

## Part I

### Two-Dimensional Turbulence

### a) The Two-Dimensional Navier-Stokes Equation

The 2D NS equation has the same form as in 3D

$$\partial_t u + (u \cdot \nabla) u = -\nabla p + v \Delta u$$

except that now  $u \in \mathbb{R}^2$  and likewise  $x \in D \subseteq \mathbb{R}^2$ . We discuss first the situations where use of this equation is justified and then its basic properties.

Justification: The most obvious use of this equation is to describe 3D "planar flows", such as plane Couette flow, plane Poiseuille flow, etc. that are independent of  $z$  and have  $w=0$ . Unfortunately, this use is limited because all such flows are unstable, either linearly or via finite-amplitude perturbations. E.g. see

P. J. Schmid & D. S. Henningson, Stability and Transition in Shear Flows, Vol. 142 of Appl. Math. Sci. (Springer, NY, 2001)

As a consequence, such 2D flows rapidly become fully 3D (and turbulent).

One might think that systems of particles in a 2D system — e.g. electrons & holes in graphene — would provide examples of 2D NS. As a matter of fact, 2D "molecular fluids" are not described by 2D NS. The basic problem is that the long-time tails in the Green-Kubo formula for the shear viscosity (and bulk viscosity) have the form  $t^{-d/2}$  in space-dimension  $d$  and thus become logarithmically divergent for  $d=2$ . This makes the viscosity system-size dependent for 2D systems of "molecules", as was verified by lattice-gas simulations in

L.P. Kadanoff, G.R. Mc Namara, and G. Zanetti, "From automata to fluid flow: comparisons of simulation and theory," Phys. Rev. A 40 4527–4541 (1989)

Hence, such systems are not described by a standard 2D Navier-Stokes equation.

Instead, 2D NS appears primarily for 3D fluids subjected to special conditions, such as

- \* rapid rotation
- \* strong stratification
- \* large aspect ratio (thin films)

In such cases, 2D NS has some asymptotic validity. Let us just discuss here, as an example, the case of rapid rotation.

The two-dimensionalization of rotating fluids is a well-known classical result also called the "Taylor-Proudman effect". For much of the classical lore, see

H. P. Greenspan, The Theory of Rotating Fluids (Cambridge, 1968)

There has been much additional work by mathematicians more recently; e.g. see

A. Babin, A. Mahalov & B. Nicolaenko,  
 "Global splitting, integrability and regularity  
 of three-dimensional Euler and Navier-Stokes  
 equations for uniformly rotating fluids,"  
 Euro. J. Mech. B / Fluids 15 291-300 (1996)

P. Embid & A.J. Majda, "Averaging over fast  
 gravity waves for geophysical flows with arbitrary  
 potential vorticity," Commun. Partial Diff. Equat.  
21 619-658 (1996)

We shall later discuss these methods for geophysical  
 applications, but here we mention them briefly  
 for rapidly rotating fluids. In the frame of rotation  
 the 3D NS equation becomes

$$\partial_t u + 2\Omega \times u = -\nabla p' + \nu \Delta u - w \times u$$

where the term containing  $\Omega = \Omega \hat{z}$  is the Coriolis acceleration. Note also that the modified pressure

$$p' = p + \frac{1}{2} |u|^2 + \frac{1}{2} |\Omega \times x|^2$$

contains a centrifugal term, but this has little consequence  
 since it is absorbed into a redefinition of pressure.

The linear system

$$\partial_t u + 2\Omega \times u = -\nabla p$$

with  $p$  chosen to guarantee  $\nabla \cdot u = 0$  supports waves called inertial (or Poincaré) waves. If one defines the wave operator

$$\mathcal{L}u = -2\Omega \times u - \nabla p$$

simple vector calculus identities show that

$$\nabla \times (\mathcal{L}u) = \mathcal{L}(\nabla \times u)$$

and thus  $\mathcal{L}$  can be diagonalized together with the curl operator. The eigenmodes of the curl operator are called helical modes and satisfy

$$ik \times h_s(k) = s|k| h_s(k), \quad s = \pm 1$$

See Greenspan (1968) or

F. Waleffe, "Inertial transfers in the helical decomposition,"  
Phys. Fluids A 5 677-685 (1993)

F. Waleffe, "The nature of triad interactions in homogeneous turbulence," Phys. Fluids A 4 350-363 (1992)

For example, for an arbitrary vector  $\hat{z}$  we can take

$$h_s(k) = \hat{z} + i\hat{k} \times \hat{z} - (\hat{z} \cdot \hat{k}) \hat{k}$$

which also satisfy  $k \cdot h_s(k) = 0$  and  $h_s^*(k) = h_s(-k)$ .

The term "helical mode" is apt since, when normalized by  $h_s(k) \cdot h_s^*(k) = 1$  they carry net helicity  $s|k|$ . If  $\hat{z}$  above is chosen to be the rotation axis it is also easy to check that

$$\sum h_s(k) = 2is \cos \theta_k h_s(k)$$

with  $\cos \theta_k = \hat{z} \cdot \hat{k}$  the angle  $\hat{k}$  makes with the rotation axis  $\hat{z}$ . Hence, the dispersion relation for internal waves is

$$\omega_s(k) = 2s \sqrt{\cos \theta_k} .$$

The zero-frequency modes or "zero-modes" are the 2D modes with  $k \perp \hat{z}$  so that  $\frac{\partial u}{\partial z} = 0$ . These are sometimes called 2D+3C, since there is no requirement that  $\hat{z} \cdot u = 0$ .

Expanding the velocity into helical modes as

$$u(x, t) = \sum_{\mathbf{k}} \sum_{s=\pm 1} a_s(\mathbf{k}, t) h_s(\mathbf{k}) e^{i \mathbf{k} \cdot \mathbf{x}}$$

the 3D Navier-Stokes equation becomes of the form

$$\begin{aligned} & (\partial_t - i \omega_s(\mathbf{k}) - v k^2) a_s(\mathbf{k}, t) \\ &= \frac{1}{2} \sum_{\mathbf{k} + \mathbf{p} + \mathbf{q} = 0} \sum_{s_p, s_q} C_{\mathbf{k} \mathbf{p} \mathbf{q}}^{s_k s_p s_q} a_{s_p}^*(\mathbf{p}, t) a_{s_q}^*(\mathbf{q}, t). \end{aligned}$$

Because the NS nonlinear term has the form  $u \times w$   
it is easy to show that, in fact,

$$C_{\mathbf{k} \mathbf{p} \mathbf{q}}^{s_k s_p s_q} = \frac{1}{2} (s_{pp} - s_{qq}) h_{sp}^*(\mathbf{p}) \times h_{sq}^*(\mathbf{q}) \cdot h_{sk}^*(\mathbf{k}).$$

Energy conservation implies that

$$C_{\mathbf{k} \mathbf{p} \mathbf{q}}^{s_k s_p s_q} + C_{\mathbf{p} \mathbf{q} \mathbf{k}}^{s_p s_q s_k} + C_{\mathbf{q} \mathbf{k} \mathbf{p}}^{s_q s_k s_p} = 0$$

and Helicity conservation implies that

$$s_k h C_{\mathbf{k} \mathbf{p} \mathbf{q}}^{s_k s_p s_q} + s_{pp} C_{\mathbf{p} \mathbf{q} \mathbf{k}}^{s_p s_q s_k} + s_{qq} C_{\mathbf{q} \mathbf{k} \mathbf{p}}^{s_q s_k s_p} = 0.$$

So far we have been completely general, but now we would like to consider the limit of very rapid rotation. This is defined by smallness of the nondimensional Rossby number

$$Ro = \frac{U/L}{2\Omega} = \frac{U}{2\Omega L}$$

where  $U$  and  $L$  are characteristic velocity and length scales of the flow. If one introduces dimensionless variables

$$\tilde{u} = u/U, \tilde{x} = x/L, \tilde{t} = Ut/L$$

then the NS equation becomes

$$\partial_t \tilde{u} + \frac{1}{Ro} \hat{\nabla} \times \tilde{u} = -\tilde{\nabla} \tilde{p} + \frac{1}{Re} \tilde{\Delta} \tilde{u} - \tilde{\omega} \times \tilde{u}$$

with  $Re = UL/v$  the corresponding Reynolds number. The equation for helicity mode coefficients becomes  
(in dimensionless variables)

$$\left( \partial_t - \frac{1}{Ro} i s u_{s_k}(k) - \frac{1}{Re} k^2 \right) a_s(k, t)$$

$$= \frac{1}{2} \sum_{k+p+q=0} \sum_{s_p s_q}^{s_p s_q} C_{kpq} a_{sp}^* a_{sq}^*$$

When  $R_0 \ll 1$ , inertial waves oscillate very rapidly.  
It is convenient to remove this fast oscillation  
by using envelope functions  $A_{S_k}(k, t)$  defined by

$$a_{S_k}(k, t) = A_{S_k}(k, t) e^{i\omega_{S_k}(k)t/R_0}$$

(similar to the "interaction picture" in quantum physics) with the fast oscillation removed.  
These envelope functions satisfy

$$\left( \partial_t - \frac{1}{R_0} k^2 \right) A_{S_k} = \frac{1}{2} \sum_{k+p+q=0} \sum_{S_q S_p} C_{k p q}^{S_k S_p S_q} e^{-i(\omega_{S_k} + \omega_{S_p} + \omega_{S_q})t/R_0} \times A_{S_p}^* A_{S_q}^*$$

In the limit  $R_0 \rightarrow 0$  it is plausible that the rapid oscillations eliminate all of the terms except those that satisfy the resonance condition

$$\omega_{S_k}(k) + \omega_{S_p}(p) + \omega_{S_q}(q) = 0.$$

For careful justification see Embid & Majda (1996).

The result is the averaged equation for  $\Omega \rightarrow 0$

$$\left( \partial_t - \frac{1}{Re} k^2 \right) A_{Sk} = \frac{1}{2} \sum_{\substack{k+p+q=0 \\ \omega_{Sk} + \omega_{Sp} + \omega_{Sq} = 0}} \sum_{Sp Sq} C_{k p q}^{S_k S_p S_q} A_{Sp}^* A_{Sq}^*$$

which keeps only the resonant terms. An important observation (Waleffe, 1992) is that the resonant terms also satisfy

$$s_k k \omega_{Sk}(k) + s_p p \omega_{Sp}(p) + s_q q \omega_{Sq}(q) = 0$$

since  $s_k k \omega_{Sk}(k) = k \cos \theta_k = k_z$  and thus the above relation is equivalent to  $k_z + p_z + q_z = 0$ . Thus, the resonant  $\omega$ 's satisfy the same relations as implied for  $C$  coefficients by conservation of energy & helicity. Since two linear relations define three quantities up to proportionality, we get an easy proof of the important fact that

$$\omega_{Sk}(k) = 0 \quad \Rightarrow \quad C_{k p q}^{S_k S_p S_q} = 0$$

$\omega_{Sp}(p) \text{ or } \omega_{Sq}(q) \neq 0$

The implication is that the slow ( $2D + 3C$ ) modes do not interact with the fast ( $w=0$ ) modes by resonant terms. Thus, the slow modes are an autonomous subspace with their own dynamics in the limit  $\Omega \rightarrow \infty$ , uninfluenced by the fast modes.

Note the same is not true of the fast modes, which are dynamically influenced by the slow modes. It is easy to see what is the evolution equation for the slow modes, because every slow-slow-slow triad is resonant (i.e.,  $0+0+0=0$ ) and thus all of the original interactions of those modes with themselves are retained. It follows that the slow dynamics is  $2D + 3C$  Navier - Stokes, or

$$\partial_t u_{2D} + (u_{2D} \cdot \nabla_{2D}) u_{2D} = -\nabla_{2D} p + v \Delta_{2D} u_{2D}$$

$$\partial_t w + (u_{2D} \cdot \nabla) w = v \Delta_{2D} w$$

with  $u = (u_{2D}, w) = (u, v, w)$ , i.e. the equation obtained from 3D NS by dropping all  $z$ -dependence. This can also be verified by explicit calculation, e.g. see

Q. Chen et al., "Resonant interactions in rotating homogeneous three-dimensional turbulence," JFM 542 139–164 (2005),

## Appendix B

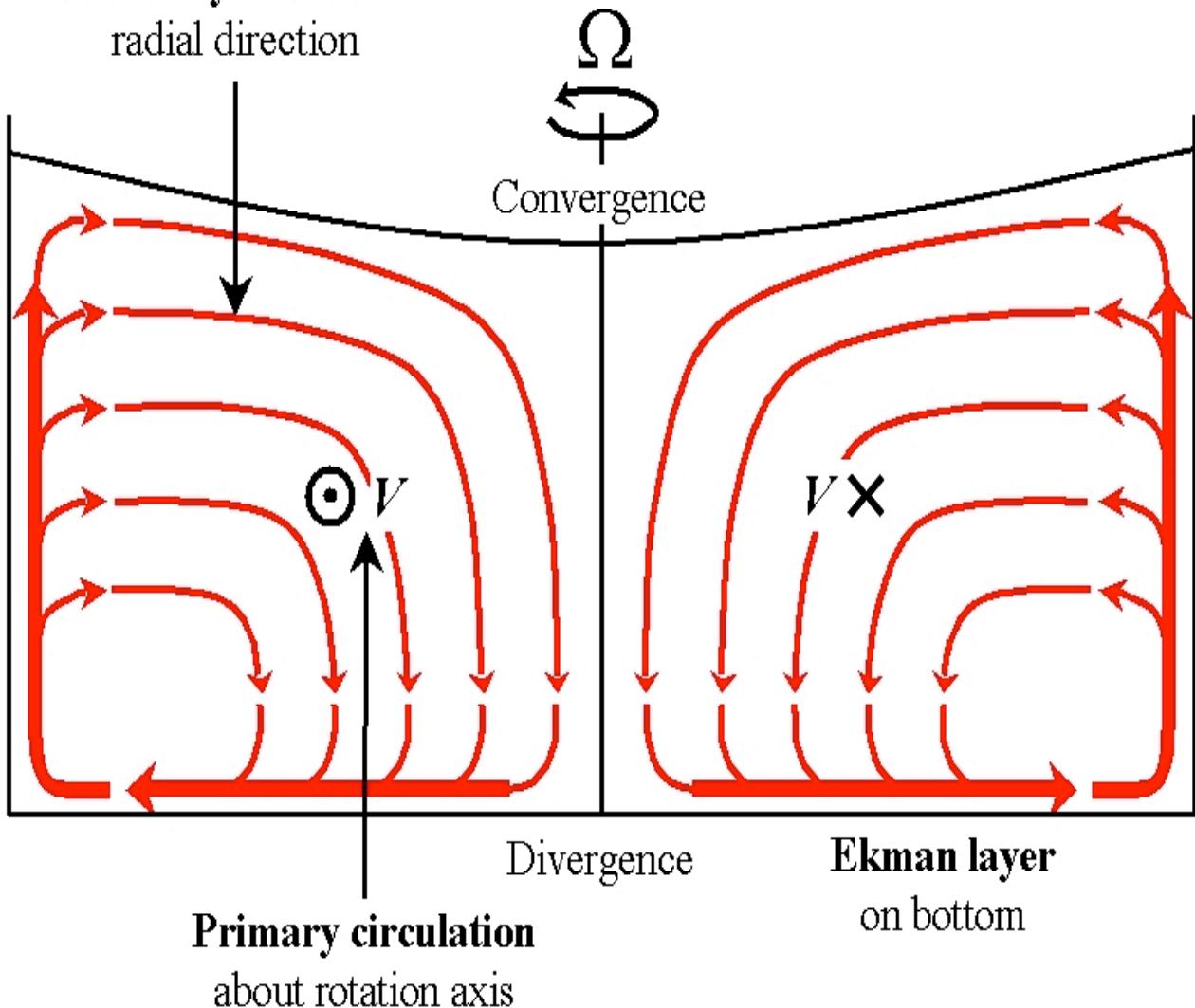
In particular,

$$u_{2D}(x, y) = \frac{1}{L_z} \int_0^{L_z} dz u_{2D}(x, y, z),$$

which corresponds to the  $k_z=0$  slow mode statistics  
the 2D Navier-Stokes equation for  $Ro \rightarrow 0$ . Note  
that it is not true, as is sometimes stated, that  
the flow becomes fully 2D for rapid rotation. For  
example, vertical velocity  $w \neq 0$  remains. In addition,  
fast 3D modes also remain which satisfy the  
averaged equation. It is only that the fast modes  
become dynamically decoupled from the 2D modes  
and do not affect them. Additional processes,  
such as forward cascade and viscous dissipation  
of the fast modes, may however deplete the energy  
in the fast 3D modes and leave only the 2D modes  
as those with conspicuous energy in the flow. (STOH,  
viscous boundary layers can produce 3D flows. See Fig.

## Secondary circulation

radial direction



## Primary circulation

about rotation axis

For a corresponding analysis of thin fluid layers, see

I. Moise, R. Temam & M. Ziane, "Asymptotic analysis of the Navier-Stokes equations in thin domains," *Topol. Meth. Nonlin. Anal.* 10 249-289 (1997)

For discussion of the effects of both rotation and stratification, see

P. Embid & A. Majda, "Low Froude number limiting dynamics for stably stratified flow with small or finite Rossby numbers," *Geophys. & Astrophys. Fluid Dyn.* 87 1-50 (1998)

A. Babin, A. Mahalov & B. Nicolaenko, "Fast singular oscillating limits of stably-stratified 3D Euler and Navier-Stokes equations and ageostrophic wave fronts," in Large-Scale Atmosphere-Ocean Dynamics, Vol. 1, eds. J. Norbury and I. Roulstone (Cambridge U Press, 2002)

In this case one does not get 2D NS for any of the standard limits, but instead related equations which we shall discuss later.

Experiments on two-dimensional turbulence are performed in the laboratory by a number of techniques, including:

\* soap films flowing between vertically suspended wires

\* electrolytic thin fluid layers driven by arrays of magnets

and others. (It seems to be an open question whether these systems are strictly described by 2D NS.)

For a review of such experiments, see:

H. Kellay & W. I. Goldburg, "Two-dimensional turbulence: a review of some recent experiments,"  
Rep. Prog. Phys. 65 845–894 (2002)

or

G. Boffetta & R. E. Ecke, "Two-dimensional turbulence," Annu. Rev. Fluid. Mech. 44  
427–451 (2012)

The latter is also a very current, general review of the current status of two-dimensional turbulence from the points of view of theory, simulation, laboratory experiment, and natural observations.

Basic properties of 2D Navier-Stokes: Now that the validity of 2D Navier-Stokes has been carefully justified, let us discuss its basic properties. We begin with kinematic relations imposed by two-dimensionality on the velocity. Every solenoid velocity field  $\mathbf{u}$  in 2D can be obtained from a stream function  $\psi$  by

$$\mathbf{u} = \nabla^\perp \psi = \begin{pmatrix} -\partial_y \psi \\ \partial_x \psi \end{pmatrix}.$$

The vector  $\mathbf{u}$  is pointwise parallel to the contour/iso-lines of  $\psi$ , which are called streamlines. Note also that the vorticity has only one component,  $\omega = \hat{\mathbf{z}} \cdot \boldsymbol{\omega}$ , given by

$$\omega = \nabla^\perp \cdot \mathbf{u} = \partial_x v - \partial_y u.$$

It follows that

$$\omega = \Delta \psi$$

and thus  $\psi(\mathbf{x}) = \int_D G(\mathbf{x}, \mathbf{y}) \omega(\mathbf{y}) d^2 y$

where  $G$  is the Green function of the Laplacian  $\Delta$  in the domain  $D$  with Dirichlet b.c. (to ensure that the boundary  $\partial D$  is a streamline). In the infinite domain  $D = \mathbb{R}^2$  one has  $G(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{y}|$ .

2D NS for  $v=0$  (i.e. 2D Euler) has all of the same constants of motion as does 3D Euler, of course. The helicity, however, is trivial since  $\mathbf{u} \cdot \mathbf{w} = 0$ . On the other hand, the kinetic energy

$$E = \frac{1}{2} \int_D d^2x |\mathbf{u}(\mathbf{x}, t)|^2$$

remains a nontrivial invariant. This can be easily written in the alternative forms

$$E = -\frac{1}{2} \int_D d^2x \psi(\mathbf{x}, t) w(\mathbf{x}, t)$$

$$= -\frac{1}{2} \int_D d^2x \int_D d^2y G(\mathbf{x}, \mathbf{y}) w(\mathbf{x}, t) w(\mathbf{y}, t)$$

There are, in addition, new conservation laws that appear only in 2D. The most important set of these can be derived by recalling the 3D Helmholtz equations for the vorticity field

$$(3D) \quad \partial_t \omega + (\mathbf{u} \cdot \nabla) \omega = (\omega \cdot \nabla) \mathbf{u} + v \Delta \omega.$$

Since  $\omega = \omega \hat{\mathbf{z}}$  in 2D and thus  $(\omega \cdot \nabla) \mathbf{u} = \omega \frac{\partial \mathbf{u}}{\partial \hat{\mathbf{z}}} = 0$ , the equation drastically simplifies to

$$(2D) \quad \partial_t \omega + (\mathbf{u} \cdot \nabla) \omega = v \Delta \omega,$$

due to the absence of vortex-stretching in 2D.

This can also be written as

$$(2D^*) \quad \partial_t \omega + J(\psi, \omega) = v \Delta \omega$$

using the Jacobian determinant

$$J(\psi, \omega) = \begin{vmatrix} \psi_x & \psi_y \\ \omega_x & \omega_y \end{vmatrix}$$

and  $u = \nabla^\perp \psi$ . These equations, (2D) and (2D<sup>\*</sup>), are equivalent formulations of 2D NS, in the so-called velocity-stream function representation, with  $\psi = \Delta \omega$ .

Equation (2D) for  $v=0$  has the remarkable implication that

$$D_t \omega = 0,$$

with  $D_t = \partial_t + u \cdot \nabla$  the material/Lagrangian time derivative. Thus, the vorticity is a Lagrangian invariant, i.e., is conserved along fluid particle trajectories. This is the most fundamental fact about 2D Euler dynamics, with profound implications.

Note that it is the consequence of the Kelvin Theorem in 2D for a loop  $C_\varepsilon(t)$  which shrinks to a point as  $\varepsilon \rightarrow 0$ .

An immediate consequence is that, for any function  $f$ ,

$$I_f(t) = \int d^2x f(\omega(x,t))$$

is an invariant of (smoother) solutions of 2D Euler, using also the volume-preserving property of the fluid flow for a solenoidal velocity. Equivalently,

$$A_b(t) = \text{area} \left\{ \mathbf{x} : \omega(\mathbf{x},t) < b \right\}$$

is conserved for all real numbers  $b$ . For discussions of 2D turbulence, a distinguished invariant of the type  $I_f(t)$  is for  $f(\omega) = \frac{1}{2}\omega^2$ :

$$\Omega(t) = \frac{1}{2} \int d^2x \omega^2(x,t).$$

This was called enstrophy by C. Leith (1968), based on the Greek word  $\sigma\tau\rho\omega\varphi\eta$ , meaning the "act of turning." As we shall see, this quantity plays a central role in the Kraichnan-Batchelor cascade theory of 2D turbulence.