b) Decaying 2D Turbulence

*It was first realized by G.I. Taylor in the years 1915-1917 that the absence of vortex-stretching in 2D implies that one of the most fundamental aspects of 3D turbulence — the non-vanishing of energy dissipation as $\nu \to 0$ — must be absent in 2D. This fact is sometimes stated, rather unfortunately, as "there is no 2D turbulence." However, there is no doubt that turbulence in 2D is drastically different than in 3D, as was emphasized not only by Taylor but also in early works of Onsager (1949), T.D. Lee (1951), R. Fjørtoft (1953), and Batchelor (1953), which we shall discuss in detail. Here we begin by presenting a simple proof of the absence of "anomalous energy dissipation," taken from the presentation of

R.W. Bray, "A study of turbulence and convection using Fourier and numerical analysis," Ph.D. dissertation at Cambridge University, under the direction of G.K. Batchelor (1966)

The starting point is the usual equation for viscous dissipation of a freely decaying NS solution:

$$\frac{dE_\nu}{dt} = -\nu \langle \omega^2 \rangle = -2\nu\Omega_\nu,$$

where $\langle \cdot \rangle$ stands for averaging over the flow domain

and we have assumed b.c. such that $\int_{\partial D} \frac{\partial \rho}{\partial n} dA = 0$, e.g. periodic b.c. It next follows from

$$\partial_t \left(\tfrac{1}{2} \omega^2 \right) + \nabla \cdot \left(\tfrac{1}{2} \omega^2 u - \nu \nabla \left(\tfrac{1}{2} \omega^2 \right) \right) = -\nu |\nabla \omega|^2$$

that, again for a homogeneous average,

$$\frac{d}{dt} \Omega_\nu = -\nu \langle |\nabla \omega|^2 \rangle \leq 0 .$$

(Note that the quantity $P = \langle |\nabla \omega|^2 \rangle$ is sometimes called the "palinstrophy".) Integrating the above inequality gives

$$\Omega_\nu(t) \leq \Omega(t_0) .$$

Since

$$E_\nu(t) = E(t_0) - 2\nu \int_{t_0}^{t} ds \, \Omega_\nu(s) ,$$

the previous inequality allows us to bound the integral by

$$\left| \int_{t_0}^{t} ds \, \Omega_\nu(s) \right| \leq (t - t_0) \Omega(t_0)$$

and thus

$$\lim_{\nu \to 0} E_\nu(t) = E(t_0) !$$

No energy is lost in 2D flow in the limit as $\nu \to 0$. This is the result well-known since Taylor (1915-1917).

* An important improvement of this result was obtained by the atmospheric scientist R. Fjørtoft in

R. Fjørtoft, "On the changes in the spectral distribution of kinetic energy for two-dimensional nondivergent flow," Tellus 5 225–230 (1953)

We shall present the proof for a periodic domain using Fourier series, although Fjørtoft derived his result on the two-dimensional sphere (surface of a ball) by an equivalent argument using spherical harmonics. Let us define the energy at time t in wavenumbers with magnitude greater than K by

$$E_{>K}(t) = \frac{1}{2} \sum_{|k|>K} |\hat{u}(k,t)|^2 .$$

The basic bound of Fjørtoft, valid for both 2D NS and 2D Euler, is that

$$\boxed{E_{>K}(t) \leq \frac{\Omega(t_0)}{K^2}}$$

(Fjørtoft energy bound)

where $\Omega(t_0)$ is the initial enstrophy. This bound sets a limit on the amount of energy which can ever reach high-wavenumbers, globally in time! Let us prove it, very simply, using the fact that

$$\hat{\omega}(\mathbf{k},t) = i\mathbf{k}^{\perp}\cdot\hat{\mathbf{u}}(\mathbf{k},t)$$

which, since $\mathbf{k}\cdot\hat{\mathbf{u}}(\mathbf{k},t)=0$, implies that

$$|\hat{\omega}(\mathbf{k},t)|^2 = |\mathbf{k}|^2|\hat{\mathbf{u}}(\mathbf{k},t)|^2.$$

Thus, by Chebyshev inequality,

$$\begin{aligned}
E_{>K}(t) &\leq \frac{1}{K^2}\sum_{|\mathbf{k}|>K} \tfrac{1}{2}|\mathbf{k}|^2|\hat{\mathbf{u}}(\mathbf{k},t)|^2 \\
&= \frac{1}{K^2}\sum_{|\mathbf{k}|>K} \tfrac{1}{2}|\hat{\omega}(\mathbf{k},t)|^2 \\
&\leq \frac{1}{K^2}\sum_{\mathbf{k}} \tfrac{1}{2}|\hat{\omega}(\mathbf{k},t)|^2 \\
&= \frac{1}{K^2}\Omega(t) \leq \frac{1}{K^2}\Omega(t_0). \quad \underline{\text{QED}}
\end{aligned}$$

Fjørtoft's 1953 paper brought a very important new idea to 2D turbulence, the notion of <u>spectral blocking</u>. Thus, a mode with energy $E_{\mathbf{k}} = \frac{1}{2}|\hat{\mathbf{u}}(\mathbf{k},t)|^2$ must have an enstrophy $\Omega_{\mathbf{k}} = |\mathbf{k}|^2 E_{\mathbf{k}}$. Modes with substantial energy $E_{\mathbf{k}}$ are not allowed at large $|\mathbf{k}|$, because this would imply a very large enstrophy $|\mathbf{k}|^2 E_{\mathbf{k}}$ much larger than the initial enstrophy, violating conservation of enstrophy. The idea of "spectral blocking" is key in 2D turbulence.

If the energy does not cascade to high wavenumbers as it does in 3D, then where does it go? Fjørtoft presented an argument that the <u>energy must be transferred toward large scales.</u> We have give a slightly modified version of his argument. Note using the NS equations in Fourier representation that

$$\frac{dE_k}{dt} = \sum_{p+q=k} T_{kpq}$$

where the <u>transfer</u> T_{kpq} is defined by

$$T_{kpq} = \frac{1}{2} \mathrm{Re}\left[(ik \cdot \hat{u}_q)(\hat{u}_p \cdot \hat{u}_k^*) \right] + (p \leftrightarrow q)$$

so that $T_{kpq} = T_{kqp}$. Just as for the coefficients in the helical decomposition in 3D, it is easy to see that energy conservation requires

$$T_{kpq} + T_{pqk} + T_{qkp} = 0$$

so-called <u>detailed energy conservation</u> and likewise enstrophy conservation gives

$$|k|^2 T_{kpq} + |p|^2 T_{pqk} + |q|^2 T_{qkp} = 0$$

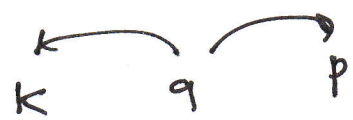
or <u>detailed enstrophy conservation</u>.

Let us assume that $|k| < |q| < |p|$, so that k corresponds to the largest scale mode, p to the smallest scale mode, and $|q|$ to intermediate scale.

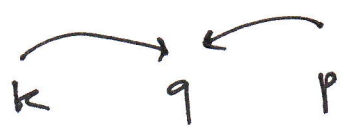
Multiplying the first equation by $|q|^2$ and subtracting the second gives

$$\frac{T_{kpq}}{T_{pqk}} = \frac{|p|^2 - |q|^2}{|q|^2 - |k|^2} > 0$$

Hence, the transfers into modes k & p have the same sign and the opposite sign as the transfer into q. There are two possibilities:



transfer out of q



transfer into q

If the dynamics is mixing in Fourier space, then the former should be more likely. Furthermore, if $|p|^2 > 2|q|^2$ then

$$|p|^2 + |k|^2 > 2|q|^2$$

implies that

$$\frac{T_{kpq}}{T_{pqk}} = \frac{|p|^2 - |q|^2}{|q|^2 - |k|^2} > 1$$

so that the energy is transferred at a faster rate to the large-scale mode $\mathbf{k}$ than to the small-scale mode $\mathbf{p}$. (Note, BTW, that necessarily $|p| = |k-q| \leq |k| + |q| \leq 2|q|$.)

* Even earlier Onsager had reached a more specific conclusion. See

L. Onsager, "Statistical hydrodynamics," Nuovo Cim. Suppl. 6 249–287 (1949)

and, for a review of Onsager's theory,

G. L. Eyink & K. R. Sreenivasan, "Onsager and the theory of hydrodynamic turbulence," Rev. Mod. Phys. 78 87–135 (2006).

Onsager asked the question: what will be the final state reached by decaying turbulence in 2D in the limit as $\nu \to 0$ ($Re \gg 1$)? Since

energy is conserved in that limit, it is plausible that the long-time state of the fluid is described by <u>thermodynamic equilibrium</u> in a <u>microcanonical ensemble</u> (fixed energy). Specifically, Onsager considered the <u>point-vortex model</u> of Kirchoff in which the vorticity is a sum of delta functions

$$\omega(\mathbf{x}, t) = \sum_{i=1}^{N} k_i \, \delta^2(\mathbf{x} - \mathbf{x}_i(t))$$

and the Euler equations reduce to the set of Hamilton's equations for the coordinates $\mathbf{x}_i = (x_i, y_i)$:

$$k_i \dot{x}_i = \frac{\partial H}{\partial y_i}$$

$$k_i \dot{y}_i = -\frac{\partial H}{\partial x_i}$$

with

$$H = -\sum_{i<j} k_i k_j \, G(\mathbf{x}_i, \mathbf{x}_j)$$

the total energy of the point-vortex array (with an infinite constant subtracted). Note that any smooth continuous solution of the Euler equation in 2D can be obtained from the point-vortex model by taking the limit of suitable spatial distributions of point vortices.

See:

C. Marchioro & M. Pulvirenti <u>Mathematical Theory of Incompressible Nonviscous Fluids</u> (Springer, 1994)

For simplicity, let us here consider only $\omega(x,t) \geq 0$, and $\int \omega \, d^2x = 1$, in which case it suffices to take

$$\omega_N(x,t) = \frac{1}{N} \sum_{i=1}^{N} \delta^2(x - x_i(t)).$$

As usual in equilibrium statistical mechanics, the available phase space for a given "macrostate" $\omega(x,t)$ is measured by the Boltzmann entropy*

$$S[\omega] = -\int_D d^2x \; \omega(x) \ln \omega(x)$$

at fixed energy

$$E[\omega] = -\frac{1}{2} \int_D d^2x \int_D d^2y \; G(x,y)\, \omega(x)\, \omega(y).$$

---

* Dividing the domain D into cells, $D = \bigcup_\alpha \Delta_\alpha$, then the "macrostate" ω is approximated by point-vortex configurations with $N_\alpha = N \int_{\Delta_\alpha} \omega(x,t)\, d^2x$ vortices in the cell Δ_α. The total number of such configurations is the number of such combinations, $\dfrac{N!}{\prod_\alpha N_\alpha!} \sim e^{+N S[\omega]}$ by Stirling's approximation $N! \sim e^{-N} N^N$. See standard Statistical Physics texts.

Assuming that the point-vortex dynamics is "sufficiently ergodic", so that vortex configurations are sampled according to the phase volume available to them, Onsager suggested that the most probable vorticity field should be that which <u>maximized entropy</u> for the given initial energy E, as usual in equilibrium statistical mechanics. However, Onsager noted that this vortex-system has the unusual feature that <u>the vorticity field with maximum entropy is NOT the field with maximum energy</u>. Indeed, the maximum entropy vorticity field is easily seen to be the <u>uniform</u> vorticity distribution

$$\omega_m(x) = \frac{1}{\text{area}(D)}$$

with entropy

$$S_m = +\ln\left(\text{area}(D)\right).$$

This corresponds to a <u>finite</u> energy E_m, $0 < E_m < +\infty$, given by

$$E_m = -\frac{1}{2}\frac{1}{[\text{area}(D)]^2}\int_D d^2x \int_D d^2y\, G(x,y).$$

But this is not the maximum possible energy for the vortex system, because

$$-G(x,y) \sim -\frac{1}{2\pi} \ln\left|\frac{x-y}{L}\right|$$

for $|x-y| \ll L = \mathrm{diam}(D)$

corresponding to a <u>repulsive</u> particle potential) (actually a 2D Coulomb potential between like-sign "charges"). Thus, $E[\omega]$ can be made arbitrarily great by squeezing more like-sign vorticity close together. Onsager argued that the most probable state for $E > E_m$ would <u>at long times be a single large-scale vortex</u> corresponding to all point vortices of the same sign clustered together. In his own words

> "... the vortices of the same sign will tend to cluster — preferably the strongest ones — so as to use up excess energy at the least possible cost in terms of degrees of freedom. It stands to reason that the large compound vortices formed in this manner will remain as the only conspicuous features of the motion; because the weaker vortices, free to roam practically at random, will yield rather erratic and disorganized contributions to the flow." — L. Onsager (1949)

It is intuitively clear that, to create the same energy as a single large vortex, two separate vortices must be more compressed and thus correspond to fewer configurations. A more quantitative estimate (Caglioti et al. 1995) compares the energy of a singular circular patch of constant vorticity with radius r_1

$$E_1 \approx -\frac{1}{4\pi} \ln r_1$$

to the energy of a pair of distantly separated circular vortex patches each of radius r_2

$$E_2 \cong -2\left(\frac{1}{2}\right)^2 \frac{1}{4\pi} \ln r_2$$

$$= -\frac{1}{8\pi} \ln r_2 \quad .$$

To give the same energy $E \gg E_m$, one must choose

$$r_1 \simeq e^{-E \cdot 4\pi}$$

but

$$r_2 \cong e^{-E \cdot 8\pi} \ll r_1$$

Comparing the entropies of the two configurations

$$S_1 \approx \ln(\pi r_1^2) \approx -8\pi E$$

whereas

$$S_2 \approx \ln(2 \cdot \pi r_2^2) \approx -16\pi E$$

so that $S_1 \gg S_2$ and the single vortex corresponds to far more many point-vortex configurations. (Note that the above heuristic arguments in fact give the correct dependence of $S(E)$ on E for $E \to \infty$.) As Onsager pointed out, the vortex system for $E > E_m$ corresponds to a system of <u>negative (absolute) temperature</u> since

$$\frac{1}{T} = \frac{dS}{dE} < 0 \quad \text{for } E > E_m.$$

Thus, the sign of the Coulomb repulsion in the Gibbs measure

$$\frac{1}{Z} e^{-H/k_B T}$$

for the point-vortex system is changed for $T < 0$, so that like-sign vortices "statistically attract". Assuming equivalence of microcanonical and canonical ensembles, this gives another argument for clustering of vortices into a single large super-vortex.

Onsager's theory furthermore makes very detailed predictions about the structure of the final equilibrium vortex structure, by calculating the maximum of $S[\omega]$ for fixed value $E[\omega] = E_0$ of the energy. This was done by Onsager in the 1940's but first published (independently) by

G. Joyce & D. Montgomery, "Negative temperature states for the two-dimensional guiding-centre plasma," J. Plasma Phys. 10 101–121 (1973); D. Montgomery & G. Joyce, "Statistical mechanics of 'negative temperature' states," Phys. Fluids 17 1139–1145 (1974)

With Lagrange multipliers to enforce the constraints $E[\omega] = E_0$ and $\int_D d^2x\, \omega(x) = 1$

$$0 = \delta\left[S[\omega] + \beta H[\omega] + \lambda \int_D d^2x\, \omega(x) \right]$$

$$= -\log \omega(x) - 1 + \beta\psi + \lambda$$

$$\Longrightarrow \quad \boxed{\Delta\psi(x) = \omega(x) = \frac{1}{Z} e^{\beta\psi}, \quad Z = \int_D e^{\beta\psi(x)} d^2x}$$

This PDE in mathematics is known as the <u>Liouville equation</u>.

The solutions give stationary solutions of the 2D Euler equations, since any relation

$$\omega(x) = f(\psi(x))$$

implies

$$J(\psi, \omega) = 0.$$

Both exact and numerical solutions show that the solutions for $\beta < 0$ (negative temperature) consist of a single, large-scale vortex. E.g. in the unit $D = \{x : |x| \leq 1\}$ the solution is

$$\omega(x) = \frac{1-A}{\pi} \frac{1}{[1 - A|x|^2]^2}, \quad A = \frac{\beta}{8\pi + \beta}$$

for $\beta > -8\pi$. (Caglioti et. al. 1992).

Finally in our historical summary, we note that G. K. Batchelor in his 1953 monograph

G. K. Batchelor, <u>The Theory of Homogeneous Turbulence</u> (Cambridge U Press, 1953)

at the very end gave arguments similar to Fjørtoft's that the moment $\int_0^\infty k E(k,t)\,dk$ should decrease, while $E(t) = \int_0^\infty E(k,t\,dk$ and $\Omega(t) = \int_0^\infty k^2 E(k,t)\,dk$ remain constant.

Batchelor concluded:

> "This net tendency for the bulk of the energy to concentrate in the small wavenumbers means that fluid elements with similarly signed vorticity must tend to group together; in no other way is it possible for the scale of the velocity distribution to increase. We expect, therefore, that from the original motion there will gradually emerge a few strong isolated vortices and that vortices of the same sign will continue to group together."
>
> — Batchelor (1953)

Batchelor then goes on to cite Onsager's 1949 paper.

* These ideas have been largely verified by subsequent empirical studies using laboratory experiments and numerical simulations. Two early studies are

J. C. McWilliams, J. Fluid. Mech. 146 21-43 (1984)

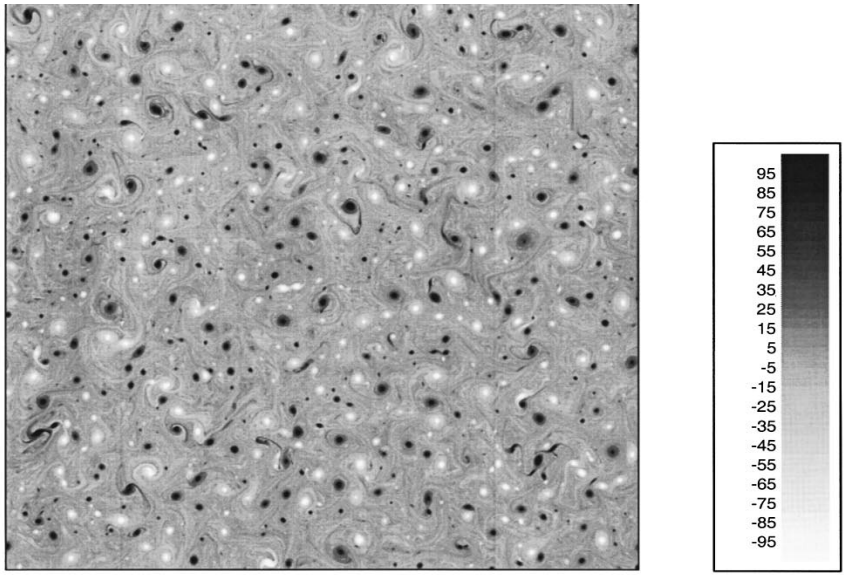
C. Basdevant et al. J. Atmos. Sci. 38 2305-2326 (1981)

Here we consider data from a more recent simulation:

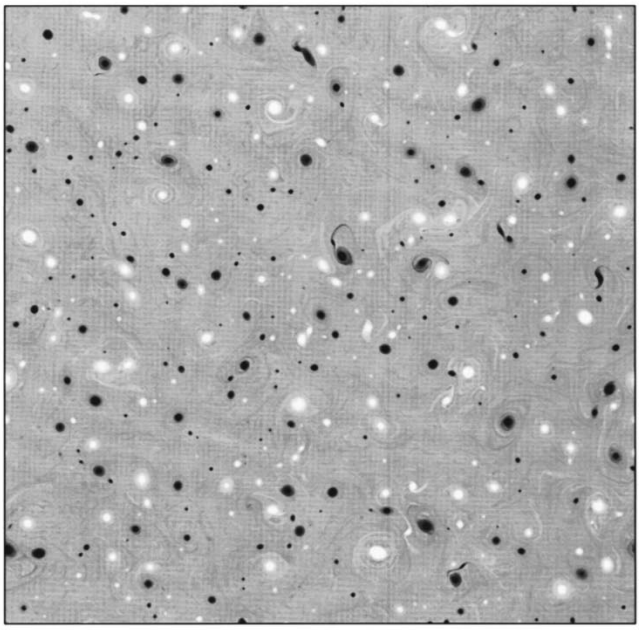
A. Bracco et al. "Revisiting freely decaying two-dimensional turbulence at millennial resolution," Phys. Fluids 12 2931–2941 (2000)

Their figure 1 (a)-(c) is reproduced on the following page, showing the vorticity fields from a high-Re 4096^2 simulation at three successive times $t = 3, 7, 11$. The plot clearly shows the coalescence of vortices by the process of vortex merger, reducing the number of vortices and increasing their scale over time. Figure 3 of the paper is next reproduced showing the energy spectra at times $t = 1, 2, \ldots, 11$. There is an obvious tendency for the energy to collect in the lowest wavenumbers. The spectra are rather steep at high wavenumbers, as expected for distributions of rather smooth vortices. Bracco et al. (2000) compare with what they call the "k^{-3} classical prediction," which is a theory of G. K. Batchelor (1969) which we shall discuss shortly. Instead the spectra are closer to a prediction of Batchelor's student Saffman:

P. G. Saffman, "On the spectrum and decay of random two-dimensional vorticity distributions at large Reynolds number," Studies in Appl. Math. 50 377–383 (1971)



(a)



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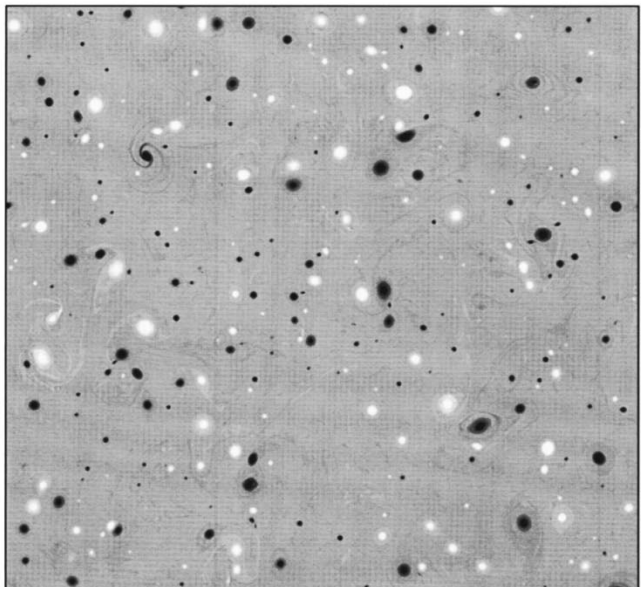


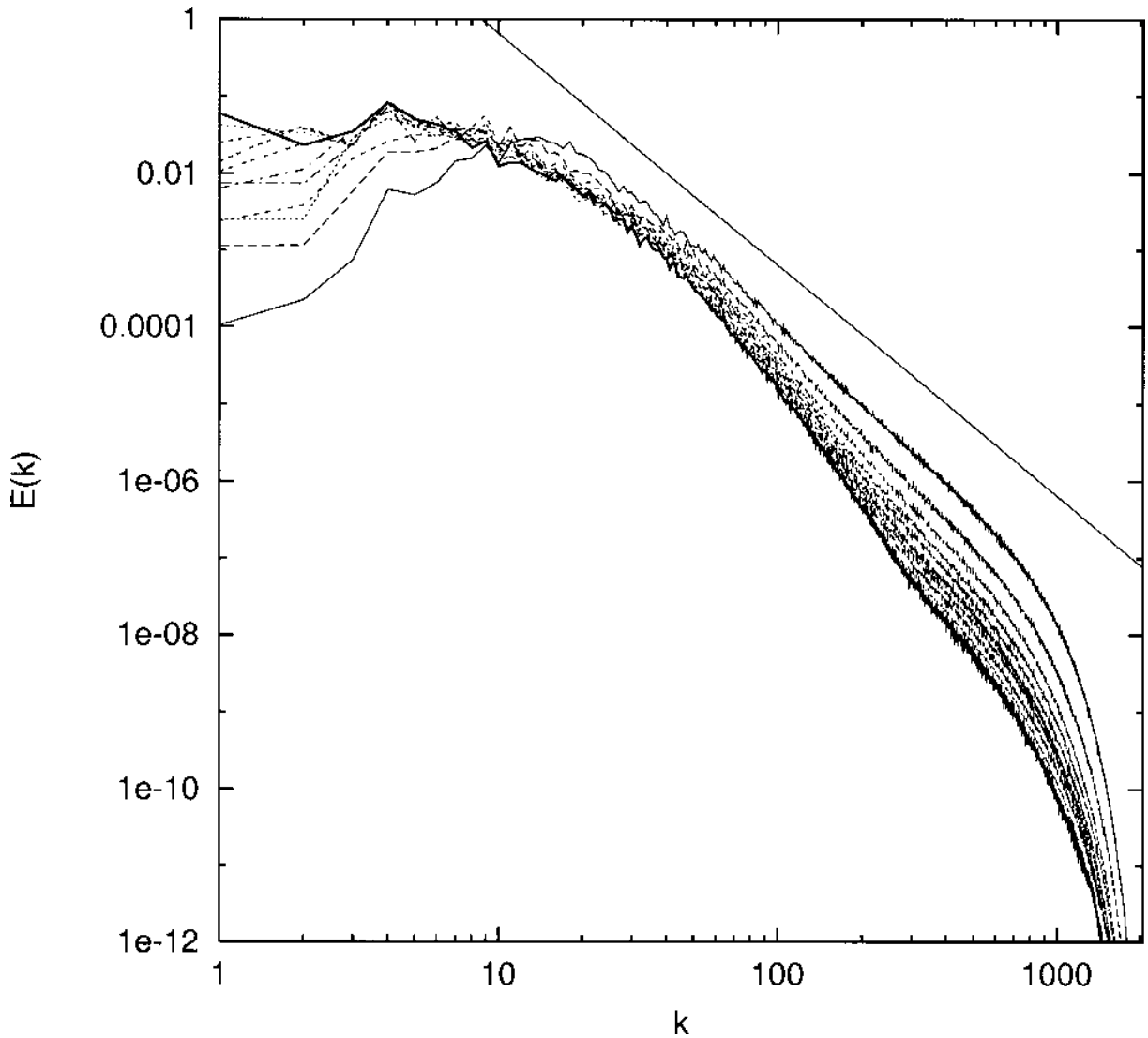


FIG. 3. Energy spectra for the solution at high Re for times $t=1,2,...,11$. Solid line shows the k^{-3} classical prediction.

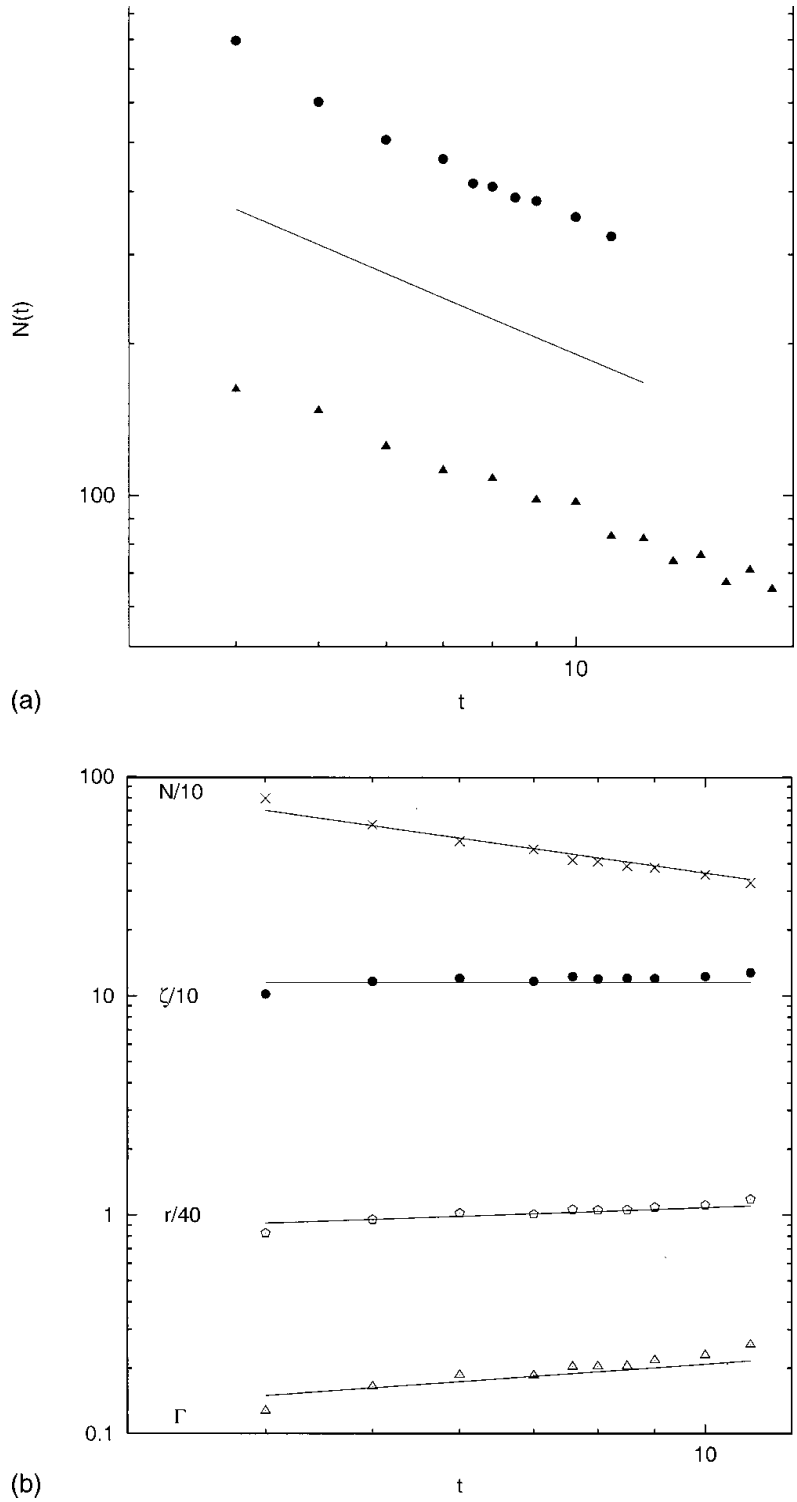


FIG. 6. Panel (a) shows the time evolution of the vortex number at high-Re (circles) and low-Re (triangles). The prediction of the scaling theory with ξ=0.72 is shown as a solid line. Panel (b) shows the time evolution of the vortex number (crosses), the average vorticity peak magnitude ζ_a (dots), the average vortex radius r_a (pentagons), and the average vortex circulation magnitude Γ_a (triangles) for the simulation at large Reynolds number. Solid lines show the slopes predicted by the scaling theory with ξ=0.72.

Saffman argued that if the boundaries of coherent vortices are rather sharp (as they seem to be in the numerical plots), then the vorticity field can be regarded to consist of roughly circular "shock" discontinuities. Analogy with Burgers led Saffman to suggest that

$$\Omega(k,t) \propto k^{-2}$$

and thus

$$E(k,t) = \frac{\Omega(k,t)}{k^2} \propto k^{-4}.$$

However steeper spectra are also observed, so one may simply be seeing the increased smoothness of vortices. We finally note that figure 6 of Bracco et. al. (2000) shows a power-law of decay of the total number of vortices as $N(t) \sim t^{-\xi}$ with $\xi \doteq 0.72$, in good agreement with an earlier scaling prediction of

J. C. McWilliams, J. Fluid Mech. 219 361-385 (1990)

G.F. Carnevale et al. PRL 66 2735-2738 (1991)

who predicted power-law scaling with the specific value $\xi = \frac{3}{4}$ of the decay exponent.

Other studies have provided more detailed evidence for predictions of Onsager's theory. Note that for simulations with periodic b.c. one must have

$$\int_D \omega(x)\, d^2x = \int_{\partial D} u^{\perp} \cdot \hat{n}\, ds = 0 \qquad \text{since } \partial D = \emptyset$$

Thus, one must compare with the version of Onsager's theory developed by Joyce & Montgomery (1973, 1974) using

$$\omega = \omega_+ - \omega_- \quad , \quad \omega_\pm \geq 0$$

and

$$S[\omega_+, \omega_-] = -\int_D d^2x\, \omega_+(x) \ln \omega_+(x) - \int_D d^2x\, \omega_-(x) \ln \omega_-(x)$$

which predicts

$$\Delta \psi = \frac{1}{Z} \sinh(\beta \psi) = \omega$$

the so-called <u>sinh-Poisson equation</u>. Numerical studies largely confirm the predictions of this theory, e.g.

D. Montgomery et al. "Relaxation in two dimensions and the sinh-Poisson equation," Phys. Fluids A <u>4</u> 3-6 (1992)

whose Figs 1-3 are reproduced on the next page. One sees a process of merger leading two large vortices of opposite sign, as predicted by the equation, and furthermore a reasonable correlation of ω with $\sinh(\beta\psi)$, increasing with time. Good agreement for a viscous NS simulation with inviscid theory!

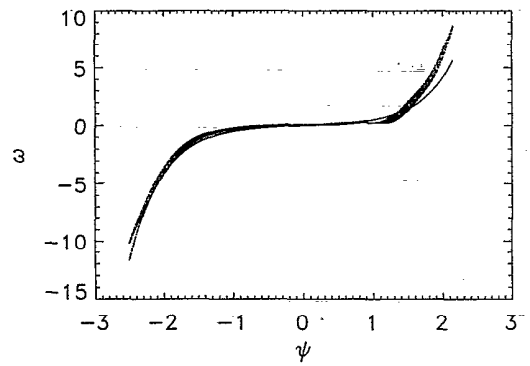


FIG. 1. Scatter plot of the streamfunction ψ versus the vorticity ω at time $t = 374$. The curve drawn through the plotted points is $c^{-1}\sinh(|\beta|\psi)$. (For a "selectively decayed" state, there is a simple proportionality between ψ and ω.)

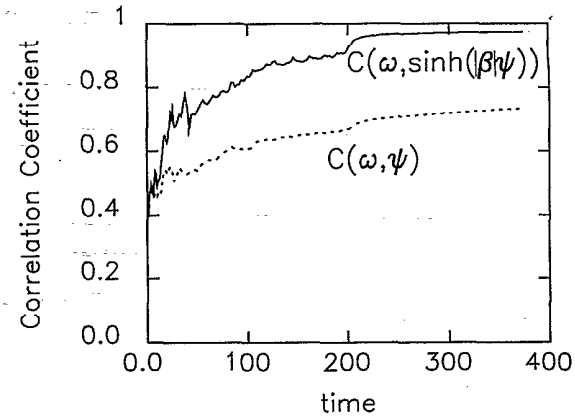


FIG. 2. Evolving spatially averaged cross-correlations between ω and $\sinh(|\beta|\psi)$ and ψ (upper and lower curves), computed as a function of time. $C = 1$ would indicate a pointwise proportionality between its arguments. (The lower curve is C for the "selective decay" hypothesis, which can be considered the best existing alternative theory.)

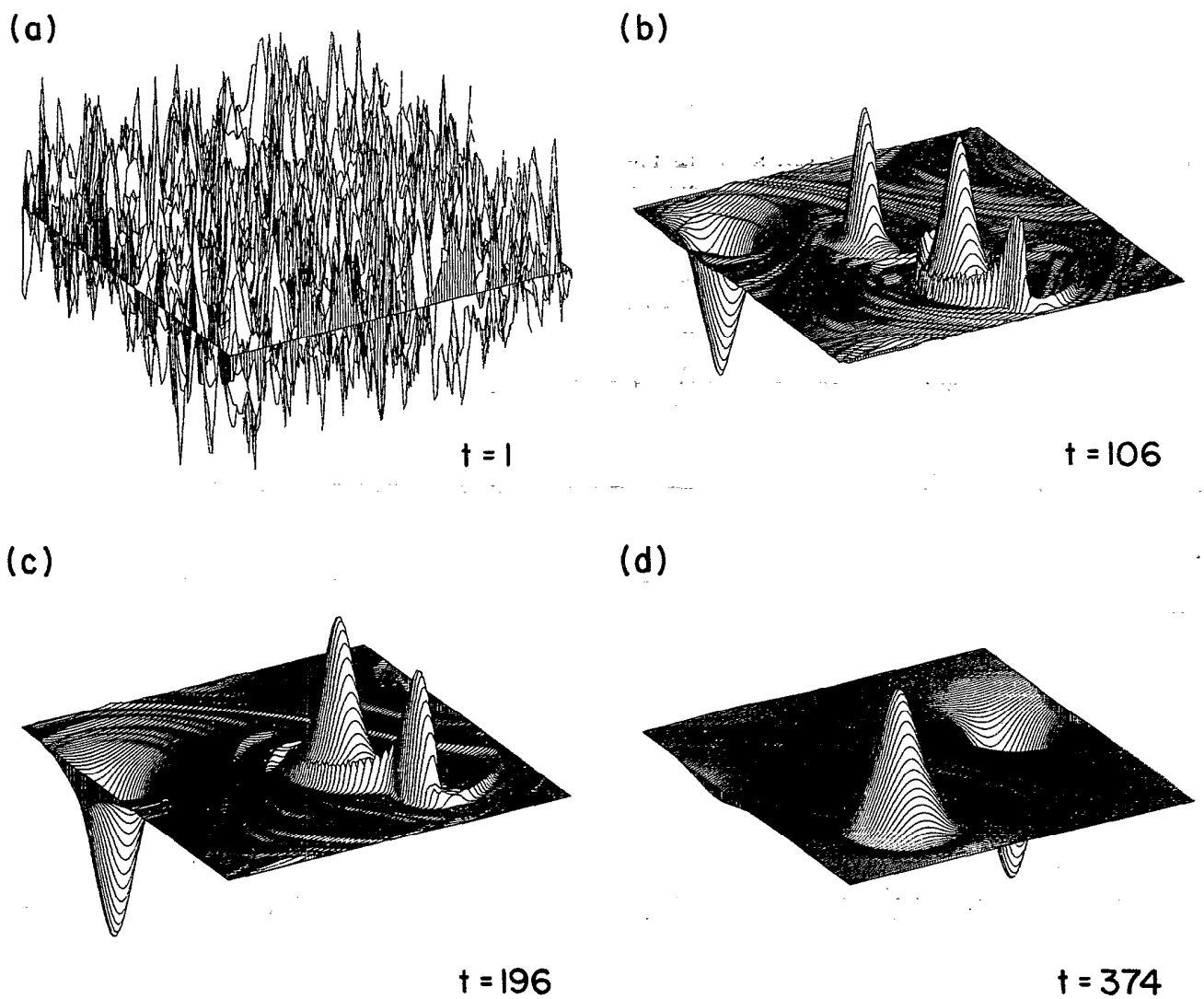


FIG. 3. Three-dimensional perspective plot of the computed vorticity versus x and y at four different times. (For clarity, the origin of coordinates has been consistently translated so that both the large vortices in the final state will lie entirely within the basic periodic box.)

Enstrophy evolution in free-decay.

We have so far focussed on energy and neglected the other inviscid invariant, enstrophy. Let's now rectify that!

We begin by noting that that there is a straightforward Fjørtoft bound on enstrophy, derived by a very similar argument as the energy bound:

$$\Omega_{<K}(t) \equiv \frac{1}{2} \sum_{|k|<K} |\hat{\omega}(k,t)|^2$$

$$\leq \frac{K^2}{2} \sum_{\substack{|k|<K \\ |k|\neq 0}} \frac{|\hat{\omega}(k,t)|^2}{|k|^2} \qquad \text{use } \hat{\omega}(0,t)=0 \text{ for periodic b.c.}$$

$$= \frac{K^2}{2} \sum_{\substack{|k|<K \\ |k|\neq 0}} |\hat{u}(k,t)|^2$$

$$\leq \frac{K^2}{2} \sum_{k} |\hat{u}(k,t)|^2$$

$$= K^2 E(t) \leq K^2 E(t_0)$$

If the initial data is spectrally localized around wavenumber k_0, so that $\Omega(t_0) \cong k_0^2 E(t_0)$, then this implies that

$$\frac{\Omega_{<K}(t)}{\Omega(t_0)} \leq (\text{const.}) \left(\frac{K}{k_0}\right)^2$$

with a constant $= O(1)$. Thus a negligible fraction of the initial enstrophy reaches wavenumbers $K \ll k_0$, globally in time.

* What happens to the initial enstrophy? It was proposed by Batchelor (1969) for decaying 2D turbulence [see also the thesis of Bray (1966)] that <u>enstrophy cascades to high wavenumbers</u> just as energy cascades to high wavenumbers in 3D. Recall that

$$\frac{d}{dt}\Omega_0(t) = -\nu \langle |\nabla\omega|^2 \rangle.$$

Batchelor proposed that for 2D decay

$$\lim_{\nu \to 0} \nu \langle |\nabla\omega|^2 \rangle = \eta(t) > 0$$

at finite times, just as believed to be true for energy dissipation $\varepsilon(t)$ in 3D decay. Batchelor's argument was that vorticity-gradients $\xi = \nabla\omega$ are stretched in 2D analogous to stretching of vorticity in 3D.

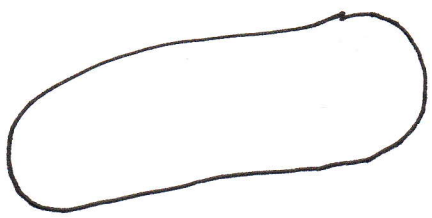
Indeed, it is straightforward to show that

$$D_t \xi = -(\nabla u)\cdot\xi + \nu\Delta\xi$$

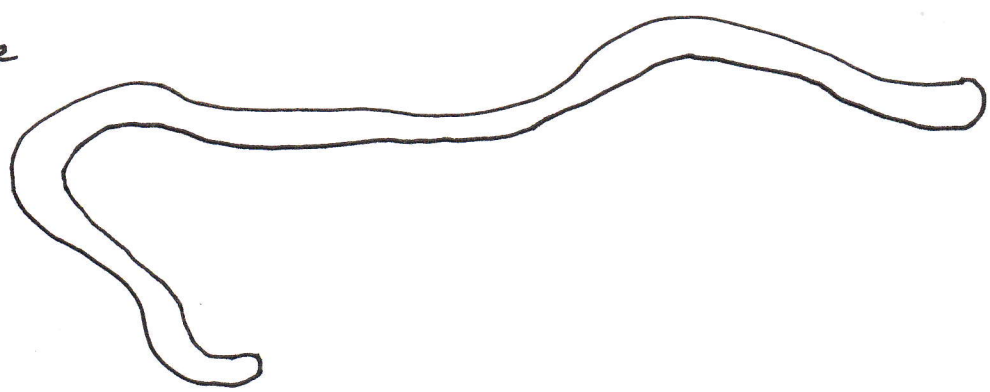
in 2D. Batchelor also noted that <u>vorticity isolines are ideal material lines</u> as a consequence of

$$D_t \omega = 0 \implies \omega(x,t) = \omega_0\left(X_t^{-1}(x)\right)$$

for 2D Euler. In analogy to Taylor's argument about vortex lines in 3D, Batchelor noted that if there is random motion of the fluid, then vorticity isolines which appear initially as



will become



i.e. will be extended and simultaneously thinned, so that $|\xi|$ will increase. This sort of mixing process is clearly seen in numerical simulations, if we zoom in. See figure next page.

Based on these hypotheses, Batchelor proposed a dimensionally determined enstrophy spectrum (a la K41)

$$\Omega_\nu(k,t) \sim C[\eta(t)]^{2/3} \frac{1}{k}, \quad k_0 \ll k \ll k_d$$

where

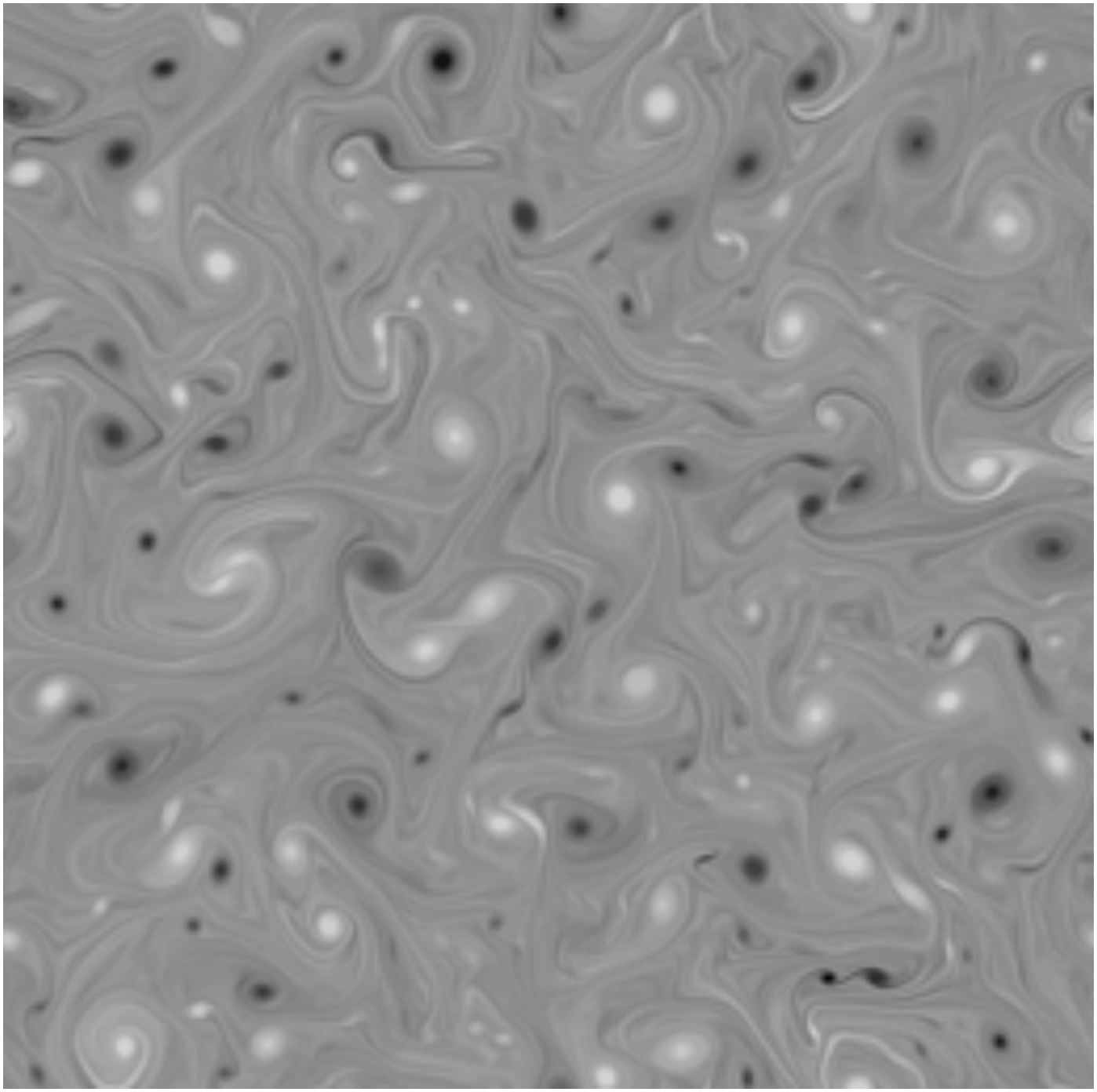
$$k_d = \eta^{1/6}/\nu^{1/2}$$

is the so-called <u>Kraichnan-Batchelor wavenumber</u> (we'll come to Kraichnan's contribution in the next section). Batchelor noted that this prediction implies a total enstrophy

$$\Omega_\nu(t) = \int_0^\infty dk\, \Omega(k,t) \geq C[\eta(t)]^{2/3} \ln\left(\frac{\eta^{1/6}(t)}{\nu^{1/2} k_0}\right)$$

$$\longrightarrow +\infty \quad \text{as} \quad \nu \to 0 \; !$$

This is clearly impossible, as it would give $\Omega_\nu(t) > \Omega(t_0)$ for sufficiently small ν. To resolve this "paradox", Batchelor suggested that the $1/k$ spectrum might extend only over a smaller range of wavenumbers $k_1(t) \ll k \ll k_2(t)$ at finite time, with $k_0 < k_1(t)$ and $k_2(t) < k_d$, with the ratio $\frac{k_2(t)}{k_1(t)}$ increasing with time but with $\eta(t)$ also decreasing in time so that the bound $\Omega_\nu(t) \leq \Omega(t_0)$ is never violated.



Batchelor's theory implies an energy spectrum

$$E(k,t) \sim C\,[\eta(t)]^{2/3}\,k^{-3}.$$

As we have seen, many simulations and also experiments report much steeper spectra k^{-4} or even k^{-5}. However, it is possible that those simulations are considering the "wrong range", i.e. that the range $[k_1(t), k_2(t)]$ with k^{-3} spectrum for very tiny ν has already reached the wavenumber k_d and been destroyed by the larger viscosity in those simulations. Indeed, several simulations report observing k^{-3} energy spectra for some intervals of wavenumber and times, e.g. simulations

M. E. Brachet et al., "The dynamics of freely decaying two-dimensional turbulence," J. Fluid. Mech. 194 333–349 (1988)

S. Fox and P. A. Davidson, "Freely decaying two-dimensional turbulence," J. Fluid. Mech. 659 351–364 (2010)

and experiments

M. Rivera, H. Aluie & R. Ecke, "The direct enstrophy cascade of two-dimensional soap film flows," arXiv: 1309.4894

Mathematically, there is an important result that sheds further light on Batchelor's "paradox". It can be shown as follows:

For a 2D decaying NS solution it can hold that

$$\lim_{\nu \to 0} \nu \langle |\nabla \omega|^2 \rangle = \eta(t) > 0$$

at finite times t, only if

$$\langle \omega_0^2 \rangle = +\infty,$$

that is, if the initial conditions have infinite enstrophy!

This is a consequence of a deep theory of "renomalized solutions" of ODE's for $W^{1,1}$ velocities by R. J. DiPerna and P. L. Lions (which was one of the works cited in Lions' award of the Fields Medal in 1994). The basic ideas of this result can be easily explained, but require notions about "enstrophy flux" that we shall discuss in the following section.

* Here let us discuss instead Batchelor's stretching argument, by showing that 2D Euler dynamics cannot produce an infinite value of $|\nabla \omega|$ in finite time if the initial data are even moderately smooth. Specifically, let us assume that $\omega_0 \in C^h(D)$, $0 < h < 1$,

that is, the initial vorticity is spatially Hölder continuous so that

$$\|\omega_0\|_n \equiv \sup_{x\in D} |\omega_0(x)| + \sup_{x,y\in D} \frac{|\omega_0(x)-\omega_0(y)|}{|x-y|^n} < +\infty$$

and show that $|\nabla\omega|$ remains bounded for finite times t. We shall sketch the main lines of the argument and more details and references can be found in

H. A. Rose & P. L. Sulem, "Fully developed turbulence and statistical mechanics," J. Physique 39 441–484 (1978), section 5.2

or A. J. Majda & A. L. Bertozzi, Vorticity and Incompressible Flow (Cambridge, 2002), Chapter 4.

We need from those works some standard "potential theory" estimates for the Green's function G of Δ:

(E0) $$|G(x,y)| \leq \frac{1}{2\pi}\left|\ln|x-y|\right|$$

(E1) $$|DG(x,y)| \leq \frac{C}{|x-y|}$$

(E2) $$|D^2G(x,y)| \leq \frac{C'}{|x-y|^2}$$

D any 1st-order derivative in x or y

(E3) $$\int_D |\nabla_x G(x,y) - \nabla_{x'} G(x',y)|\, d^2y \leq C|x-x'| \ln\left(\frac{eL}{|x-x'|}\right)$$

$L = \mathrm{diam}(D)$

Because $\xi = \nabla\omega$ obeys

$$D_t \xi = -\nabla u \cdot \xi$$

we clearly need to get some estimate on $|\nabla u|$! At the initial time this comes from

$$\nabla u(x) = \nabla\nabla^\perp \psi(x) = \int_D \nabla_x \nabla_x^\perp G(x,y)\, \omega(y)\, d^2y$$

But also $\int_D \nabla_x \nabla_x^\perp G(x,y)\, d^2y = 0$, since for constant vorticity in D the Dirichlet problem

$$\Delta u = \nabla^\perp \omega = 0 \quad \text{on } D$$

$$u = 0 \quad \text{on } \partial D$$

has only the solution $u \equiv 0$. Thus,

$$\nabla u(x) = -\int_D \nabla_x \nabla_x^\perp G(x,y)\, [\omega(x) - \omega(y)]\, d^2y$$

$$= -\int_D \nabla_x \nabla_x^\perp G(x,y)\, \frac{\omega(x) - \omega(y)}{|x-y|^h}\, \frac{d^2y}{|x-y|^{1-h}}$$

by (E2) implies

$$\sup_{x\in D} |\nabla u(x)| \leq C' \int_D \frac{|\omega(x) - \omega(y)|}{|x-y|^h}\, \frac{d^2y}{|x-y|^{2-h}}$$

$$\leq C' \|\omega\|_h \int_D \frac{d^2y}{|x-y|^{2-h}} \leq \frac{C''}{h} \|\omega\|_h$$

(In fact, this estimate can be improved to give $\|\nabla u\|_h \leq C_h \|\omega\|_h$ a Calderón–Zygmund-type inequality in Hölder spaces for the singular integral operator with kernel $\nabla\nabla^\perp G$.)

To extend this estimate to later times t we need information about the Hölder continuity of $\omega(\cdot,t)$. We shall obtain this information from a Lagrangian argument about particle separation which is interesting in itself. From

$$u(x,t) - u(x',t) = \int_D \left[\nabla_x^\perp G(x,y) - \nabla_{x'}^\perp G(x',y) \right] \omega(y,t)\, d^2y$$

and estimate (E3) we obtain

$$|u(x,t) - u(x',t)| \leq C \sup_{y\in D} |\omega(y,t)| \times |x-x'| \ln\left(\frac{eL}{|x-x'|}\right)$$

Consider the flow maps solving

$$\frac{d}{dt} X(a,t) = u(X(a,t),t)$$

$$X(a,0) = a$$

and set

$$\rho(t) = |X(a,t) - X(a',t)| \qquad \text{fixed } a, a' \in D$$

Then we obtain the differential inequality

$$\frac{d}{dt}\rho(t) \leq C\,\|\omega\|_\infty\,\rho(t)\ln\left(\frac{eL}{\rho(t)}\right)$$

or

$$-\frac{d\rho}{\rho\ln\left(\frac{\rho}{eL}\right)} \leq C\|\omega\|_\infty\,dt$$

which by integration gives

$$\frac{\rho}{eL} \leq \left(\frac{\rho_0}{eL}\right)^{e^{-C\|\omega\|_\infty t}} = \exp\left[-C_0\,e^{-C\|\omega\|_\infty t}\right]$$

with $C_0 = \left|\ln\left(\frac{\rho_0}{eL}\right)\right|$. This estimate gives a limit to the possible separation of the particles in time t. Notice that, unlike in Richardson dispersion in 3D, $\rho(t) \to 0$ as $\rho_0 \to 0$! We can also get a similar estimate by reversing the roles of $\rho_0, 0$ and $\rho(t), t$ and integrate backward in time to get

$$\frac{\rho_0}{eL} \leq \left(\frac{\rho(t)}{eL}\right)^{e^{-C\|\omega\|_\infty t}}$$

or

$$\frac{\rho(t)}{eL} \geq \left(\frac{\rho_0}{eL}\right)^{e^{C\|\omega\|_\infty t}} = \exp\left[-C_0\,e^{+C\|\omega\|_\infty t}\right]$$

which likewise limits how close particles may approach.

These results imply Hölder regularity of $\omega(\cdot,t)$. In fact, using $\rho_0 = |a-a'|$, $\rho(t) = |X(a,t) - X(a',t)|$ we see that

$$\|\omega_0\|_h = \|\omega_0\|_\infty + \sup_{a,a'} \frac{|\omega_0(a) - \omega_0(a')|}{|a-a'|^h}$$

$$= \|\omega(t)\|_\infty + \sup_{a,a'} \frac{|\omega(X(a,t),t) - \omega(X(a',t),t)|}{|a-a'|^h}$$

$$\geq \|\omega(t)\|_\infty + \sup_{a,a'} \frac{|\omega(X(a,t),t) - \omega(X(a',t),t)|}{|X(a,t) - X(a',t)|^{h(t)}}$$

with $h(t) \equiv e^{-C\|\omega\|_\infty t} h$ by using the bound from below for $\rho(t)$. But then by changing from a to x variables

$$\|\omega_0\|_h \geq \|\omega(t)\|_\infty + \sup_{x,x'} \frac{|\omega(x,t) - \omega(x',t)|}{|x-x'|^{h(t)}}$$

$$= \|\omega(t)\|_{h(t)}$$

and we see that $\omega(t)$ is Hölder continuous with exponent at least $h(t)$. We thereby also obtain the bound on $\|\nabla u(t)\|_\infty \equiv \sup_{x\in D} |\nabla u(x,t)|$ given by

$$\|\nabla u(t)\|_\infty \leq \frac{C''}{h(t)} \|\omega(t)\|_{h(t)} \leq \frac{C''\|\omega_0\|_h}{h} \cdot e^{+C\|\omega_0\|_\infty t}$$

Now, going back to

$$D_t \xi = -\nabla u \cdot \xi$$

we note that

$$\|\xi(t)\|_\infty = \sup_{x \in D} |\xi(x,t)|$$

$$= \sup_{a \in D} |\xi(X(a,t),t)| = \|\xi(t) \circ X(t)\|_\infty$$

with $\frac{d}{dt} \xi \circ X = (D_t \xi) \circ X$. Hence,

$$\frac{d}{dt} \|\xi(t)\|_\infty = \frac{d}{dt} \|\xi(t) \circ X(t)\|_\infty$$

$$\leq \left\| \frac{d}{dt} (\xi(t) \circ X(t)) \right\|_\infty$$

$$= \| -(\nabla u(t) \cdot \xi(t)) \circ X(t) \|_\infty$$

$$= \| \nabla u(t) \cdot \xi(t) \|_\infty \leq \|\nabla u(t)\|_\infty \|\xi(t)\|_\infty$$

which integrates to

$$\|\xi(t)\|_\infty \leq \|\xi_0\|_\infty \exp\left(\int_0^t ds \, \|\nabla u(s)\|_\infty \right)$$

$$\leq \|\xi_0\|_\infty \exp\left(\int_0^t \frac{C''}{h} \|\omega_0\|_h e^{C\|\omega_0\|_\infty s} ds \right)$$

$$= \|\xi_0\|_\infty \exp\left(C' \frac{\|\omega_0\|_h}{\|\omega_0\|_\infty} e^{C\|\omega_0\|_\infty t} \right) < +\infty.$$

QED

Remark. The mathematical theory presented above was developed, in fact, long before Batchelor's paper by

W. Wolibner, "Un theorème sur l'existence du mouvement plan d'un fluide parfait, homogène, incompressible, pendant un temps infiniment long," Math. Z. 37 698–726 (1933)

E. Hölder, "Über die unbeschränkte Fortsetzbarkeit einer stetigen ebenen Bewegung in einer unbegrenzten inkompressiblen Flüssigkeit," Math. Z. 37 727–738 (1933)

More recently, these results have been extended to vorticity in Besov spaces. We recall that

$$\|\omega\|_{B_p^{\sigma,\infty}} = \|\omega\|_p + \sup_{|\ell|<L} \frac{\|\delta\omega(\ell)\|_p}{|\ell|^\sigma}$$

defines the norm in Besov space $B_p^{\sigma,\infty}$ with $p \geq 1$ and $0<\sigma<1$, where

$$\|\omega\|_p = \left[\int_D d^2x \, |\omega(x)|^p \right]^{1/p}$$

and

$$\delta\omega(x;\ell) = \omega(x+\ell) - \omega(x).$$

Thus, $\omega \in B_p^{\sigma,\infty} \Longrightarrow$

$$\langle |\delta\omega(\ell)|^p \rangle = O(|\ell|^{\zeta})$$

with $\zeta = \sigma p$ and these spaces appear naturally in a <u>multifractal</u> description of the vorticity field (see section III). It has been proved by

H. Bahouri & J.-Y. Chemin, "Equations de transport relatives à des champs de vecteurs non-lipschitziens et mécanique des fluides," Arch. Rat. Mech. Anal. <u>127</u> 159–181 (1994)

Those authors have proved that if $\omega_0 \in L^\infty(\mathbb{T}^2) \cap B_p^{\sigma,\infty}(\mathbb{T}^2)$ then $\omega(\cdot,t) \in L^\infty(\mathbb{T}^2) \cap B_p^{\sigma(t),\infty}(\mathbb{T}^2)$ with $\sigma(t) = e^{-C\|\omega\|_\infty t}\sigma$.

* The above results on $\xi = \nabla\omega$ showing that $\|\xi(t)\|_\infty < +\infty$ if initially $\|\xi_0\|_\infty < +\infty$ prove that <u>there is no 2D Euler singularity in finite time</u>. If the initial vorticity-gradients are bounded, then they remain so at all finite times, because, of course, $\|\xi_0\|_\infty < +\infty \Longrightarrow \omega_0 \in C^h$ for any $0 < h < 1$. The results can also be interpreted in a more physically vivid way by the statements there is <u>a nonzero minimum length-scale to the vorticity for finite times</u>:

$$\ell_\infty(t) \equiv \frac{\|\omega(t)\|_\infty}{\|\xi(t)\|_\infty}$$

$$\geq \ell_\infty(0) \exp\left(-C' \frac{\|\omega_0\|_h}{\|\omega_0\|_\infty} e^{+C\|\omega_0\|_\infty t}\right)$$

Notice that this essentially coincides with the lower <u>Wolibner bound</u> on particle separations

$$\frac{\rho(t)}{\varepsilon L} \geq \exp\left[-C_0 e^{+C\|\omega_0\|_\infty t}\right]$$

and is indeed derived from it. Notice that the Wolibner upper and lower bounds are sharp, as there are simple examples of flows with $\|\omega_0\|_\infty < +\infty$ for which the bounds are equalities (see Bahouri & Chemin, 1994).

These results also give support to Batchelor's conjecture that his predicted spectrum of anisotropy

$$\Omega(k,t) \sim [\eta(t)]^{2/3} k^{-1}$$

should be restricted to a finite (but growing) range of wavenumbers

$$k_1(t) < k < k_2(t),$$

with $k_2(t)$ independent of viscosity. Indeed,

such a spectral range would lead to

$$\|\xi(t)\|_2^2 \sim [\eta(t)]^{2/3} \int_{k_1(t)}^{k_2(t)} k\,dk$$

$$\sim [\eta(t)]^{2/3} [k_2(t)]^2$$

If we also use Batchelor's suggest to relate "the rate of transfer to higher wavenumbers" to "the reservoir of mean-square vorticity" then we may take $[\eta(t)]^{2/3} \sim \|\omega(t)\|_2^2$ and obtain

$$k_2(t) \sim \frac{\|\xi(t)\|_2}{\|\omega(t)\|_2},$$

with $l_2(t) = \dfrac{1}{k_2(t)}$ similar to $l_\infty(t)$ rigorously estimated from below above, independent of viscosity. Since l_∞ is defined by sup-norms and l_2 by L^2-norms, $l_2(t)$ may be considerably larger than $l_\infty(t)$, which focusses on the worst singularity. Based on arguments of Kraichnan for the Batchelor $1/k$-range of a passive scalar,

R. H. Kraichnan, "Convection of a passive scalar by a quasi-uniform random straining field," JFM 64 737–762 (1974)

it has often been suggested that

$$l_2(t) \sim l_2(0)\, e^{-C\|\omega\|_\infty t},$$

rather than the super-exponential smallness of $l_\infty(t)$ possible.

* This very fine-scale mixing of vorticity leads to a problem for the Onsager theory of long-time equilibrium vortex structures in 2D, however, Onsager's theory predicts that, in some sense,

$$\omega(x,t) \xrightarrow[t\to\infty]{} \omega_{eq}(x).$$

If $\|\omega_0\|_\infty < +\infty$, then this always holds in the sense that

$$\lim_{n\to\infty} \int_D d^2x\, \varphi(x)\, \omega(x,t_n) = \int_D d^2x\, \varphi(x)\, \bar{\omega}(x)$$

for all $\varphi \in L^1(D)$, for some subsequence $t_n \nearrow \infty$ and for some $\bar{\omega} \in L^\infty(D)$*. However, large oscillations can remain in this limit! Consider the time-sequence of photographs of mixing ink on the following page:

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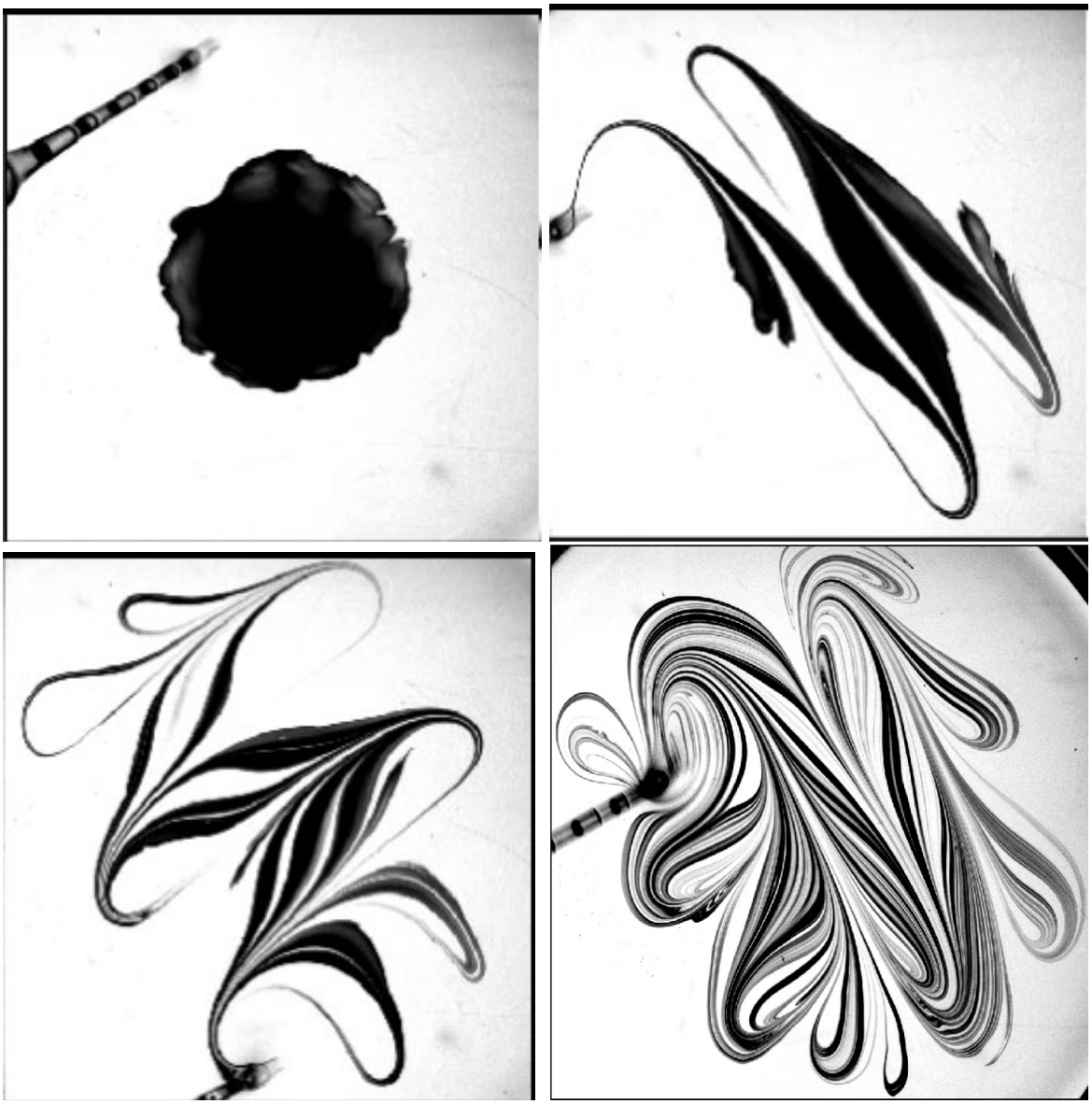
* By the Banach–Alaoglu theorem, since $[L^1(D)]^* = L^\infty(D)$.



Consider some $x \in D$ and some ball $B_R(x)$ of radius R around that point. Then define the <u>distribution of ω-values inside $B_R(x)$ at time t_n</u> as

$$\nu^{(n)}_{x,R}\left(\{\omega < m\}\right) = \frac{\text{area}\left\{\{y : \omega(y,t_n) < m\} \cap B_R(x)\right\}}{\text{area}\left(B_R(x) \cap D\right)}$$

It is intuitively clear that if one <u>first</u> fixes t_n and <u>then</u> takes the limit $R \to 0$, it must be true that

$$\nu^{(n)}_{x,R}(d\omega) \xrightarrow[R \to 0]{} \delta(\omega - \omega(x,t_n))\, d\omega .$$

The only value will be the unique value of $\omega(\cdot, t_n)$ at point x. However, if one takes <u>first</u> the limit $t_n \to \infty$, then fine-scale mixing of vorticity will produce a nontrivial distribution of ω-values inside a ball $B_R(x)$ for any $R > 0$. Even if one subsequently takes the limit $R \to 0$ a nontrivial measure

$$\nu_x(d\omega) = \lim_{R \to 0} \lim_{t_n \to \infty} \nu^{(n)}_{x,R}(d\omega)$$

may result. This is called a <u>Young measure</u>.

This can be put another way by noting that for any continuous function f on $[-\|w\|_\infty, \|w\|_\infty]$

$$\int f(w)\,\nu^{(n)}_{x,R}(dw) = \frac{1}{\text{area}(B_R(x)\cap D)} \int_{B_R(x)\cap D} f(w(y,t_n))\,d^2y$$

$$= \int_D \hat{\chi}_{B_R(x)}(y)\, f(w(y,t_n))\,d^2y$$

where $\hat{\chi}_{B_R(x)}$ is the characteristic function of $B_R(x)$ normalized so that $\int_D \hat{\chi}_{B_R(x)}(y)\,d^2y = 1$. Taking the limit $R \to 0$ at fixed t_n

$$\lim_{R\to 0} \int f(w)\,\nu^{(n)}_{x,R}(dw) = \lim_{R\to 0} \frac{\int_{B_R(x)\cap D} f(w(y,t_n))\,d^2y}{\text{area}(B_R(x)\cap D)}$$

$$= f(w(x,t_n)) \quad \left(\begin{array}{c}\text{Lebesgue}\\ \text{theorem}\end{array}\right)$$

$$= \int f(w)\,\delta(w - w(x,t_n))\,dw$$

However

$$\lim_{R\to 0}\lim_{t_n\to\infty} \int f(w)\,\nu^{(n)}_{x,R}(dw) = \int f(w)\,\nu_x(dw).$$

Mathematically, the problem is that with $\|w(\cdot,t_n)\|_\infty \leq M < \infty$

$$\lim_{n\to\infty} \int_D \varphi(x)\, w(x,t_n)\, d^2x = \int_D \varphi(x)\, \bar{w}(x)\, d^2x$$

for all $\varphi \in L^1(D)$ [weak* convergence] does <u>not</u> imply for all continuous functions f on $[-M,M]$ that $f(w(x,t_n))$ converges to $f(\bar{w}(x))$ in the same sense, i.e.

$$\int_D \varphi(x) f(w(x,t_n))\, d^2x \underset{n\to\infty}{\not\longrightarrow} \int_D \varphi(x) f(\bar{w}(x))\, d^2x$$

for all $\varphi \in L^1(D)$. However, the fundamental theorem on Young measures guarantees that

$$\lim_{n\to\infty} \int_D \varphi(x) f(w(x,t_n)) = \int_D \varphi(x) \int_{[-M,M]} f(w)\, \nu_x(dw)$$

E.g. see

J. M. Ball, "A version of the fundamental theorem for Young measures," <u>PDE's and Continuum Models of Phase Transitions</u> (Springer, 1989), pp. 207-215

Two authors, independently, have suggested that the long-time vorticity distribution resulting from free decay in 2D is a Young measure $\nu_x(d\omega) = \rho(x,\omega)\,d\omega$:

J. Miller, "Statistical mechanics of Euler equations in two dimensions," PRL 65 2137–2140 (1990)

R. Robert, "Etats d'équilibre statistique pour l'écoulement bidimensionnel d'un fluide parfait," C. R. Acad. Sci. Paris, Sen I 311: 575-578 (1990)

In that case,

$$\int_D \varphi(x)\,\omega(x,t_n)\,d^2x \to \int_D \varphi(x)\,\bar{\omega}(x)\,d^2x$$

as in Onsager's theory, but with

$$\bar{\omega}(x) = \int_{[-M,M]} \omega\,\nu_x(d\omega).$$

Note, furthermore, that

$$\text{area}\left(\{x\in D : \omega(x,t_n) < m\}\right) = \int_D \nu_x^{(n)}\left(\{\omega < m\}\right) d^2x$$

with $\nu_x^{(n)}(d\omega) = \delta(\omega - \omega(x,t_n))\,d\omega$. Since the left side is independent of t_n but the right converges to $\int_D \nu_x(\{\omega<m\})\,d^2x$, one gets

$$\int_D \nu_x\left(\{\omega < m\}\right) d^2x = \text{area}\{x \in D : \omega_0(x) < m\}$$

Thus, the Young measure preserves the knowledge about area of all the level sets of the initial vorticity.

(The Onsager theory, because it started with point vortices, had the artificial feature that all these areas are zero, as Onsager himself pointed out.)

Put another way, for all continuous functions f on $[-M, M]$,

$$\int_D d^2x \int_{[-M,M]} f(\omega)\,\nu_x(d\omega) = \int_D f(\omega_0(x))\, d^2x$$

so that one can recover from ν_x all of the invariants $I_f = \int_D f(\omega_0(x))\, d^2x$ or, equivalently,

$$\int_D d^2x\ \rho(x,\omega) = \int_D \delta(\omega_0(x) - \omega)\, d^2x$$

$$\equiv g(\omega).$$

with $\nu_x(d\omega) = \rho(x,\omega)\, d\omega$.

By a Boltzmann counting argument, Miller and Robert showed that the entropy associated with a given density function $\rho(\mathbf{x},\omega)$ of the Young measure is given by

$$S[\rho] = -\int_D d^2x \int_{-M}^{M} d\omega\, \rho(\mathbf{x},\omega) \ln \rho(\mathbf{x},\omega).$$

Miller and Robert suggested to maximize $S[\rho]$ subject to the constraints

$$\int_{-M}^{M} \rho(\mathbf{x},\omega)\, d\omega = 1$$

and

$$\int_D \rho(\mathbf{x},\omega)\, d^2x = g(\omega)$$

and also fixed energy of $\overline{\omega}$

$$-\frac{1}{2}\int_D d^2x \int_D d^2x' \int_{-M}^{M} d\omega \int_{-M}^{M} d\omega'\, \omega\omega' G(\mathbf{x},\mathbf{x}')\, \rho(\mathbf{x},\omega)\rho(\mathbf{x}',\omega')$$

$$= -\frac{1}{2}\int_D d^2x \int_D d^2x'\, G(\mathbf{x},\mathbf{x}')\, \omega_0(\mathbf{x})\omega_0(\mathbf{x}')$$

$$= E_0,$$ the initial energy

This gives the maximizer

$$\rho(\mathbf{x},\omega) = \frac{1}{Z(\mathbf{x})} \exp\left\{ +\beta\left[\omega\,\bar{\Psi}(\mathbf{x}) + \mu(\omega)\right]\right\}$$

with $Z(\mathbf{x})$, $\mu(\omega)$ and β the Lagrange multipliers to enforce the previous three constraints. The stream function satisfies the equation

$$\Delta\bar{\Psi}(\mathbf{x}) = \bar{\omega}(\mathbf{x}) = \frac{1}{Z(\mathbf{x})}\int d\omega\; \omega\, e^{\beta\left[\omega\bar{\Psi}(\mathbf{x})+\mu(\omega)\right]}$$

with $Z(\mathbf{x}) = \int_{-M}^{M} e^{\beta\left[\omega\bar{\Psi}(\mathbf{x})+\mu(\omega)\right]}\, d\omega$. The above set of predictions are called the <u>Robert-Miller theory</u>. It is worth remarking that an identical theory was proposed much earlier by the astronomer Donald Lynden-Bell:

D. Lynden-Bell, "Statistical mechanics of violent relaxation in stellar systems," Mon. Not. R. Astr. Soc. <u>136</u> 101–121 (1967)

to predict the structure of elliptical galaxies described by the Vlasov–Poisson equations of Newtonian gravity. The distribution function $f(\mathbf{x},\mathbf{v},t)$ in that theory shows a similar small-scale mixing in the velocity ($\mathbf{v}$) space, due to Landau damping effects.

Some simulations have shown that this theory gives better agreement with results of numerical simulations than the Onsager - Joyce - Montgomery theory, when the initial conditions have vorticity supported on a finite area, e.g.

J. Sommeria et al., "Final equilibrium state of a two-dimensional shear layer," JFM 233 661-689 (1991)

There are questions, however, that remain about the domain of applicability of any such equilibrium theory. One question concerns the ergodic properties of 2D Euler and whether they are sufficiently chaotic in phase space to justify entropy maximization. There is also concern whether the theory should apply to 2D NS solutions, even with small viscosities. The fine-scale mixing of the vorticity field implies that high-order invariants of the form

$$I_n = \int_D |\omega(x)|^n d^2x \ , \quad n > 2$$

should be efficiently dissipated by viscosity. In that case, it may not be appropriate to include them as constraints in entropy maximization.

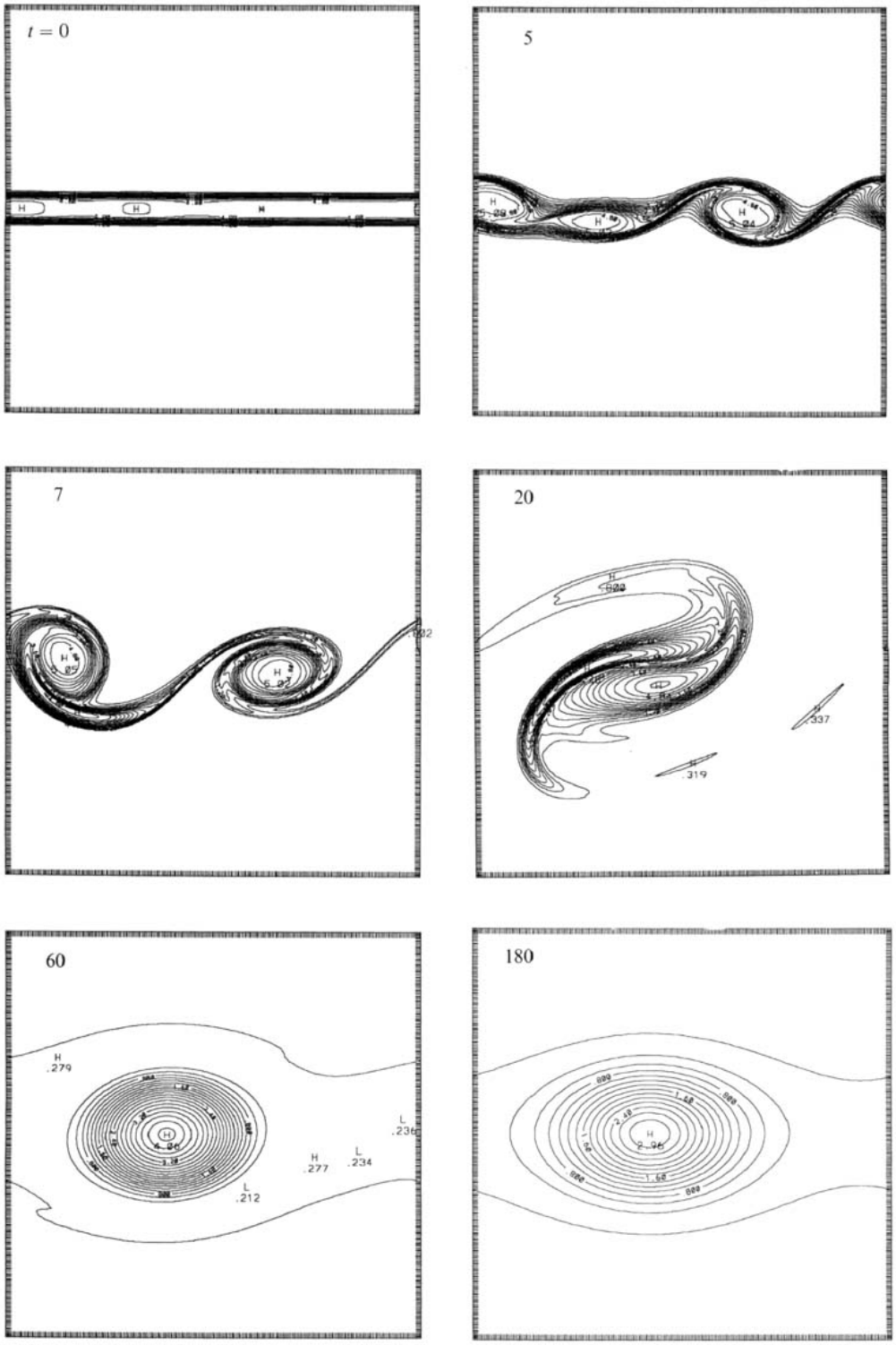


FIGURE 2. Successive snapshots of the vorticity field ($\delta = 0.215$, $Re = 1500$). The contour interval is 0.3 for $t = 0$ to 20, and 0.2 for $t = 60$ and 180.

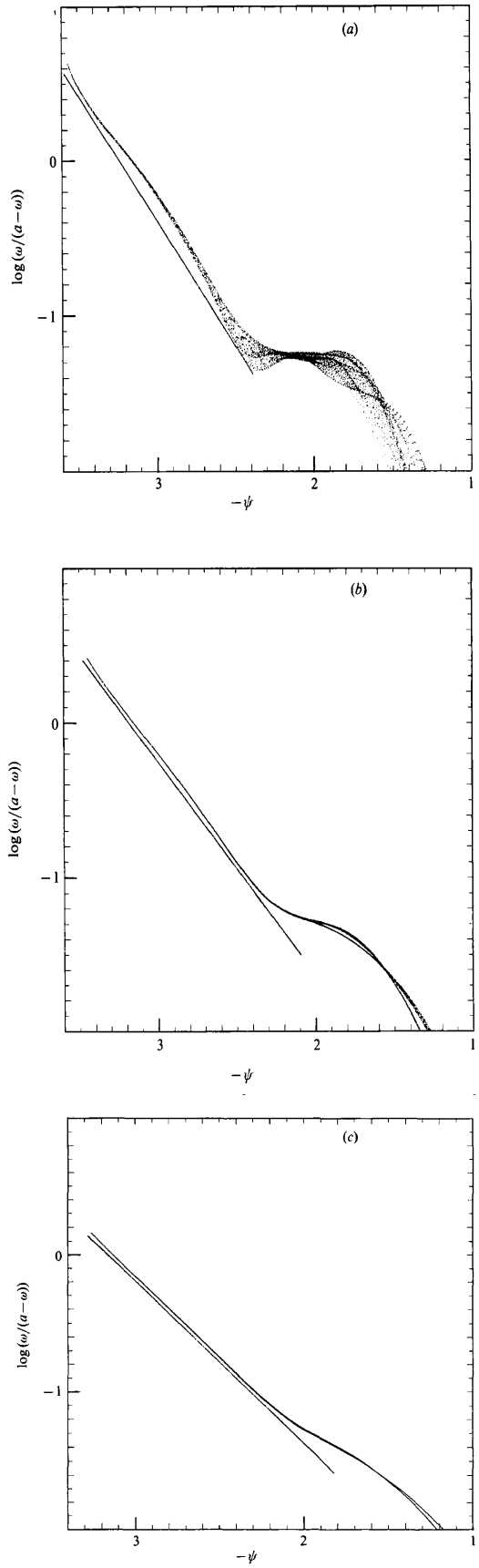


FIGURE 9. Scatterplot of log $(\omega/a-\omega))$ versus stream function ψ at different times in the equilibrium regime for $\delta = 0.215$: (a) $t = 60$, the fitted parameters are $\mu = 1.2\times10^{-5}$ and $C = 5.96$; (b) $t = 100$, fitting $\mu = 3.7\times10^{-5}$ and $C = 5.0$; (c) $t = 180$, fitting $\mu = 19.9\times10^{-5}$ and $C = 2.35$. (To reduce the size of the figures, we have removed the points with a stream function between 0 and -1, where the vorticity is virtually zero.)

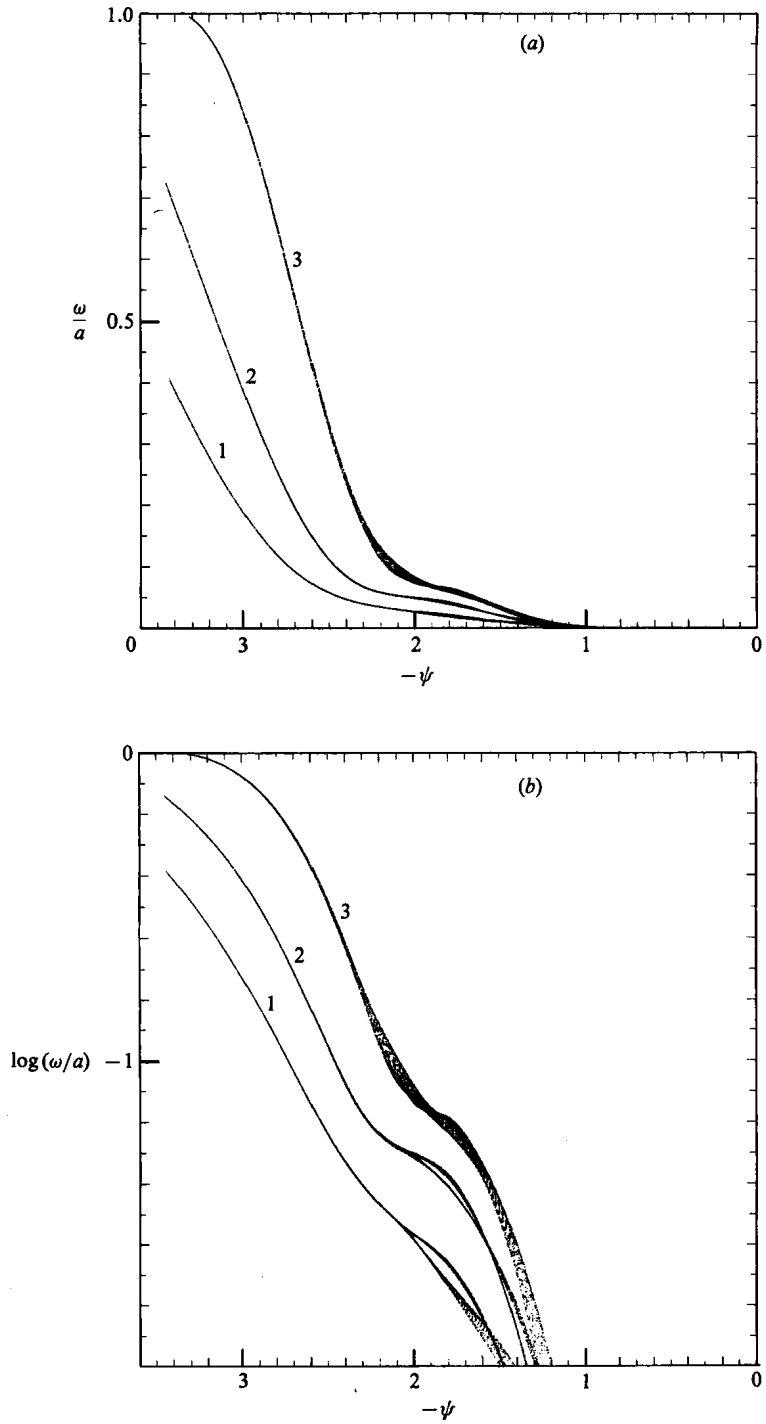


FIGURE 8. Scatterplot of vorticity versus stream function, with different representations (at $t = 100$): curve 1, $\delta = 0.1075$; 2, $\delta = 0.215$; 3, $\delta = 0.43$. (*a*) Linear coordinates ω/a and ψ, (*b*) log ω/a versus ψ.

* A simpler approach is the <u>enstrophy minimization theory</u> proposed by

F. P. Bretherton & D. B. Haidvogel, "Two-dimensional turbulence above topography," J. Fluid. Mech. 78 129–154 (1976)

This theory is based on the fact that energy is very well conserved for small viscosity, while enstrophy remains highly damped. On the other hand, because $\Omega(k,t) = k^2 E(k,t)$, vanishing enstrophy $\Omega(t) = \int_0^\infty k^2 E(k,t)\,dk = 0$ implies also vanishing energy $E(t) = \int_0^\infty E(k,t)\,dk = 0$! Thus, a non-vanishing energy implies that there must be at least some small amount of enstrophy. Bretherton & Haidvogel proposed that the final ~~state~~ of decay should be the vorticity field $\omega(x)$ which minimizes Ω for fixed value of E:

$$\omega(x) \equiv \underset{E[\omega] = -\frac{1}{2}\iint G(x,x')\,\omega(x)\,\omega(x') = E}{\arg\min} \left\{ \frac{1}{2} \int \omega^2(x)\,d^2x \right\}$$

which leads to the equation

$$\Delta\psi(x) = \omega(x) = \beta\psi(x)$$

where β is the Lagrange multiplier to enforce the energy constraint. This is called a <u>selective decay theory</u> because certain of the invariants are selected to decay (here, enstrophy) and others to be preserved (here, energy) in the presence of damping. This theory has also received some support from numerical simulations, e.g.

W. H. Matthaeus et al., "Decaying, two-dimensional Navier-Stokes turbulence at very long times," Physica D <u>51</u> 531-538 (1991)

Recent work has attempted to reconcile the enstrophy minimization theory with the entropy maximization approach of Onsager, Miller, Robert, etc. by showing that minimum enstrophy emerges as a stability criterion in the Miller-Robert theory with the enstrophy as the only inviscid invariant:

A. Naso, P.H. Chavanis & B. Dubrulle, "Statistical mechanics of two-dimensional Euler flows and minimum enstrophy states," Eur. Phys. J. B <u>77</u> 187-212 (2010)

This is still a very active field with great geophysical interest!