direction (small scale to large scale).

Furthermore, Kraichnan argued that there should be an direct enstrophy cascade range at scales smaller than the forcing scale ℓ_f, with an energy spectrum

$$E(k) \sim C' \eta^{2/3} k^{-3}, \quad k_f \ll k \ll k_{uv}$$

just as Batchelor proposed independently for the case of decaying 2D turbulence. This is the celebrated dual cascade picture of 2D turbulence.

In the original Kraichnan 1967 paper, the infrared cutoff wavenumber was supposed to decrease with time as

$$k_{ir}(t) \sim (\text{const.})\, \varepsilon^{-1/2} t^{-3/2}$$

corresponding to a total energy in the $-5/3$ range growing as $\sim \varepsilon t$. In a steady state this infrared cutoff is instead obtained by balancing the damping rate to the turnover rate, as

$$\alpha_p k_{ir}^{-2p} \sim \varepsilon^{1/3} k_{ir}^{2/3}$$

giving

$$\underline{k_{ir} \sim \left(\frac{\alpha_p^3}{\varepsilon}\right)^{\frac{1}{6p+2}}}$$

e.g. $k_{ir} \sim (\alpha^3/\varepsilon)^{1/2}$ for the case of linear friction with $p=0$.

Likewise, the ultraviolet cutoff associated to viscosity can be obtained by balancing the viscous damping rate to the turnover rate, as

$$\nu k_{uv}^2 \sim \eta^{1/3}$$

giving

$$k_{uv} \sim \frac{\eta^{1/6}}{\nu^{1/2}}$$

the so-called <u>Kraichnan - Batchelor wavenumber</u>.

It is possible to make a more precise estimate of the amounts of input which go to low and high wavenumbers, as

$$\varepsilon = \varepsilon_{ir} + \varepsilon_{uv}$$

$$\eta = \eta_{ir} + \eta_{uv}$$

and using the assumption that the inputs and the dissipation are all spectrally well-localized so that

$$\eta \cong C k_f^2 \varepsilon, \quad \eta_{ir} \cong C_{ir} k_{ir}^2 \varepsilon_{ir}, \quad \eta_{uv} \cong C_{uv} k_{uv}^2 \varepsilon_{uv}$$

with some constants C, C_{ir}, C_{uv} that may depend upon the details of the forcing and damping. Using these relations it is easy to calculate that

$$\varepsilon_{uv} = \frac{Ck_f^2 - C_{ir}k_{ir}^2}{C_{uv}k_{uv}^2 - C_{ir}k_{ir}^2}\varepsilon, \quad \varepsilon_{ir} = \frac{C_{uv}k_{uv}^2 - Ck_f^2}{C_{uv}k_{uv}^2 - C_{ir}k_{ir}^2}\varepsilon$$

and

$$\eta_{ir} = \frac{\frac{1}{Ck_f^2} - \frac{1}{C_{uv}k_{uv}^2}}{\frac{1}{C_{ir}k_{ir}^2} - \frac{1}{C_{uv}k_{uv}^2}}\eta, \quad \eta_{uv} = \frac{\frac{1}{C_{ir}k_{ir}^2} - \frac{1}{Ck_f^2}}{\frac{1}{C_{ir}k_{ir}^2} - \frac{1}{C_{uv}k_{uv}^2}}$$

One can see that for $k_{uv} \gg k_f$,

$$\varepsilon_{ir} \cong \varepsilon, \quad \varepsilon_{uv} \cong \left(\frac{C}{C_{uv}}\right)\left(\frac{k_f}{k_{uv}}\right)^2 \varepsilon \longrightarrow 0$$

However, when $k_{uv} \gtrsim k_f$, the "leakage flux" ε_{uv} can be a sizable fraction of ε. Likewise, for $k_{ir} \ll k_f$

$$\eta_{uv} \cong \eta, \quad \eta_{ir} \cong \left(\frac{C_{ir}}{C}\right)\left(\frac{k_{ir}}{k_f}\right)^2 \eta \longrightarrow 0$$

but when $k_{ir} \lesssim k_f$, then η_{ir} can be a sizable fraction of the input η. The above estimates are all clearly consistent with the exact Fjørtoft-type bounds on the fluxes. The relation $\varepsilon_{uv} \cong (k_f/k_{uv})^2\varepsilon$ was already noted by Kraichnan (1967) but the above more detailed accounting is from

V. Borue, "Spectral exponents of enstrophy cascade in stationary two-dimensional turbulence," Phys. Rev. Lett. <u>71</u> 3967–3970 (1993)

How much of the above picture is firmly established?, Are there theoretical alternatives? We address first this latter question, before we turn to a review of the empirical evidence from simulations and laboratory experiments.

The basic dual cascade picture, while not yet rigorously established *a priori*, seems to be true almost without a doubt. Just using the spectral balance equations of energy and enstrophy, it is possible to show that, if $\frac{k_f}{k_{ir}} \gg 1$ and $\frac{k_{uv}}{k_f} \gg 1$, then there are long ranges of wavenumber with

$$Z(k) \cong \eta, \quad k > k_f$$

and

$$\Pi(k) \cong -\varepsilon, \quad k < k_f$$

<u>assuming only that the total energy remains finite in the limit</u>. See Eyink (1996) and the homework. As we shall see, all numerical simulations support both the hypothesis of this theorem and its conclusion.

However, there are many theoretical alternatives which have been proposed to the Kraichnan-Batchelor scaling theories. Let us mention here just a few.

There have been recurrent claims in the literature that the energy spectrum in the inverse energy cascade range is steeper than $k^{-5/3}$, say, k^{-2} or even k^{-3}. These proposals have been made based largely on the basis of numerical simulations, for a recent review of which, see:

J. Fontane, D. G. Dritschel & R. K. Scott, "Vortical control of forced two-dimensional turbulence," Phys. Fluids 25 015101 (2013)

The deviations from Kraichnan's predicted $k^{-5/3}$ spectrum are generally attributed to the appearance of a set of large-scale coherent vortices.

There are also competing proposals for the spectrum in the direct enstrophy cascade range. We have already mentioned the Saffman (1971) proposal of a k^{-4} spectrum based on vorticity fronts, which he proposed for decaying 2D turbulence but which some have suggested applies also to forced steady-states. Another proposal is from

H. K. Moffatt, "Geophysical and astrophysical turbulence," in Advances in Turbulence, eds. G. Comte-Bellot & J. Mathieu (Springer, 1987)

who proposed a $k^{-11/3}$ spectrum associated to a "spiral range," arising from patches of vorticity wound into tight spiral structures.

A much more sophisticated approach was proposed more recently by the field & string theorist A. Polyakov:

A. Polyakov, "Conformal turbulence," arXiv: hep-th/9209046

and

A. Polyakov, "The theory of turbulence in two dimensions," Nucl. Phys. B 396 367-385 (1993)

This theory attempted to construct exact solutions to the analog of the Friedman-Keller hierarchy equations for the vorticity field in 2D

$$\sum_{i=1}^{n} \langle \omega(x_1) \cdots \omega(x_{i-1}) J(\psi,\omega)(x_i)\, \omega(x_{i+1}) \cdots \omega(x_n) \rangle = 0$$

which should hold for a stationary ensemble of Euler solutions.

Polyakov proposed to find suitable solutions by using methods of conformal field theory. The basis of his approach is the <u>operator product expansion</u> (OPE) for composites of a primary operator $\psi(x)$ in a suitable non-unitary conformal field theory:

$$\epsilon_{ij}\,\partial_i\psi(x+a)\,\partial_j\partial^2\psi(x)$$

$$\underset{a\to 0}{\sim}\ (\text{const.})\,(a\bar{a})^{\Delta_\phi-2\Delta_\psi}$$

$$\times\left[L_{-2}\bar{L}_{-1}^2-\bar{L}_{-2}L_{-1}^2\right]\phi(x)$$

where $a=a_x+ia_y$, L_n are generators of the Virasoro algebra, and $\phi(x)$ is the leading operator in the OPE $\psi(x+a)\psi(x)\sim(a\bar{a})^{\Delta_\phi-2\Delta_\psi}\phi(x)$. As observed by Polyakov, the constraint $J(\psi,\omega)=0$ can be achieved in the hierarchy equations either by looking for cases with $[L_{-2}\bar{L}_{-1}^2-\bar{L}_{-2}L_{-1}^2]\phi=0$ or more simply by cases with

$$\Delta_\phi > 2\Delta_\psi$$

We consider only this latter possibility, in which case $\Delta_{\dot{\omega}}=\Delta_{J(\psi,\omega)}=\Delta_\phi+2$. A second constraint arises from the condition of constant enstrophy flux

$$Z(K) = -\frac{d}{dt}\left(\sum_{|k|<K} \langle |\hat{\omega}(k)|^2 \rangle \right)\Bigg|_{t=0}$$

$$= \eta$$

or

$$\sum_{|k|<K} \mathrm{Re}\left(\langle \hat{\omega}^*(k)\, \dot{\hat{\omega}}(k) \rangle \right) = \eta$$

which requires

$$\Delta_\omega + \Delta_{\dot{\omega}} = 0 .$$

Since also $\Delta_\omega = \Delta_\psi + 1$, since $\omega = \Delta\psi$, one obtains finally the constraint that

$$\Delta_\phi + \Delta_\psi = -3 .$$

In principle any conformal theory which meets these constraints is a "constant enstrophy flux" solution of the stationary hierarchy equations. These have energy spectra

$$E(k) = k^2 E_\psi(k) \sim (\mathrm{const.})\, k^{4\Delta_\psi + 1} .$$

In general, $\Delta_\psi < 0$, so that these have spectra steeper than the Kraichnan-Batchelor k^{-3} prediction.

For example, Polyakov pointed out that the simplest solution is given by the (2,21) minimal conformal model with $\Delta_\psi = -\frac{8}{7}$ and thus

$$E(k) \sim k^{-25/7},$$

as well as other possibilities.

What, then, is the verdict of experiments and simulations? The answer is a bit cloudy. There are a large number of simulations and laboratory experiments which support the Kraichnan-Batchelor predictions, but others which contradict it. One of the former is the work

G. Boffetta & S. Musacchio, "Evidence for the double cascade scenario in two-dimensional turbulence," Phys. Rev. E 82 016307 (2010)

which performs numerical simulations of 2D NS up to resolutions of $32{,}768^2$ or 2^{15} grid points in both directions.

These simulations use linear damping with coefficient α to remove energy at large scales, except for the largest simulation at smallest value of ν, where $\alpha = 0$ and the simulation was stopped before steady-state was reached. As can be seen, there is inverse flux of energy and direct flux of enstrophy, with fluxes becoming more nearly constant as the scaling ranges increase (by decreasing both α and ν). See their Fig. 1, reproduced on the following page. The energy spectra in these simulations match very well the Kraichnan prediction $k^{-5/3}$ in the inverse cascade range, but have generally steeper spectra than k^{-3} in the forward enstrophy cascade range. In addition to the theoretical predictions of such steeper spectra which we have already discussed (Saffman, Moffatt, Polyakov), there are other possible reasons for such steeper spectra which we shall discuss in detail later. One of these is a logarithmic correction to the k^{-3} spectrum predicted by Kraichnan (1971) and another is the disruptive effect of linear damping on the 2D enstrophy cascade predicted by Bernard (2000).

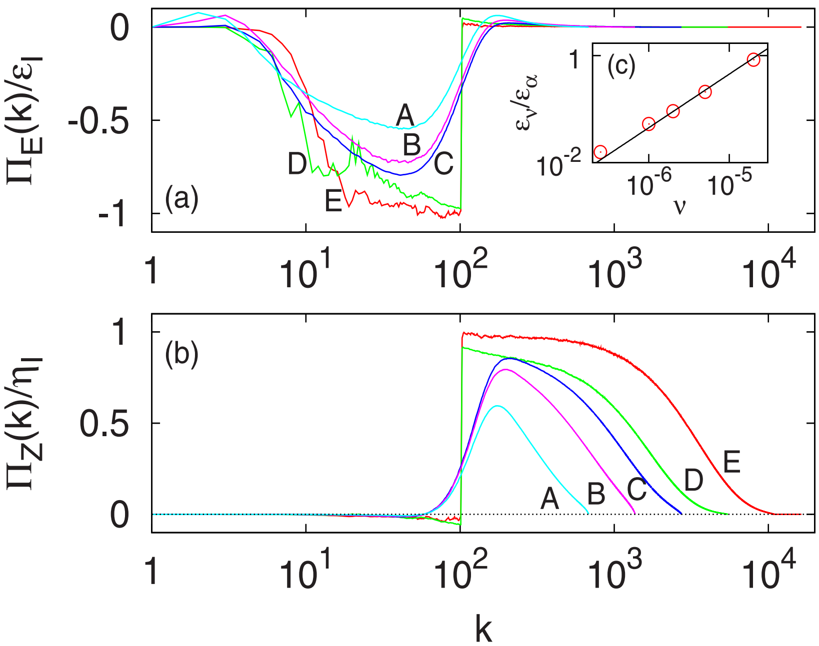


FIG. 1. Energy and enstrophy fluxes in Fourier space for the runs of Table I. Fluxes for runs D and E are computed from a single snapshot. Inset (c): ratio of viscous over friction energy dissipation versus kinematic viscosity for the 5 runs, the line is a linear fit.

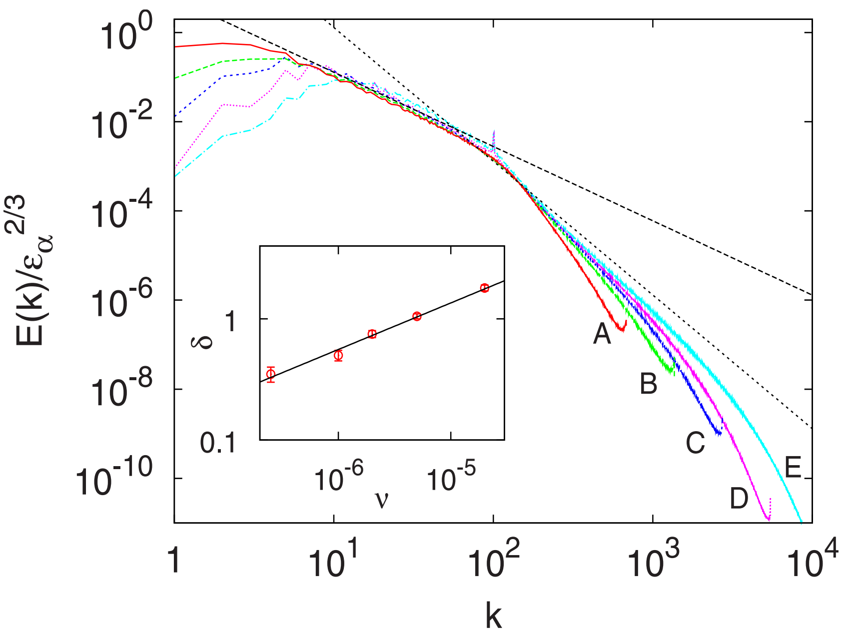


FIG. 2. Energy spectra for the simulation of Table I compensated with the inverse energy flux. Lines represent the two Kraichnan spectra $Ck^{-5/3}$ (dashed) with $C = 6$ and k^{-3} (dotted). The inset shows the correction δ to the Kraichnan exponent for the direct cascade 3 obtained from the minimum of the local slope of the spectra in the range $k_f \leq k \leq k_\nu$ as a function of the viscosity. Error bars are obtained from the fluctuations of the local slope. The line has a slope 0.38 and is a guide for the eyes.

Other recent simulations reach quite different conclusions, however! Consider the results of Fontane et al. (2013), who perform simulations using a novel "combined Lagrangian advection model."

They consider the transient growth problem with no large-scale damping. They prefer to present their results in terms of the enstrophy spectrum $\Omega(k,t)$ [which they denote $Z(k,t)$]. As seen in their Fig. 3 there is a roughly k^{-1} spectrum at wavenumbers $k > k_f$, but the spectrum in the inverse cascade range $k < k_f$ is more consistent with a k^0 spectrum than the Kraichnan $k^{1/3}$ spectrum. This translates into a k^{-2} energy spectrum. They argue that this discrepancy is due to large-scale coherent vortices (see their Fig. 6). When the decompose the vorticity into a coherent part and an incoherent part, they find that the incoherent part's spectra scales as the Kraichnan prediction $k^{1/3}$ but the coherent vortices give the k^0 spectrum, which dominates at small k. See their Fig. 7.

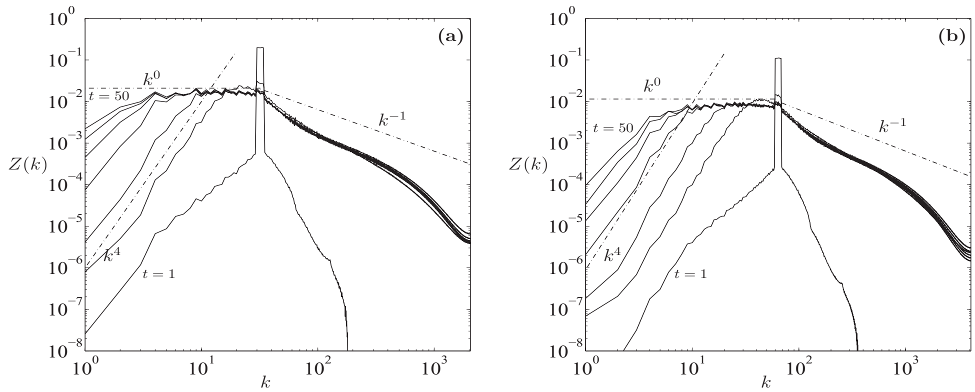


FIG. 3. Enstrophy spectra at increasing times $t = 1$, 5, 10, 20, 30, 40, and 50 (from right to left) for set A (a) and set B (b). Spectra are normalised by total enstrophy.

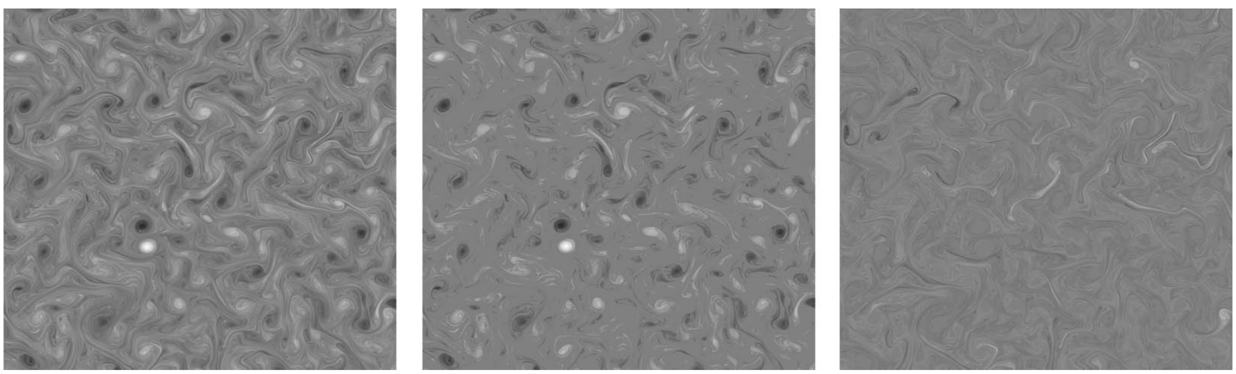


FIG. 6. Decomposition of the vorticity field (left) into coherent (middle) and incoherent part (right). The images are screen-shots of one of the simulations in set B at time $t = 10$. Only one sixteenth of the domain is shown.

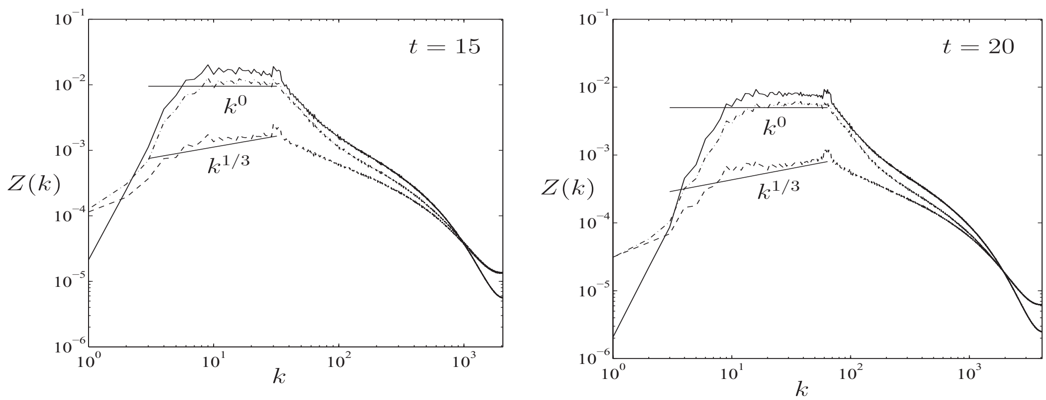


FIG. 7. Decomposition of the enstrophy spectra (solid) into coherent (dashed-dotted) and incoherent (dashed) parts. These figures correspond to set A at $t = 15$ (left) and set B at $t = 20$ (right). Spectra are normalised by total enstrophy.

These results are in flat contradiction to the earlier ones of Boffetta & Musacchio (2010)! Fontane et al. (2013) attempt to explain the discrepancy on the basis that their simulation has no large-scale damping, which could destroy the large-scale vortices. However, this is also the case for the largest-scale simulation of Boffetta & Musacchio with $\alpha = 0$. Other claims exist in the literature which further complicate the picture. For example,

A. Mizuta, T. Matsumoto & S. Toh, "Transition of the scaling law in inverse energy cascade range caused by a nonlocal excitation of coherent structures observed in two-dimensional turbulent fields," Phys. Rev. E 88 053009 (2013)

use pseudo-spectral simulations up to 4096^2 resolution with a very narrow enstrophy cascade range, and claim to see the $k^{-5/3}$ range robustly in the inverse cascade range for deterministic forcing, but steeper spectra at higher effective Reynolds corresponding to smaller large-scale damping when stochastic forcing is employed! This effect is attributed to coherent vortices that appear with stochastic forcing (which are not destroyed by the hyperviscosities employed for large-scale damping).

In the enstrophy cascade range there are also several studies which observe in numerical simulations the k^{-3} spectrum of Kraichnan - Batchelor, e.g.

A. Vallgren & E. Lindborg, "The enstrophy cascade in forced two-dimensional turbulence," J. Fluid. Mech. <u>671</u> 168-183 (2011)

These simulations are pseudospectral at 8192^2 resolution and avoid using a linear drag ($p=0$). The simulations are performed either with no large-scale drag at all, in a transient regime as originally considered by Kraichnan (1967), or in a steady-state with a $p=2$ inverse-Laplacian drag at large scales. Their results presented in their Figs. 4-5 (next page) show almost 2 decades of constant enstrophy flux with an enstrophy spectrum close to the prediction k^{-3} of Kraichnan - Batchelor. As we shall discuss later, this difference from the results of Boffetta & Musacchio (2010) is explainable by the latter's use of a linear damping.

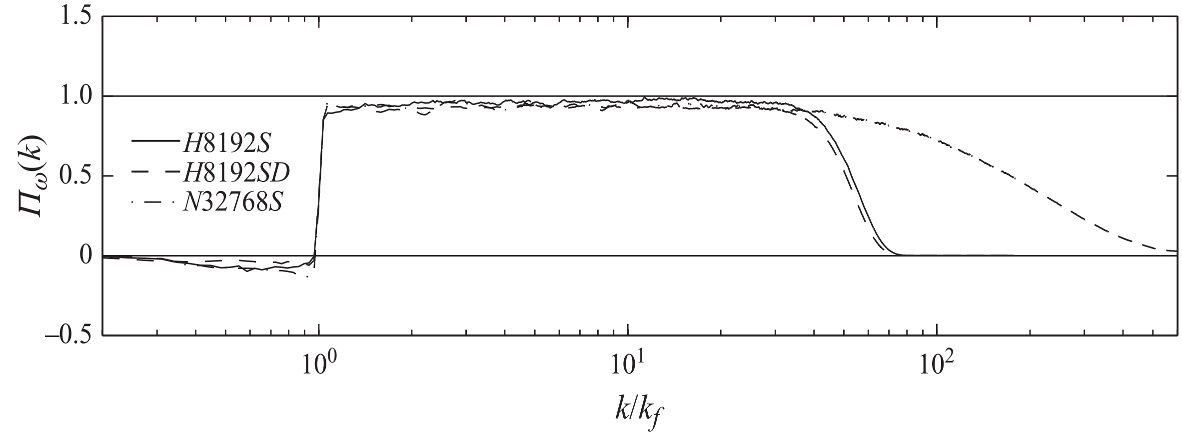


FIGURE 4. Mean enstrophy fluxes averaged over the quasi-stationary time periods for $H8192S$ ($t \in [48, 73]$), $H8192SD$ ($t \in [92, 200]$) and $N32768S$ ($t \in [63, 73]$). The abscissa is the wavenumber scaled by the forcing wavenumber.

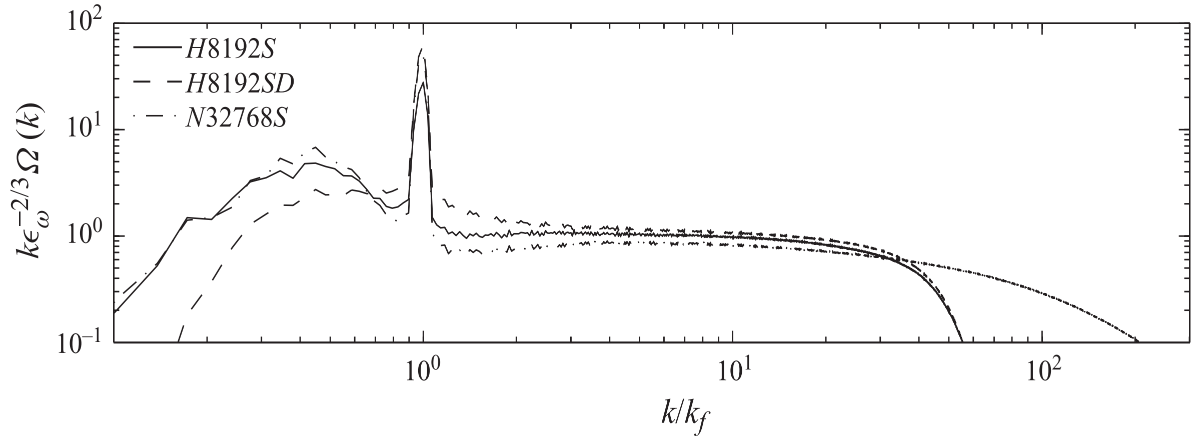


FIGURE 5. Mean compensated enstrophy spectra for $H8192S$, $H8192SD$ and $N32768S$, over the same time intervals as in figure 4, where the abscissa is the wavenumber scaled by the forcing wavenumber.

As should be clear from the above very, very abbreviated review of the literature, the picture presented by current numerical simulations (and also laboratory experiments) is quite complex and not even obviously self-consistent. Some of the seeming contradictions can be explained by rather well-understood effects (e.g. linear damping in the enstrophy range). Other conflicting results are on the face direct contradictions for which there is no generally accepted explanation, e.g. the spectrum ($k^{-5/3}$, k^{-2} or other) in the inverse energy cascade range. It is probably not possible for numerical simulations or laboratory experiments alone to resolve this issue of the spectral power-laws, even with further great improvements in computing power or experimental techniques, because it is very hard to untangle "finite-size" effects of forcing and damping (also numerical artefacts!) from putative "universal" behavior with only a few decades of scaling. Better analytic tools are required: let's start to develop some!

Spatial scaling of vorticity and velocity fields

We now discuss the definition of fluxes of energy and enstrophy in real space and the physical mechanisms of their cascades. We must first make some important preliminary remarks about the scaling/regularity of vorticity and velocity fields, especially in the forward enstrophy cascade.

Most alternatives to the Kraichnan-Batchelor theory, as we have seen, postulate an enstrophy spectrum steeper than k^{-1}, of the form

$$\Omega(k) \sim k^{-(1+2s)}, \quad k_f \ll k \ll k_{uv}$$

for some $0<s<1$. By the Wiener-Khinchin theorem (see U. Frisch (1995), Section 4.5) this implies that the <u>2nd-order vorticity structure function</u> scales as

$$S_2^{\omega}(r) \equiv \langle (\omega(x+r)-\omega(x))^2 \rangle$$

$$\sim r^{2s}, \quad r_{uv} \ll r \ll r_f$$

with the correspondence $k \sim 2\pi/r$ of wavenumbers k and length-scales r. Recall from the Turbulence

Coursenotes, Section II(C), that this means that the vorticity field ω belongs the Besov space $B_2^{s,\infty}(D)$, or $\omega \in B_2^{s,\infty}(D)$ where generally

$$B_p^{s,\infty}(D) = \{ f \in L^p(D) : \ \|\delta\omega(r)\|_p = O(|r|^s) \}$$

with $\delta\omega(r;x) = \omega(x+r) - \omega(x)$ [and we have assumed for simplicity that $D = \mathbb{T}^2$]. In fact, if

$$\Omega(k) \sim k^{-(1+2s)}, \quad k \gg k_f$$

then s is the <u>maximal Besov index</u> of order 2, so that for any $\epsilon > 0$

$$\text{holds } \omega \in B_2^{s-\epsilon,\infty}(D) \text{ but } \omega \notin B_2^{s+\epsilon,\infty}(D).$$

With an appropriate definition of the enstrophy spectrum using Paley-Littlewood decomposition, it is even true that

$$\Omega(k) = O(k^{-(1+2s)}) \Longleftrightarrow \omega \in B_2^{s,\infty}(D).$$

See:

P. Constantin, "The Littlewood-Paley spectrum in two-dimensional turbulence," Theor. Comp. Fluid. Dyn. <u>9</u> 183-189 (1997)

This should all sound very familiar, as it is analogous to the results that hold for velocity in 3D turbulence. However, now consider the velocity in the enstrophy cascade range, where the energy spectrum is

$$E(k) \sim k^{-(3+2s)}, \quad k_f \ll k \ll k_{uv}.$$

or $E(k) \sim k^{-(1+2s')}$ with $s' = 1+s \in (1,2)$. However, the Wiener-Khinchin theorem does not apply with $s' \in (1,2)$ and instead the 2nd-order velocity structure function

$$S_2^u(r) = \langle |u(x+r) - u(x)|^2 \rangle \sim r^2$$

rather than $r^{2s'}$. This is easy to understand, because the velocity field is differentiable (with derivative in $B_2^{s,\infty}$) and thus, heuristically,

$$\delta u(r) \sim (r \cdot \nabla) u + O(r^{1+s})$$

in L^2-sense. If one wants to cancel the trivial leading term and detect the "true" scaling, one must use a higher-order difference, e.g. 2nd-order

$$\delta^2 u(r; x) = u(x+r) + u(x-r) - 2u(x).$$

It is then indeed true that

$$E(k) \sim k^{-(3+2s)}, \quad k_f \ll k \ll k_{uv}$$

$$\Longrightarrow \quad S_2^{2,u}(r) = \langle |\delta^2 u(r)|^2 \rangle \sim r^{2(1+s)},$$

$$r_{uv} \ll r \ll r_f$$

for $0<s<1$. In fact, the correct definition of Besov spaces for $s' \in [1,2)$ is

$$p \geq 1$$

$$u \in B_p^{s',\infty}(D) \Longleftrightarrow u \in L_p(D) \;\&\; \|\delta^2 u(r)\|_p = O(r^{s'})$$

See H. Triebel, Theory of Function Spaces (Birkhäuser, 2010). With this definition it is also true that

$$E(k) = O(k^{-(1+2s')})$$

$$\Longleftrightarrow \quad u \in B_2^{s',\infty} \qquad s' \in [1,2)$$

using Paley-Littlewood spectrum and

$$\omega \in B_p^{s,\infty}(D) \Longleftrightarrow u \in B_p^{1+s,\infty}(D), \quad p \geq 1$$

(The reverse implication involves the Calderón-Zygmund inequality for singular integral operators on Besov spaces.)

We conclude that the structure functions of 2nd-order differences

$$S_p^{2,u}(r) = \langle |\delta^2 u(r)|^p \rangle$$

are now more important than structure functions for first-order differences, and observe that

$$S_2^{2,u}(r) = 6\langle |u(0)|^2 \rangle + 2\langle u(2r)\cdot u(0) \rangle$$
$$- 8\langle u(r)\cdot u(0) \rangle$$

for a homogeneous average.

The situation is even more complex for the scaling of the Kraichnan-Batchelor theory. Consider first the vorticity field. The general relation in 2D between the vorticity 2nd-order correlation function, defined by

$$B_2^{\omega}(r) \equiv \langle \omega(r)\omega(0) \rangle$$

for a homogeneous average, and enstrophy spectrum is

$$B_2^{\omega}(r) = 2\int_0^{\infty} J_0(kr)\,\Omega(k)\,dk$$

assuming also isotropy. Here, $J_0(x)$ is the Bessel function of 0 index.

It then follow easily that for a Batchelor–Kraichnan range

$$B_2^\omega(r) = 2\int_{k_f}^{k_{uv}} C\eta^{2/3} J_0(kr)\,\frac{dk}{k}$$

$$\sim \begin{cases} 2C\eta^{2/3}\left[(\text{const.}) - \ln(r/\ell_f)\right], & \ell_{uv} \ll r \ll \ell_f \\ 2C\eta^{2/3}\left[\ln\left(\frac{\ell_f}{\ell_{uv}}\right) + O\left(\left(\frac{r}{\ell_{uv}}\right)^2\right)\right], & r \ll \ell_{uv} \end{cases}$$

In particular, note that

$$\Omega = \frac{1}{2} B_2^\omega(0) = C\eta^{2/3} \ln\left(\frac{\ell_f}{\ell_{uv}}\right)$$

which diverges as $\ell_{uv} \to 0$ (i.e. $\nu \to 0$), because the KB-range then contains infinite enstrophy. However, because the 2nd-order structure function of vorticity is defined by

$$S_2^\omega(r) = 2\left[B_2^\omega(0) - B_2^\omega(r)\right]$$

the vorticity structure functions also diverge as $k_{uv} \to \infty$ or $\nu \to 0$! In particular,

$$S_2^\omega(r) \sim \begin{cases} 4C\eta^{2/3}\left[\ln\left(\frac{r}{\ell_{uv}}\right) - (\text{const.})\right], & \ell_{uv} \ll r \ll \ell_f \\ (\text{const.})\,\eta^{2/3}\left(\frac{r}{\ell_{uv}}\right)^2, & r \ll \ell_{uv} \end{cases}$$

and clearly $S_2^\omega(r) \longrightarrow \infty$ as $\ell_{uv} \to 0$ in the enstrophy cascade range. Because of the Hölder inequality $S_2^\omega(r) \leq \left[S_{2p}^\omega(r)\right]^{1/p}$ the same is true for all higher-order structure functions as well. Thus, one must conclude that vorticity structure functions are not useful to study the inviscid limit of the KB range.

Instead, one must use general vorticity correlation functions of the form

$$\langle \omega(x_1)\,\omega(x_2)\cdots\omega(x_p)\rangle \equiv B_p^\omega(x_1, x_2, \ldots, x_p)$$

These may be expected to exist in the limit $\nu \to 0$ if $x_i \neq x_j$ for all $i \neq j$. The example of $p=2$ shows, however, that these correlation functions have logarithmic divergences $\sim \ln\left(\frac{r_{ij}}{\ell_f}\right)$ if $r_{ij} = |x_i - x_j| \longrightarrow 0$ for any $i \neq j$. This means the vorticity field does not exist as an ordinary function in the limit $\nu \to 0$ but instead only as a distribution (i.e. generalized function).

The types of divergence one sees at the limit of coinciding points ($r_{ij} \to 0$) are typical of those in the operator product expansions of quantum field theory, which arise because one is not generally permitted to multiply distributions pointwise.

We can be more precise. As a matter of fact, if enstrophy spectrum is interpreted in the Littlewood-Paley sense, then

$$\Omega_{LP}(k) = O(k^{-(1+2s)})$$
$$\iff \omega \in B_2^{s,\infty}$$

for all real s, including $s \lesssim 0$. This, Littlewood-Paley spectrum is defined precisely as

$$\Omega_{LP}(k) = \frac{1}{k}\|\omega_N\|_2^2, \quad k \in [2^N, 2^{N+1})\cdot k_0$$

where ω_N is a suitable band-pass filtered vorticity field in the range $(2^{N-1}, 2^{N+1})$. See Turbulence Notes, Section II D or the paper of Constantin (1997)*.

---

* Note, in fact, that $f \in B_p^{s,\infty} \iff \|f_N\|_p = O(2^{-sN})$ for all N, for general real $s \in \mathbb{R}$.

Thus, we see that <u>the KB spectrum $\Omega_{Lp}(k) \sim k^{-1}$ corresponds to the zeroth-order Besov space $B_2^{0,\infty}$</u>.

This is, in fact, a space of distributions rather than ordinary functions and, in general, $\omega \in B_2^{0,\infty}$ does <u>not</u> satisfy $\omega \in L_2$ (finite enstrophy).

It is interesting to recall here the result of Bahouri & Chemin (1994) on 2D free decay that if $\omega_0 \in B_2^{s,\infty}$ for $0 < s < 1$, then $\omega(t) \in B_2^{s(t),\infty}$ with $s(t) = e^{-C\|\omega\|_\infty t} s$. Since $s(t) \longrightarrow 0$ as $t \longrightarrow \infty$, the space $B_2^{0,\infty}$ is "natural" to describe the long-time steady-state.

What about the velocity field in the enstrophy cascade range of KB theory? Since $\omega \in B_2^{0,\infty}$, then $\mathbf{u} \in B_2^{1,\infty}$ and $\mathbf{u}$ is an ordinary function (of course, since it has finite energy and $\mathbf{u} \in L^2$). Also, by definition of $B_2^{1,\infty}$,

$$\| \delta^2 \mathbf{u}(\mathbf{r}) \|_2 = O(r).$$

However, it need not be true that $\| \delta \mathbf{u}(\mathbf{r}) \|_2 = O(r)$! As a matter of fact, it is easy to check that for $E(k) = C\eta^{2/3} k^{-3}$ in $k_f < k < k_{uv}$

$$S_2^u(r) = 4C\eta^{2/3} \int_{k_f}^{k_{uv}} \left[1 - J_0(kr)\right] \frac{dk}{k^3}$$

$$\sim 2C\eta^{2/3}\left[r^2 + \frac{1}{4}r^2 \ln\left(\frac{\ell_f}{r}\right) + O\left(\frac{r^4}{\ell_{uv}^2}\right)\right]$$

for $\ell_{uv} \ll r \ll \ell_f$

whereas

$$S_2^{2,u}(\mathbf{r}) = \langle |\delta^2 u(\mathbf{r})|^2 \rangle \sim C'\eta^{2/3} r^2$$

for the same range $\ell_{uv} \ll r \ll \ell_f$. Thus suggests that in a KB enstrophy cascade range, the scaling properties of $\delta^2 u(\mathbf{r})$ will be better than those of $\delta u(\mathbf{r})$.

Finally, what about the inverse energy cascade range? All current theories of that range assume an energy spectrum $E(k) \sim Ck^{-(1+2s)}$ with $0 < s < 1$. Thus, the situation is quite similar to the 3D case, w

$$u \in B_2^{s,\infty}, \quad 0 < s < 1.$$

The Kraichnan $k^{-5/3}$ spectrum corresponds to $s = \frac{1}{3}$, just as the K41 spectrum in 3D. However, we shall see that there is a major difficulty in the 2D case that we do not yet know the analogue of the "zeroth law of turbulence" for the 2D inverse energy cascade!

Subscale fluxes in physical space. We now turn to the problem of deriving and estimating physical-space expressions for subscale fluxes (transfers) of energy and enstrophy.

We begin with the 2D inverse energy cascade range. This is fairly straightforward, because it was essentially already done in the Turbulence I course, for any space dimension d. We discussed there two approaches, one based on spatial coarse-graining / filtering / mollification and another approach based on spatial point-splitting. The spatial coarse-graining approach was carried out in the Turbulence I Coursenotes, Section II (c). In that approach one defines a large-scale velocity

$$\overline{u}_\ell(x,t) = \int d^dr\, G_\ell(r)\, u(x+r,t)$$

where $G_\ell(r) = \frac{1}{\ell^d} G\left(\frac{r}{\ell}\right)$ is a smooth, rapidly decaying filter kernel. It is then straightforward to derive a balance equation for the large-scale energy density

$$\overline{e}_\ell(x,t) = \frac{1}{2}|\overline{u}_\ell(x,t)|^2,$$

as follows ;

$$\partial_t \bar{e}_\ell + \nabla\cdot\left[(\bar{e}_\ell + \bar{p}_\ell)\bar{u}_\ell + \tau_\ell\cdot\bar{u}_\ell - \nu\nabla\bar{e}_\ell \right]$$

$$= -\Pi_\ell - \nu|\nabla\bar{u}_\ell|^2 + \bar{u}_\ell\cdot\bar{f}_\ell - 2\alpha\bar{e}_\ell$$

where one defines the <u>subscale stress</u>

$$\tau_\ell = \tau_\ell(u,u) = \overline{(uu)}_\ell - \bar{u}_\ell\bar{u}_\ell$$

and the <u>subscale energy flux</u>

$$\Pi_\ell = -\nabla\bar{u}_\ell : \tau_\ell = -\bar{S}_\ell : \tau_\ell = -\bar{S}_\ell : \mathring{\tau}_\ell$$

which represents "deformation work" of the large/resolved scales on the small/unresolved scales.

A second approach is to introduce a point-split kinetic energy

$$e_r(x,t) = \frac{1}{2}u(x,t)\cdot u(x+r,t).$$

with the abbreviations

$$u = u(x,t), \quad u' = u(x+r,t)$$

$$p = p(x,t), \quad p' = p(x+r,t)$$

$$\delta u = u' - u = u(x+r,t) - u(x,t)$$

it is again straightforward to show that (Turbulence I Notes, Section III(c))

$$\partial_t e_r + \nabla_x \cdot \left[e_r u + \frac{1}{2}(pu' + p'u) + \frac{1}{4}|u'|^2 \delta u - \nu \nabla_x e_r \right]$$

$$= - \Pi_r - \nu \nabla u : \nabla u' + \frac{1}{2}(f \cdot u' + f' \cdot u) - 2\alpha e_r$$

with the <u>point-splitting flux</u>

$$\Pi_r = -\frac{1}{4} \nabla_r \cdot [\delta u |\delta u|^2]$$

This is the deterministic (no averaging over an ensemble of random velocities) version of the <u>Kolmogorov-Monin relation</u>, originally derived to express mean energy flux.

Of course, in a statistical ensemble with constant mean flux $\langle \Pi(k) \rangle$ = constant over a long range of wavenumbers, it is straightforward to show also that

$$\langle \Pi_\ell \rangle = \langle \Pi_r \rangle = \langle \Pi(k) \rangle,$$

for $\ell \sim r \sim 1/k$ and k in the range of wavenumbers where $\langle \Pi(k) \rangle$ is constant. For $\langle \Pi_r \rangle$ this is shown by Frisch (1995), section 6.2.2. For $\langle \Pi_\ell \rangle$

it was shown by

G. Eyink, "Energy dissipation, Euler equations, and the multifractal model of turbulence," in: Proceedings of the XIth International Congress of Mathematical Physics, Paris, France, 1994. D. Iagolnitzer, ed. (International Press, Cambridge, 1995)

using the simple observation that

$$\frac{1}{|D|}\int_D d^dx\, \Pi_\ell = \int_0^\infty dk\, P_\ell(k)\, \Pi(k)$$

where $P_\ell(k) = -\frac{d}{dk}|\hat{G}(\ell k)|^2$ is a normalized density localized at wavenumbers around $k \sim 1/\ell$.

In the inverse cascade range of 2D turbulence, these results imply that for a long inertial range

$$\langle \Pi_\ell \rangle = \langle \Pi_r \rangle = -\varepsilon_{ir}$$

where also $\varepsilon_{ir} \approx \varepsilon$, when the viscosity is sufficiently small. For simplicity, we shall often assume this when we discuss the inverse cascade range (although this assumes that the inverse cascade properties are independent of viscosity ν, which is contested by some!)

For isotropic statistics, it therefore follows from $\langle \Pi_r \rangle = -\varepsilon$ that

$$\langle \delta u_L^3(r) \rangle \sim +\frac{3}{2}\varepsilon r,$$

which is the "$\frac{4}{5}$th-law" for space dimension $d=2$. Notice the opposite sign compared with $d=3$!

It is then straightforward to estimate these energy fluxes in magnitude, as we did in the Turbulence I Coursenotes, Section II(c). For example, using the identities

$$\nabla \bar{u}_\ell(x) = -\frac{1}{\ell}\int d^d r\, (\nabla G)_\ell(r)\, \delta u(r;x)$$

and

$$\tau_\ell = \int d^d r\, G_\ell(r)\, \delta u(r;x)\, \delta u(r;x)$$
$$- \left(\int d^d r\, G_\ell(r)\, \delta u(r;x)\right)\left(\int d^d r\, G_\ell(r)\, \delta u(r;x)\right)$$

which are valid in any space dimension, we can develop estimates of $\Pi_\ell(x)$ both pointwise:

$$|\Pi_\ell(x)| \leq (\text{const.}) \frac{\delta u^3(\ell;x)}{\ell}$$

with $\delta u(\ell;x) \equiv \sup_{|r|\leq \ell} |\delta u(r;x)|$, or for space-averages

$$|\langle \Pi_\ell \rangle_{space}| \leq \|\Pi_\ell\|_1 \leq \frac{(\text{const.})}{\ell}\left[\sup_{|r|\leq\ell} \|\delta u(r)\|_3\right]^3.$$

When there are power-law bounds of the form

$$\delta u(\ell;x) = O(\ell^h)$$

pointwise at x, or

$$\sup_{|r|\leq\ell} \|\delta u(r)\|_3 = O(\ell^{\sigma_3})$$

globally, then we obtain

$$|\Pi_\ell(x)| = O(\ell^{3h-1})$$

or

$$|\langle \Pi_\ell \rangle_{space}| = O(\ell^{3\sigma_3 - 1}),$$

respectively.

Exactly the same bounds can be obtained in the point-splitting approach for the "smeared" flux Π_ℓ^* defined by

$$\Pi^*_\ell(x) = \int d^dr\, G_\ell(r)\, \Pi_r(x)$$

$$= +\frac{1}{4\ell}\int d^dr\, (\nabla G)_\ell(r)\cdot \delta u(r;x)\, |\delta u(r;x)|^2$$

which appears in the balance equation for the "smeared" point-split energy $\frac{1}{2}u(x,t)\cdot\bar{u}_\ell(x,t)$. See Turbulence I. Coarsenotes, section III(C).

To this point, the development is identical to that carried out for 3D forward energy cascade in Turbulence I. However, the further interpretation of these results in 2D requires a discussion of "intermittency" in the inverse cascade, which we present in detail later!

For the 2D forward enstrophy cascade, one may employ both of these approaches, spatial coarse-graining or point-splitting, to define a pointwise subscale flux of enstrophy. In the coarse-graining approach one defines a large-scale (resolved-scale vorticity

$$\bar{\omega}_\ell(x,t) = \int d^2r\, G_\ell(r)\,\omega(x+r,t)$$

which satisfies

$$\partial_t \bar{\omega}_\ell + \nabla \cdot [\bar{u}_\ell \omega_\ell + \sigma_\ell] = \nu \Delta \bar{\omega}_\ell + \bar{g}_\ell - \alpha \bar{\omega}_\ell$$

with $g = \nabla^\perp \cdot f$ and the subscale vorticity transport (in space)

$$\sigma_\ell = \overline{(u\omega)}_\ell - \bar{u}_\ell \bar{\omega}_\ell .$$

It follows easily from this that the large-scale enstrophy density

$$h_\ell(x,t) = \frac{1}{2} |\bar{\omega}_\ell(x,t)|^2$$

satisfies

$$\partial_t h_\ell + \nabla \cdot [\bar{u}_\ell h_\ell + \bar{\omega}_\ell \sigma_\ell - \nu \nabla h_\ell]$$
$$= - Z_\ell - \nu |\nabla \bar{\omega}_\ell|^2 + \bar{\omega}_\ell \bar{g}_\ell - 2\alpha h_\ell$$

where we define the subscale enstrophy flux

$$Z_\ell(x,t) = - \nabla \bar{\omega}_\ell(x,t) \cdot \sigma_\ell(x,t).$$

Notice that $Z_\ell(x,t) \geq 0$ precisely when the subscale enstrophy transport is "down-gradient," i.e when σ_ℓ is in the direction of decreasing values of $\bar{\omega}_\ell$. This means that there is forward cascade of enstrophy when the small scales tend to mix and homogenize the large-scales.

The point-splitting approach can also be applied, by defining a <u>point-split enstrophy density</u>

$$h_r(x,t) = \frac{1}{2}\omega(x,t)\omega(x+r,t).$$

which satisfies the balance equation (with similar notations as before)

$$\partial_t h_r + \nabla_x \cdot \left[h_r u + \frac{1}{4}(\omega')^2 \delta u - \nu \nabla_x h_r \right]$$
$$= Z_r - \nu \nabla\omega \cdot \nabla\omega' + \frac{1}{2}(g\omega' + g'\omega) - 2\alpha h_r$$

and the <u>point-split enstrophy flux</u> is

$$Z_r(x,t) = -\frac{1}{4}\nabla_r \cdot \left[\delta u(r;x) |\delta\omega(r;x)|^2 \right].$$

Now just as for energy cascade, in a long enstrophy cascade range with ensemble average $\langle Z(k) \rangle$ nearly independent of k, then

$$\langle Z_\ell \rangle = \langle Z_r \rangle = \langle Z(k) \rangle \cong \eta_{uv}$$

for $\ell \sim r \sim 1/k$ and k in the range where $\langle Z(k) \rangle$ is nearly constant. As before, we shall generally take $\eta_{uv} \cong \eta$ under the assumption that $\alpha \longrightarrow 0$ (and taking for granted that the enstrophy cascade range properties in a very long interval are independent of the length of the energy cascade range.

when the ensemble statistics are also isotropic, it is not hard to derive from $\langle Z_r \rangle \cong \eta$ that

$$(*) \qquad \langle \delta u_L(r) (\delta \omega(r))^2 \rangle \sim -2\eta r$$

for $l_{uv} \ll r \ll l_f$. This is exactly analogous to the Yaglom relation for passive scalar cascades, but applied to an active scalar (vorticity) and specialized to $d=2$.

The above "4/5th-law" for 2D enstrophy cascade is quite simple, but has some disadvantages. For laboratory and natural observations it requires the measurement of vorticity as well as velocity, to good spatial resolution in the enstrophy cascade range. This may be done with multi-point hot-wire techniques, but it would be advantageous to have an expression which only involves the velocity. A mathematical difficulty with $(*)$, as we shall later, is that it is hard to give meaning (and may not be true) in the limit as $\nu \to 0$. Fortunately, there is another approach which was discovered, independently, by

E. Lindborg, "Can the atmospheric kinetic energy spectrum be explained by two-dimensional turbulence," J. Fluid Mech. 388 259-288 (1999) and

D. Bernard, "Three-point velocity correlation functions in two-dimensional forced turbulence," Phys. Rev. E 60 6184 – 6187 (1999)

These authors showed that the point-split enstrophy balance can be written in an alternate form, as

$$\partial_t h_r + \nabla_x \cdot J_r = -\Delta_r \Pi_r - \nu \nabla\omega : \nabla\omega'$$

(★) $$+ \frac{1}{2}(g\cdot\omega' + g'\cdot\omega) - 2\alpha h_r$$

which holds in 3D as well as 2D, for a suitable enstrophy transport vector J_r and $g = \nabla\times f$. Hence, one may take as an alternate definition of the local point-split enstrophy flux the expression

$$Z_r(x) = \frac{1}{4}\Delta_r \nabla_r \cdot \left[\delta u(r;x)\,|\delta u(r;x)|^2\right],$$

which involves only velocity increments. To derive (★), one can start with the point-split energy balance for $e_r = \frac{1}{2}u\cdot u'$ and use the expression

$$\nabla_x \cdot (u\times\omega') = \omega'\cdot\omega + u\cdot\Delta_x u'$$
$$= \omega'\cdot\omega + u\cdot\Delta_r u',$$

which follows from the vector calculus identities

$$\nabla\cdot(a\times b) = b\cdot(\nabla\times a) - a\cdot\nabla\times b$$

$$\nabla\times(\nabla\times a) = -\Delta a \quad \text{for } \nabla\cdot a = 0$$

using $a = u$, $b = \omega'$. Likewise, for different choices of a, b, one can derive the expressions

$$\nabla_x\cdot(\partial_{x_i}u \times \partial_{x_i}\omega')$$
$$= \partial_{x_i}\omega'\cdot\partial_{x_i}\omega + \partial_{x_i}u\cdot\Delta_r\,\partial_{x_i}u'$$

$$\nabla_x\cdot(f\times\omega') = \omega'\cdot g + f\cdot\Delta_r u'$$

$$\nabla_x\cdot(u\times g') = g'\cdot\omega + u\cdot\Delta_r f'.$$

The result (☆) then follows by applying $-\Delta_r$ to the balance for the point-split energy density $\frac{1}{2}u\cdot u'$. QED!

It is worth remarking that the "enstrophy flux" $-\Delta_r\Pi_r$ for smooth 3D flows is not zero in the limit $r\to 0$, but instead yields the usual vortex-stretching term

$$\omega_i S_{ij}\omega_j.$$

Of course, this term is absent for 2D and then $Z_r^* = -\Delta_r\Pi_r$ gives a vanishing contribution for smooth 2D flows.

Averaging the flux Z_r^* over a homogeneous ensemble with constant mean spectral flux $\langle Z(k) \rangle = \eta$ gives

$$\Delta_r \nabla_r \cdot \langle \delta u(r) |\delta u(r)|^2 \rangle = 4\eta$$

and for isotropic statistics,

$$\langle \delta u_L^3(r) \rangle = +\frac{1}{8}\eta r^3.$$

The one disadvantage of this expression is that $\langle \delta u_L^3(r) \rangle \propto r^3$ also in the viscous dissipation range, whereas $\langle \delta u_L(r)\, \delta \omega^2(r) \rangle \propto r$ in the enstrophy cascade range but is $\propto r^3$ in the viscous dissipation range, Thus, one cannot infer from the scaling $\langle \delta u_L^3(r) \rangle \propto r^3$ alone that r lies in an enstrophy cascade range of scales. This is inconvenient for laboratory experiments where viscosity may be significant, However, we shall see that the above relations have very important applications in atmospheric studies, where they allow measurements of cascade rates and directions using synoptic scale measurements from single aircraft flights (see Cho & Lindborg, J. Geophys. Res. 106 10,223–10,232 (2001)).

Now let us make instead an important mathematical application of these formulas, by using them to derive limits on non-KB theories of the enstrophy cascade range. We have seen that those theories posit enstrophy spectra of the form $\Omega(k) \sim k^{-(1+2s)}$, with $0 < s < 1$, or $\omega \in B_2^{s,\infty}$, whereas KB theory corresponds to $\omega \in B_2^{0,\infty}$. Let us now make the additional

HYPOTHESIS: $\omega \in B_p^{\sigma_p,\infty}$, $0 < \sigma_p < 1$
for all $p \geq 1$

This will be true e.g. if the vorticity structure functions show scaling

$$S_p^{\omega}(r) = \langle |\delta\omega(r)|^p \rangle \sim r^{\zeta_p}, \quad 0 < \zeta_p < p$$

so that $\sigma_p = \zeta_p / p$. We shall now show that such assumptions are inconsistent with a constant enstrophy flux range over arbitrarily long scales. The argument is similar to that given by Onsager in the 1940's and also to the bounds on energy flux in 2D given earlier.

The starting point of the argument is the enstrophy flux $Z_\ell(x) = -\nabla\bar{\omega}_\ell(x)\cdot\sigma_\ell(x)$, written using

$$\nabla\bar{\omega}_\ell(x) = -\frac{1}{\ell}\int d^2r\,(\nabla G)_\ell(r)\,\delta\omega(r)$$

and

$$\sigma_\ell(x) = \int d\,r\, G_\ell(r)\,\delta\omega(r)\,\delta u(r)$$
$$-\left(\int d^2r\, G_\ell(r)\,\delta\omega(r)\right)\left(\int d^2r\, G_\ell(r)\,\delta u(r)\right)$$

We could also use the "smeared" point-split flux

$$Z_\ell^*(x) = \frac{1}{4\ell}\int d^2r\,(\nabla G)_\ell(r)\cdot\delta u(r)\,(\delta\omega(r))^2.$$

E.g. using the latter we can estimate the $L_{p/3}$-norm of the flux for $p\geq 3$ by using the Hölder inequality

$$\|Z_\ell^*\|_{p/3} \leq \frac{1}{4\ell}\int d^2r\,|(\nabla G)_\ell(r)|$$
$$\times\|\delta u(r)\|_p\,\|\delta\omega(r)\|_p^2.$$

Recall that our assumption $\omega\in B_p^{\sigma_p,\infty}$ implies also that $\nabla u\in B_p^{\sigma_p,\infty}$ (by Calderón–Zygmund inequality) and thus that $\nabla u\in B_p^{\sigma_p}\subset L_p$.

Hence

$$\delta u(r;x) = \int_0^1 d\lambda \frac{d}{d\lambda} u(x+\lambda r)$$

$$= \int_0^1 d\lambda\, (r\cdot\nabla) u(x+\lambda r)$$

gives $\|\delta u(r)\|_p \leq \int_0^1 d\lambda \cdot r \cdot \|\nabla u\|_p = O(r)$.

Putting all these estimates together gives

$$\|Z_\ell^*\|_{p/3} \leq \frac{\text{const.}}{\ell} \int d^2r\, |(\nabla G)_\ell(r)| \cdot |r|^{2\sigma_p+1}$$

$$= O(\ell^{2\sigma_p})$$

Hence, if $\sigma_p > 0$ for $p \geq 3$ then

$$|\langle Z_\ell^* \rangle_{space}| \leq \|Z_\ell^*\|_1$$

$$\leq \|Z_\ell^*\|_{p/3} = O(\ell^{2\sigma_p})$$

$$\longrightarrow 0$$

as $\ell \to 0$. Hence the enstrophy flux must tend toward zero in a sufficiently long range and no asymptotic enstrophy cascade is possible.

These results give some support to KB theory, since they show that

$$\sigma_p \leqslant 0 \quad (\text{or } \zeta_p \leqslant 0) \quad \text{for } p \geqq 3$$

is <u>required</u> for enstrophy cascade. $\sigma_p = 0$ is the KB dimensional prediction, whereas most competitors to the KB theory have assume <u>smoother</u> fields or <u>larger</u> exponents. We can derive a result directly for spectra by recalling from Turbulence I, Coursenotes Section III(E) that

$$\sigma_p - \frac{d}{p} \leqslant \sigma_{p'} - \frac{d}{p'}, \quad p' \geqq p$$

as a consequence of the Parisi-Frisch multifractal formalism (or, rigorously, from Besov embedding theorems). Applying this for $d=2$, $p=2$, $p'=3$ gives

$$\sigma_2 - \frac{1}{3} \leqslant \sigma_3 .$$

Hence, $\sigma_3 > 0$ if $\sigma_2 > \frac{1}{3}$. This means that <u>there can be no enstrophy spectrum steeper than $\Omega(k) \sim k^{-5/3}$ in a long constant enstrophy flux range</u>. In terms of energy spectrum, one cannot have spectrum steeper than $E(k) \sim k^{-11/3}$. It is amusing that Moffatt's spectrum is marginal for this argument. However, Saffman's k^{-4} spectrum and many of the spectra in Polyakov's conformal models are clearly ruled out by these arguments. The previous results were obtained in:

G. L. Eyink, "Exact results on scaling exponents in the 2D enstrophy cascade," Phys. Rev. Lett. 74 3800–3803 (1995)

But more is true. It can be shown using the 1989 Fields-Medal winning work of DiPerna & Lions that even for $p=2$

$$\sigma_2 \leq 0.$$

This means that the energy spectrum in the enstrophy cascade range of arbitrary length must be no steeper* than the KB spectrum k^{-3}. This rules out Moffatt's spectrum and all of the spectra proposed by Polyakov, and gives moreover strong support to the original proposal of Kraichnan–Batchelor. We will discuss this application of the DiPerna–Lions theory in detail later.

---

* up to logarithms.

Scale-locality and physical mechanisms. Based on the previous results we now discuss the scale-locality and physical mechanisms of the dual 2D cascades.

The discussion of scale-locality of the 2D inverse energy cascade is very similar to the discussion for the 3D forward energy cascade in Turbulence I, Coursenotes, Section II(E). We shall review this here briefly, but for full details see

G. L. Eyink, "Locality of turbulent cascades," Physica D **207** 91–116 (2005)

The essential points are that

(i) Quantities $\nabla\bar{u}_\ell$, τ_ℓ, Π_ℓ and Π_r can be written in terms of velocity-increments $\delta u(r)$, as we have seen in the previous section.

(ii) In a long range of length scales r where $\delta u(r) \sim r^h$, the velocity increments are scale-local.

This latter property means, precisely, that the following two conditions hold:

$$\underline{\text{IR scale-locality}}: \quad \frac{|\delta \bar{u}_\Delta(r)|}{|\delta u(r)|} = O\left(\left(\frac{r}{\Delta}\right)^{1-h}\right) \quad r<\Delta$$

$$\underline{\text{UV scale-locality}}: \quad \frac{|\delta u'_\delta(r)|}{|\delta u(r)|} = O\left(\left(\frac{\delta}{r}\right)^{h}\right) \quad r>\delta$$

where the prime " ′ " has a different meaning than in the previous section and now denotes

$$u'_\delta(x) = u(x) - \bar{u}_\delta(x)$$

the <u>small-scale/unresolved velocity</u> at length-scales $<\delta$.

Thus, IR locality of the increment means that length-scales $\Delta \gg r$ make a negligible contribution for $h<1$. Likewise, UV locality of the increment means that length-scales $\delta \ll r$ make a negligible contribution for $h>1$. These results hold both pointwise in terms of Hölder exponents h, or for pth-order moments (L_p-means) in terms of scaling exponents $\sigma_p = \zeta_p/p$ (Besov indices).

Because all theories of 2D inverse cascade assume exponents h or σ_p between 0 and 1 (e.g. the Kraichnan theory proposes $h = \sigma_p = \frac{1}{3}$ for all p), the fluxes Π_ℓ and Π_p are scale-local in both the IR and the UV, just as in 3D. There is just one important difference, which was pointed out by

R. H. Kraichnan, "Inertial-range transfer in two- and three-dimensional turbulence," J. Fluid Mech. 47 525-535 (1971)

Kraichnan pointed out that the spectral flux depends only upon the shell transfers

$$T(k,p,q) = \langle T_{k,p,q} \rangle_{avg.} \cdot (4\pi k^2)(4\pi p^2)(4\pi q^2)$$

where $\langle \cdot \rangle_{avg}$ denotes the angular average over all possible directions of the three wave-vectors satisfying $|\mathbf{k}| = k$, $|\mathbf{p}| = p$, $|\mathbf{q}| = q$. In fact,

$$\Pi(K) = -\int_{k_0}^{K} dk \int_{k_0}^{\infty} dp \int_{k_0}^{\infty} dq \, T(k,p,q)$$

and the issue of scale-locality can be framed as the question whether the wavenumber magnitudes k,p,q which contribute most to $\Pi(K)$ are those with $k,p,q \approx K$ within a factor of a few. However, detailed conservation of energy implies

$$T(k,p,q)+T(p,q,k)+T(q,k,p)=0$$

and detailed conservation of enstrophy implies

$$k^2T(k,p,q)+p^2T(p,q,k)+q^2T(q,k,p)=0.$$

If any two of the wavenumber magnitudes are equal then there are only two independent transfers (e.g. $T(k,p,k)=T(k,k,p)$ and $T(p,k,k)$ for $q=k$) and the above two relations imply that both must <u>vanish</u>. It follows by continuity that in 2D the transfers $T(k,p,q)$ must vanish if any two of k,p,q approach each other, and the "super-local" contributions to the flux vanish in 2D. This is very different from 3D. Using his TFM closure, Kraichnan predicted that most flux in 2D arises from interactions of K with wavenumbers about 8 times larger in 2D. See Figs. 1 & 2 from Kraichnan (1971), reproduced on the next page.

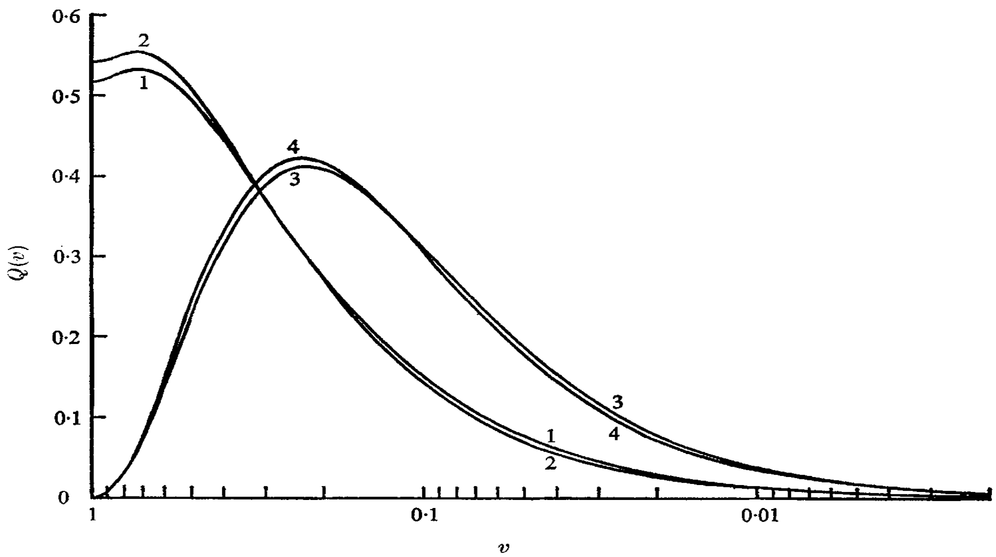


FIGURE 1. Localness of energy transfer. Curves 1 and 2: three dimensions, with (2.17a) and (2.17b), respectively. Curves 3 and 4: two dimensions, with (2.17a) and (2.17b), respectively.

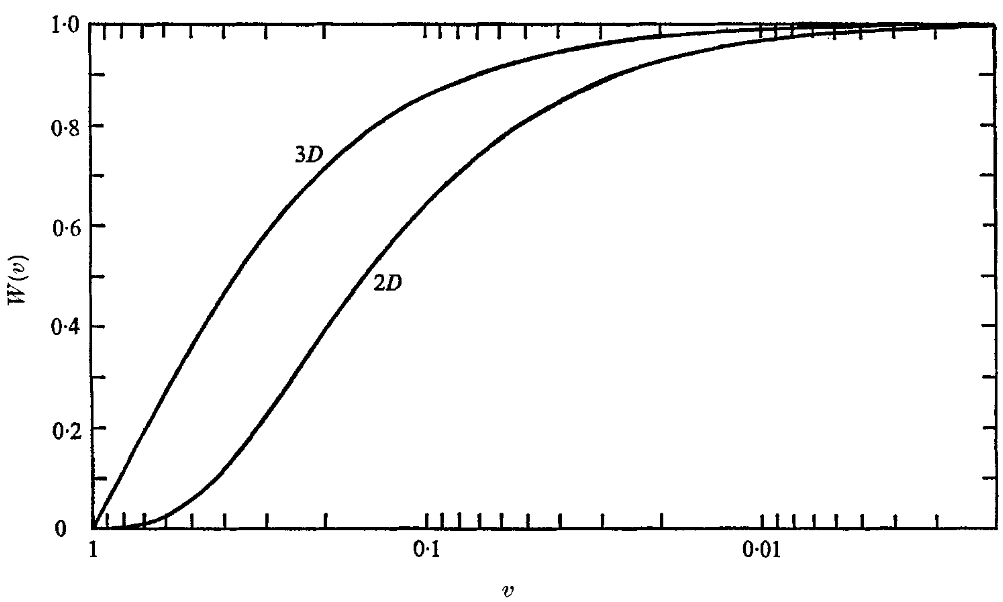


FIGURE 2. The function $W(v)$ in three dimensions (3D) and two dimensions (2D).

While the 2D inverse energy cascade is scale-local (if just a bit less than 3D), the 2D forward enstrophy cascade is a very different story. Neither IR nor UV locality are obvious for quantities such as $\nabla\bar{\omega}_\ell$, σ_ℓ and Z_ℓ in the KB theory of the 2D enstrophy cascade. In fact, these quantities depend upon both $\delta\omega(r)$ and $\delta u(r)$, and in the KB theory $h_\omega = 0$ and $h_u = 1$. Thus, UV locality is marginal for $\delta\omega(r)$, while IR locality is marginal for $\delta u(r)$! Thus, any kind of locality whatsoever appears questionable for the enstrophy flux Z_ℓ in the KB theory.

On the other hand, the "smeared" point-split enstrophy flux

$$Z^*_\ell(x) = \int d^2r\, G_\ell(r)\, Z_r(x)$$
$$= -\frac{1}{4\ell^3}\int d^2r\, (\nabla\Delta G)_\ell(r)\cdot\delta u(r)\,|\delta u(r)|^2$$

depends only on the velocity increment $\delta u(r;x)$! Thus, this quantity is strongly UV local. In fact, since in the KB theory $\delta u(r) \sim r \ln^{1/2}\left(\frac{\ell_f}{r}\right)$, it is not hard to show that

$$\frac{|Z_\ell^{*<\delta}(x)|}{|Z_\ell^{*}(x)|} = O\left(\frac{\delta}{\ell}\right) \quad \text{(modulo log's)}$$

where $Z_\ell^{*<\delta}$ is the contribution to Z_ℓ^{*} in which at least one velocity mode is restricted to length scales $< \delta$. The IR locality of Z_ℓ^{*} is quite a different story. Because of the scaling properties in the KB range

$$\frac{|Z_\ell^{*>\Delta}(x)|}{|Z_\ell^{*}(x)|} \cong O\left(\frac{1}{\ln^{1/2}(\Delta/r)}\right)$$

and $Z^{*>\Delta}$ is virtually the same as Z_ℓ^{*}. This is believed to be correct, not merely a limitation of the mathematical estimates. In fact, it was already proposed by Kraichnan (1967) that the <u>2D enstrophy cascade is IR-nonlocal</u>. We shall discuss the source of this nonlocality in more detail shortly.

It turns out that all the same locality statements carry over to the enstrophy flux Z_ℓ defined by spatial coarse-graining. This quantity can also be written entirely in terms of velocity-increments $\delta u(r)$!
Note first that

$$\nabla \bar{\omega}_\ell(x) = \nabla \nabla^\perp \cdot \bar{u}_\ell(x) = \frac{1}{\ell^2} \int d^2r \, (\nabla \nabla^\perp G)_\ell(r) \, \delta u(r).$$

Furthermore, from the standard relation

$$\omega u^\perp = \omega \times u = \nabla \cdot (uu) - \nabla\left(\tfrac{1}{2}|u|^2\right)$$

it follows easily that

$$\sigma_\ell^\perp = \tau_\ell(\omega, u^\perp) = \nabla \cdot \tau_\ell(u,u) - \nabla k_\ell$$

with $k_\ell = \frac{1}{2}\tau_\ell(u_i,u_i)$. Finally, we use the "shift trick" from Turbulence I, Coarsenotes, Section II(D), to see that

$$(\nabla \cdot \tau_\ell(u,u))_i = -\frac{1}{\ell}\Big[\int d^2r \, (\partial_j G)_\ell(r) \, \delta u_i(r) \delta u_j(r)$$

$$- \left(\int d^2r \, (\partial_j G)_\ell(r) \, \delta u_i(r)\right)\left(\int d^2r \, G_\ell(r) \, \delta u_j(r)\right)\Big]$$

and likewise

$$\partial_{x_i} k_\ell = -\frac{1}{2\ell}\left[\int d^2r\, (\partial_i G)_\ell(\mathbf{r})\, |\delta u(\mathbf{r})|^2 \right.$$

$$\left. -2\left(\int d^2r\, (\partial_i G)_\ell(\mathbf{r})\, \delta u(\mathbf{r}) \right)\cdot\left(\int d^2r\, G_\ell(\mathbf{r})\, \delta u(\mathbf{r}) \right) \right]$$

From these formulas one can see that Z_ℓ possesses all the same locality properties as Z_ℓ^* does.

UV scale locality is a very important property with numerous implications. For one thing, it suggests that the properties of the 2D enstrophy cascade range should be independent of the precise mechanism of small-scale dissipation. E.g. nothing should change for very long enstrophy ranges if a hyperviscosity is used rather than ordinary viscosity. Furthermore, UV scale locality can be made the basis of quantitative approximations to quantities such as σ_ℓ and Z_ℓ. (This is reminiscent of the way in which spatial locality can be employed as the basis of quantitative approximations in the theory of critical phenomena; cf. K. Wilson, Rev. Mod. Phys. (1975)). This scheme of approximation was already pursued for the 3D energy cascade in Turbulence I, Coursenotes, Section IV(A). Here we pursue the same approach in 2D.

The idea of the approximation is to appeal to UV locality to make first the replacement

$$\delta u(r;x) \longrightarrow \delta\overline{u}_\ell(r;x).$$

Next, because $\overline{u}_\ell$ is a smooth field, one can Taylor expand to leading-order to get

$$\delta\overline{u}_\ell(r;x) \longrightarrow r\cdot\nabla\overline{u}_\ell(x).$$

The physical assumption here is space-locality, since the integrals over r are confined to a domain $|r| \lesssim \ell$. The results of these approximations are algebraic combinations of $\nabla\overline{u}_\ell(x)$ and possibly higher-gradients.

Consider this approximation for the subscale vorticity transport vector $\sigma_\ell(x)$. It is not hard to see that one gets the same answer using either the original expression

$$\begin{aligned}\sigma_\ell &= \overline{(\omega u)}_\ell - \overline{\omega}_\ell\overline{u}_\ell \\ &= \int d^2r\, G_\ell(r)\,\delta\omega(r)\,\delta u(r) \\ &\quad - \left(\int d^2r\, G_\ell(r)\,\delta\omega(r)\right)\left(\int d^2r\, G_\ell(r)\,\delta u(r)\right)\end{aligned}$$

or the alternative expression which only involves $\delta u(r)$ (and which justifies UV locality). If one chooses, for simplicity, a filter kernel $G(r)$ which depends only on $r = |\mathbf{r}|$, the result is

$$(*) \qquad \sigma_\ell^{NL}(\mathbf{x}) = \frac{1}{2} C \ell^2 \overline{\mathbf{D}}_\ell(\mathbf{x}) \cdot \nabla \overline{\omega}_\ell(\mathbf{x})$$

where
$$\overline{D}_{\ell ij} = \frac{\partial \overline{u}_{\ell i}}{\partial x_j}$$

is the large-scale velocity-gradient / deformation matrix, and $C = 2\pi \int_0^\infty r^3 G(r)\,dr$. The approximation $(*)$ is the analogue of the so-called nonlinear model or tensor eddy-viscosity model used in 3D for τ_ℓ.

This approximation gives support to Batchelor's ideas on the physical-space mechanisms of the k^{-3} range. Because the flow is area-preserving,

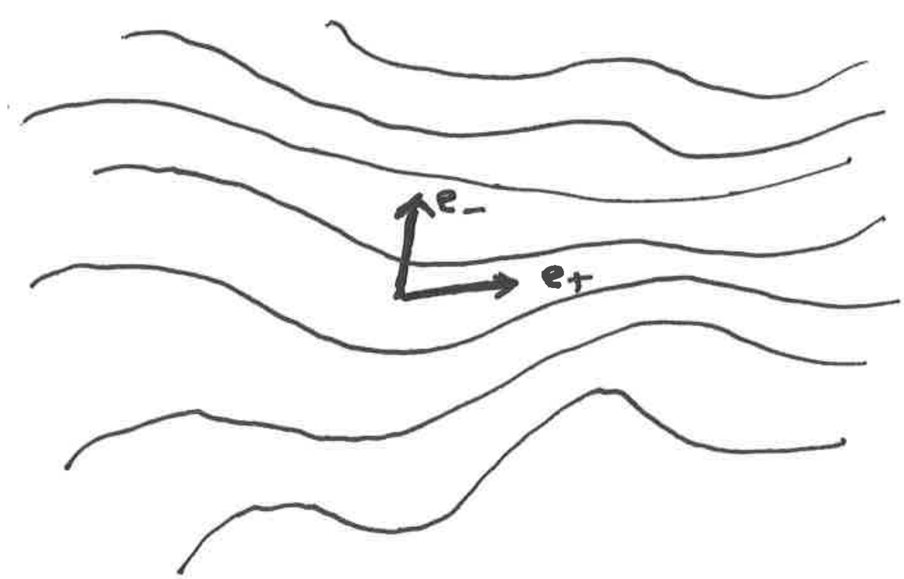
$$\mathrm{tr}(\overline{\mathbf{D}}_\ell) = \nabla \cdot \overline{\mathbf{u}}_\ell = 0 .$$

Thus, the eigenvalues of $\overline{\mathbf{D}}_\ell$ are either:

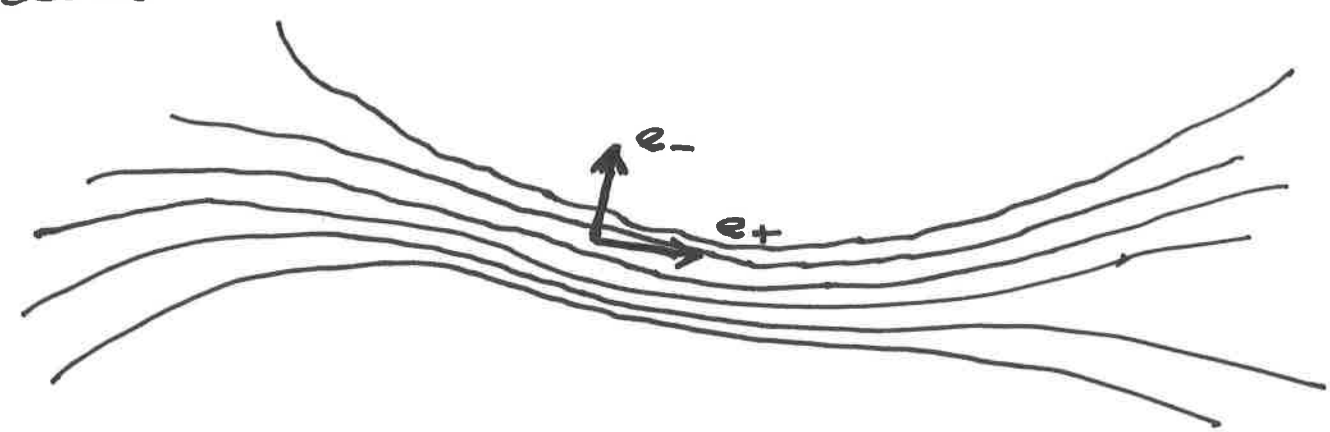
(i) pure imaginary ($\pm i \overline{\beta}_\ell$), as expected in vorticity dominated regions (vortices), or

(ii) real pair ($\pm \overline{\gamma}_\ell$), as expected in strain dominated regions (outside vortices)

Consider now what happens in such latter strain dominated regions to vorticity isolines. These are compressed in the eigendirection e_- corresponding to $-\overline{\gamma}_\ell$ and stretched in the eigendirection e_+ corresponding to $+\overline{\gamma}_\ell$ (assuming that the isolines of $\overline{\omega}_\ell$ are "nearly" material lines). The picture is as below:



becomes



This process of squeezing creates large vortizity-gradients along e_- and reduces gradients along e_+, so that with enhanced probability,

$$(**) \qquad \nabla\bar{\omega}_\ell(\mathbf{x}) \propto \mathbf{e}_-(\mathbf{x}).$$

This alignment is not exact, because $\bar{\omega}_\ell$-lines are not exactly material lines of $\bar{\mathbf{u}}_\ell$ and because $\bar{\mathbf{D}}_\ell$ is also changing in time on a scale comparable, just slightly slower, than $\nabla\bar{\omega}_\ell$. However, when $(**)$ holds, the approximation $(*)$ becomes

$$\boldsymbol{\sigma}_\ell^{NL}(\mathbf{x}) \cong -\frac{1}{2}C\ell^2\bar{\gamma}_\ell(\mathbf{x})\nabla\bar{\omega}_\ell(\mathbf{x})$$

so that the vorticity transport becomes <u>down-gradient</u>. In fact, this is equivalent to

$$\boldsymbol{\sigma}_\ell^{NL}(\mathbf{x}) \cong -\nu_\ell(\mathbf{x})\nabla\bar{\omega}_\ell(\mathbf{x})$$

with an <u>eddy-viscosity</u> $\nu_\ell(\mathbf{x}) = \frac{1}{2}C\ell^2\bar{\gamma}_\ell(\mathbf{x})$.

Finally, using just $(*)$, and no further approximation, notice that

$$Z_\ell(\mathbf{x}) \cong \frac{1}{2}C\ell^2(\nabla\bar{\omega}_\ell(\mathbf{x}))^\top \bar{\mathbf{S}}_\ell(\mathbf{x})(\nabla\bar{\omega}_\ell(\mathbf{x}))$$

or the rate of <u>vorticity-gradient stretching</u> by the large-scale strain field $\bar{\mathbf{S}}_\ell(\mathbf{x})$. This is an exact statement about the UV-local contribution to enstrophy flux.

To see how well these approximations represent reality, let us examine the results of

S. Chen et al. "Physical mechanism of the two-dimensional enstrophy cascade," Phys. Rev. Lett. 91 214501 (2003)

who simulated the 2D enstrophy cascade using hyperviscosity at high wavenumbers and inverse Laplacian damping at low wavenumbers, at 2048^2 resolution. Those simulations found that 2D plots of $Z_\ell(x)$ and $Z_\ell^{NL}(x)$ side-by-side are essentially identical and cannot be distinguished by eye! This is an indication that UV locality holds very well.

In Fig. 2 of that paper (next page) are plotted spatial PDF's of Z_ℓ for different ℓ values in the enstrophy cascade range [(a)] and a PDF comparison of Z_ℓ and Z_ℓ^{NL} for a fixed such ℓ [(b)]. First, it can be seen that the PDF's of Z_ℓ and Z_ℓ^{NL} are indeed indistinguishable. In part (a) one can also see that the PDF of Z_ℓ becomes increasingly skewed to the right as ℓ decreases. Nevertheless, Z_ℓ remains negative at nearly 40% of the space points! Hence, the positive average over space $\langle Z_\ell \rangle = \eta$ arises from a small difference between two large, nearly cancelling contributions of opposite sign.

,

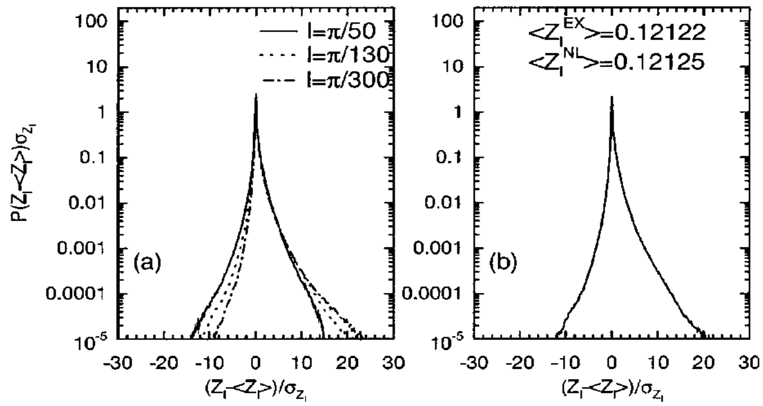


FIG. 2. (a) PDF of $(Z_\ell - \langle Z_\ell\rangle)/\sigma_Z$ with $Z_\ell(\mathbf{r}, t)$ enstrophy flux and $\sigma_Z^2 = \langle (Z_\ell - \langle Z_\ell\rangle)^2\rangle$, at different filter lengths ℓ. (b) PDF's of the true flux (solid line) and the "nonlinear model" (dashed line) at $\ell = \pi/130$. The two lines are indistinguishable.

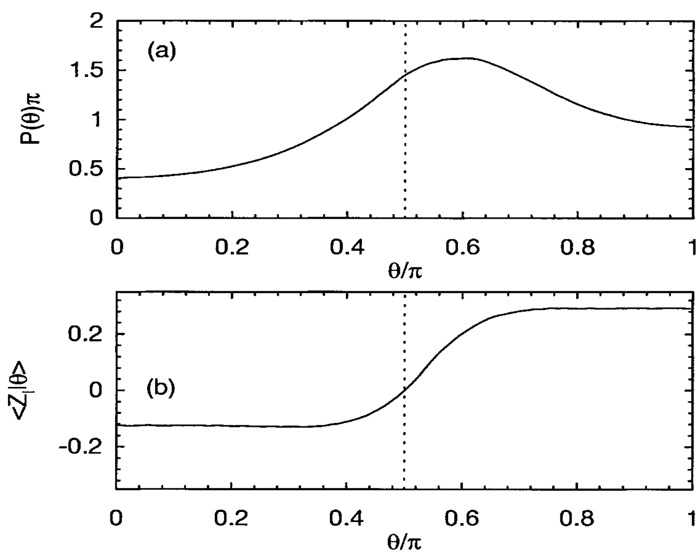


FIG. 3. (a) PDF of the angle θ between $\boldsymbol{\sigma}_\ell$ and $\nabla\overline{\varpi}_\ell$ and (b) conditional mean $\langle Z_\ell|\theta\rangle$, for $\ell = \pi/130$.

Further information comes from Fig. 3 of that paper, which plots in part (a) the PDF of the angle θ between σ_ℓ and $\nabla\bar{\omega}_\ell$. The most probable angle is, very interestingly, close to $\frac{\pi}{2}$. Hence, σ_ℓ and $\nabla\bar{\omega}_\ell$ are perpendicular with high probability! There is only a somewhat greater tendency for σ_ℓ to be anti-parallel to $\nabla\bar{\omega}_\ell$ $(\theta \cong \pi)$ than to be parallel $(\theta \cong 0)$. In part (b) of that plot one can see that the magnitude of the conditional expectation $\langle Z_\ell | \theta \rangle$ tends to be somewhat greater for $\theta > \frac{\pi}{2}$ where it is positive, than for $\theta < \frac{\pi}{2}$ where it is negative. These two facts explain why $\langle Z_\ell \rangle > 0$.

Fig. 4 of the paper provides information about the spatial location of forward enstrophy cascade. In panel (a) is plotted $\phi_\ell = \det(\bar{D}_\ell) = \frac{1}{2}\bar{\omega}_\ell^2 - \bar{S}_\ell^2$ to identify regions as either vorticity-dominated or strain-dominated. In panel (b) is plotted the spatial field of $Z_\ell(x)$. One can clearly see that the positive regions occur mainly in the strain-dominated regions of the flow. Notice also how the flux has intense streaks that are clearly associated with Batchelor's suggested mechanism of isoline compression and stretching.

,

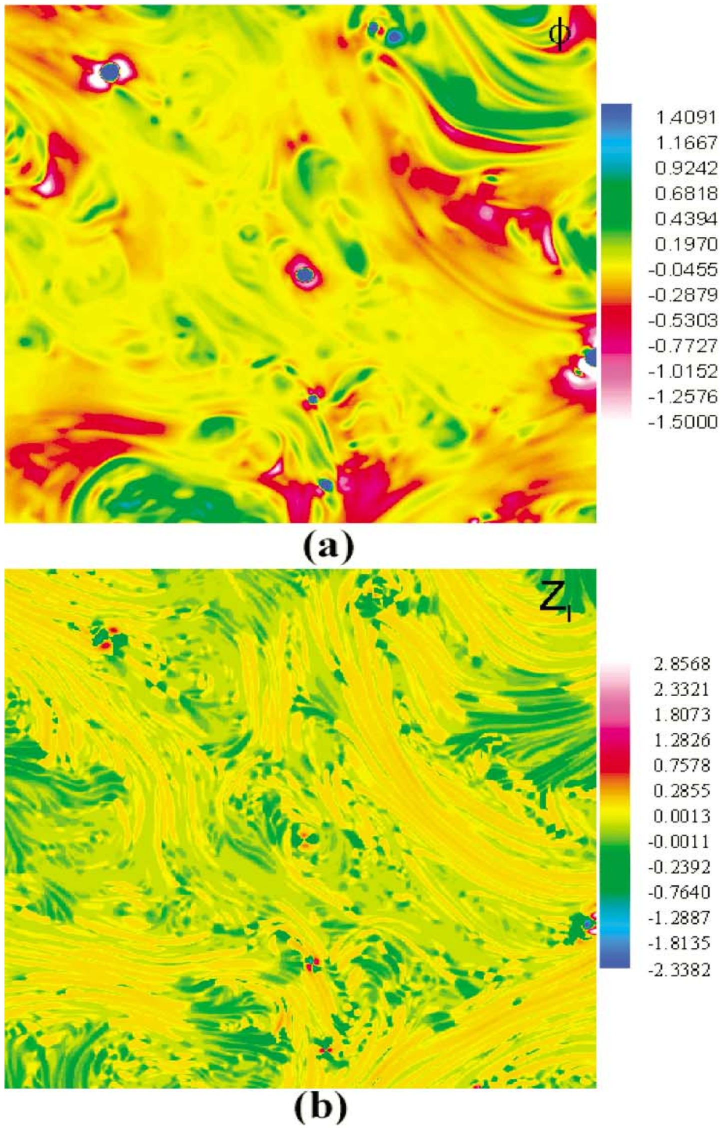


FIG. 4 (color). Instantaneous snapshot of (a) $\phi_\ell(\mathbf{r}, t)$ and (b) $Z_\ell(\mathbf{r}, t)$, for $\ell = \pi/130$ in a 512^2 subdomain. The red regions in (a) are dominated by strain and the green by vorticity.

Fig. 5 of the paper provides more quantitative information about the relative contribution of the enstrophy cascade from strain and vortical regions. In panel (a) is plotted the PDF of Z_ℓ conditioned on both $\phi_\ell > 0$ (vortical) and on $\phi_\ell < 0$ (strain). It can be seen that the PDF of Z_ℓ is nearly symmetrical in the vortex regions, with a quite small mean, whereas the PDF in the strain dominated region is distinctly skewed to the right. In panel (b) is plotted the PDF of the angle θ conditioned upon $\phi_\ell > 0$ and $\phi_\ell < 0$. The tendency for σ_ℓ to be perpendicular to $\nabla\omega_\ell$ is quite large in the vortices, but more muted in the straining regions outside. In addition, the probabilities to be parallel and anti-parallel are nearly the same inside vortices, but anti-parallel configurations dominate outside.

Finally, Fig. 6 of the paper plots the alignment angle between $\nabla\bar{\omega}_\ell$ and the right eigenvector of $\bar{\mathbf{D}}_\ell$ with negative eigenvalue. (Note the paper defines $\bar{\mathbf{D}}_\ell \Rightarrow \bar{\mathbf{D}}_\ell^T$!) There is an increasing tendency for these to be aligned as ℓ decreases, which supports again Batchelor's picture of the dynamics of the k^{-3} range.

,

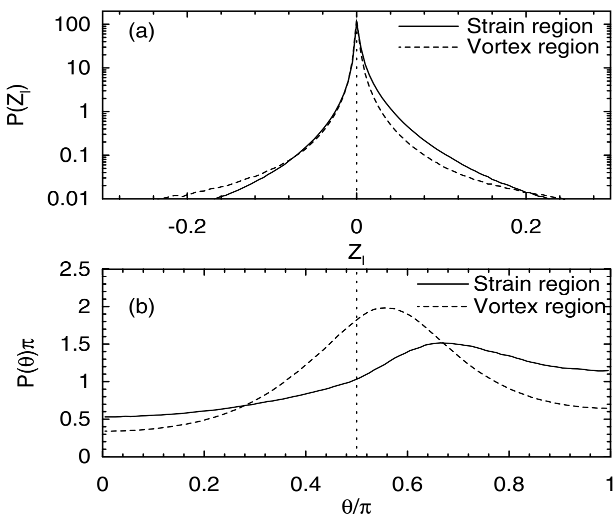


FIG. 5. (a) PDF's of $Z_\ell(\mathbf{r}, t)$ and (b) PDF's of the angle θ between $\boldsymbol{\sigma}_\ell$ and $\boldsymbol{\nabla}\overline{\omega}_\ell$, $\ell = \pi/130$. The solid line indicates strain, and the dashed line indicates the vorticity region. $\langle Z_\ell \rangle_{\mathrm{strain}} = 3.7 \times 10^{-3}$ and $\langle Z_\ell \rangle_{\mathrm{vortex}} = 9.5 \times 10^{-4}$.

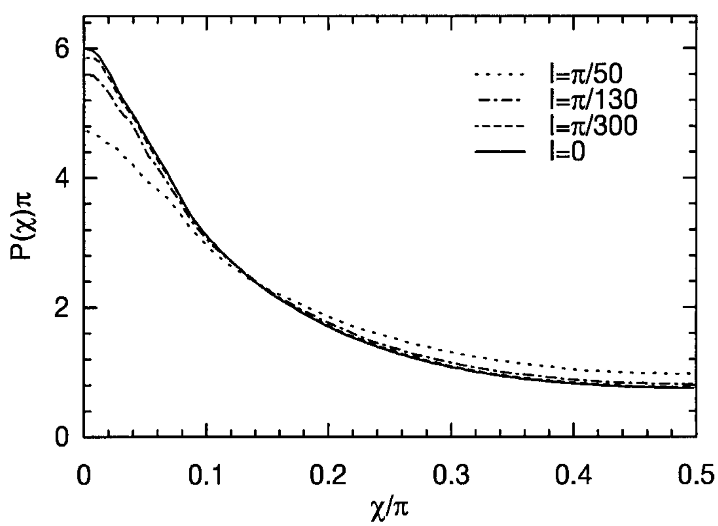


FIG. 6. PDF's of the angle χ between $\boldsymbol{\nabla}\overline{\omega}_\ell$ and $\boldsymbol{l}_\ell^-$, the left eigenvector of $\mathbf{D}_\ell$ for the negative eigenvalue.

These empirical results support the accuracy of the approximation (*), the assumptions underlying it, and further conclusions that can be drawn from it. In particular, the 2D enstrophy cascade is strongly UV local but IR nonlocal (or, at best, marginally local). The latter property should lead one to doubt that all of the properties of the 2D enstrophy cascade range are "IR universal," i.e. independent of the particular large-scale driving in the steady-state (or initial) conditions in free decay).

Another possible implication of the IR nonlocality is that was suggested by Kraichnan (1971) is that there may be a <u>logarithmic correction</u> to the k^{-3} energy spectrum. More specifically, Kraichnan (1971) proposed that $C\eta^{2/3}k^{-3}$ be replaced by

$$E(k) \sim C\eta^{2/3} k^{-3} \ln^{-1/3}(k/k_f).$$

We can easily understand Kraichnan's reasoning on the basis of the approximation (*). If the traditional KB k^{-3} spectrum held, then it is easy to see that

$$(\overline{S}_\ell)_{rms} \cong \left(\int_{k_f}^{2\pi/\ell} k^2 E(k)\,dk \right)^{1/2}$$

$$= \left(C\eta^{2/3} \int_{k_f}^{2\pi/\ell} \frac{dk}{k} \right)^{1/2} \cong C'\eta^{1/3} \ln^{1/2}\left(\frac{1}{k_f \ell}\right)$$

and

$$(\nabla\overline{\omega}_\ell)_{rms} \cong \left(\int_{k_f}^{2\pi/\ell} k^4 E(k)\,dk \right)^{1/2}$$

$$= \left(C\eta^{2/3} \int_{k_f}^{2\pi/\ell} k\,dk \right)^{1/2} \cong C'' \frac{\eta^{1/3}}{\ell}$$

Hence, ignoring alignment angles, we estimates

$$Z_\ell \cong \frac{1}{2} C \ell^2 (\overline{S}_\ell)_{rms} (\nabla\overline{\omega}_\ell)^2_{rms} \cong (\text{const.})\, \eta \ln^{1/2}\left(\frac{1}{k_f \ell}\right).$$

This is not independent of ℓ! Kraichnan (1971) was worried about this inconsistency and suggested that it could be fixed by the logarithmic correction. Indeed, in that case

$$(\overline{S}_\ell)_{rms} \cong \left(C\eta^{2/3} \int_{k_f}^{2\pi/\ell} \frac{dk}{k \ln^{1/3}(k/k_f)} \right)^{1/2} \cong C'\eta^{1/3} \ln^{1/3}\left(\frac{1}{k_f \ell}\right)$$

and

$$(\nabla \bar{\omega}_\ell)_{rms} \cong \left(C\eta^{2/3} \int_{k_f}^{2\pi/\ell} \frac{k\,dk}{\ln^{1/3}(k/k_f)} \right)^{1/2} \cong C''\eta^{1/3} \frac{1}{\ell \ln^{1/6}(1/k_f \ell)}.$$

In that case, one finds that

$$Z_\ell \cong \eta,$$

self-consistently. Thus, the logarithmic correction is one way to remove the (weak) divergence in the estimate of Z_ℓ. Of course, it is also possible that the estimate is simply too large, because it ignores the reduction in Z_ℓ due to misalignment of angles. The evidence from numerical simulations is mixed. Some works such as Borue (1993), Chen et al. (2003) claim consistency with the log-correction, while other studies such as Vallgren & Lindborg (2011) claim it is absent. Detecting or ruling out a logarithmic factor is exceedingly difficult numerically! We just note here that the total enstrophy with the log-correction

$$\Omega = \int_{k_f}^{k_{uv}} C\eta^{2/3} \frac{dk}{k \ln^{1/3}(k/k_f)} = (\text{const.})\, \eta^{2/3} \ln^{2/3}\left(\frac{k_{uv}}{k_f}\right)$$

still diverges as $k_{uv} \to \infty$. Hence, the Kraichnan (1971) proposal is consistent with the Di Perna-Lions rigorous theory that we discuss later.