Homework #6

Problem 1. (a) Note that

$$\frac{\partial}{\partial x_i}\begin{pmatrix} \cos(k_0\cdot x) \\ \sin(k_0\cdot x)\end{pmatrix} = k_i^{(0)}\begin{pmatrix} -\sin(k_0\cdot x) \\ \cos(k_0\cdot x)\end{pmatrix}$$

which is $O(k_0)$ in the L^∞-norm, for example. On the other hand,

$$\frac{\partial}{\partial x_i} e^{-r^2/2D^2} = \frac{x_i}{D^2} e^{-r^2/2D^2}$$

which is $O\left(\frac{1}{D}\right)$, achieving its maximum when $r \approx D$. Hence, any derivative of $e^{-r^2/2D^2}$ is smaller than the corresponding derivative of $\cos(k_0\cdot x)$ or $\sin(k_0\cdot x)$ by a factor of $O\left(\frac{1}{k_0 D}\right)$. It follows that we may differentiate only the trigonometric functions and not the "shape function" $e^{-r^2/2D^2}$ to get the leading-order expressions. From

$$\psi_0(x) = \frac{\omega_0}{k_0^2} e^{-r^2/2D^2} \cos(k_0\cdot x)$$

it follows that

$$\begin{aligned} u_0(x) &= \nabla^\perp \psi_0(x) \\ &= \frac{k_0^\perp}{k_0^2} e^{-r^2/2D^2}\left[\sin(k_0\cdot x) + O\left(\frac{1}{k_0 D}\right)\right] \end{aligned}$$

and

$$\omega_0(\mathbf{x}) = \nabla^{\perp} \cdot \mathbf{u}_0(\mathbf{x}) = \omega_0 e^{-r^2/2D^2}\left[\cos(\mathbf{k}_0 \cdot \mathbf{x}) + O\left(\frac{1}{k_0 D}\right)\right].$$

To calculate the enstrophy and energy, it suffices to take the vector $\mathbf{k}_0 = k_0 \hat{\mathbf{x}}$. Thus, to leading order in $\frac{1}{k_0 D}$,

$$\Omega_0 = \frac{1}{2} \int d^2x \; \omega_0^2(\mathbf{x})$$

$$= \frac{\omega_0^2}{2} \int_0^{\infty} r\,dr\; e^{-r^2/D^2} \int_0^{2\pi} d\theta \; \cos^2(k_0 r \cos\theta)$$

Using the trig identity $\cos^2\phi = \dfrac{1+\cos(2\phi)}{2}$, one obtains a sum of two integrals. The first is

$$I_1 = \frac{\omega_0^2}{2} \int_0^{\infty} r\,dr\; e^{-r^2/D^2} \cdot \int_0^{2\pi} d\theta \; \frac{1}{2}$$

$$= \frac{\pi \omega_0^2}{4} \int_0^{\infty} 2r\,dr\; e^{-r^2/D^2} = \frac{\pi \omega_0^2 D^2}{4},$$

while the second is

$$I_2 = \frac{\omega_0^2}{4} \int_0^{\infty} r\,dr\; e^{-r^2/D^2} \int_0^{2\pi} d\theta \; \cos(2k_0 r \cos\theta)$$

Using the integral representation of the Bessel function

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} \cos(z\cos\theta)\, d\theta,$$

this becomes

$$I_2 = \frac{\pi w_0^2}{2}\int_0^{\infty} r\, dr\, e^{-r^2/D^2}\, J_0(2k_0 r)$$

and changing variables to $u = r^2/D^2$

$$I_2 = \frac{\pi w_0^2 D^2}{4}\int_0^{\infty} du\, e^{-u}\, J_0(2k_0 D\sqrt{2u})$$

$$= \frac{\pi w_0^2 D^2}{4} \exp\left(-2k_0^2 D^2\right)$$

which is transcendentally small compared with the first integral. Thus,

$$\Omega_0 = \frac{\pi w_0^2 D^2}{4}$$

up to corrections of order $O\left(\frac{1}{k_0 D}\right)$. Exactly the same argument applies to the energy integral

$$E_0 = \frac{1}{2}\int d^2x \; |u_0(x)|^2$$

$$= \frac{\omega_0^2}{2k_0^2}\int_0^\infty r\,dr\; e^{-r^2/D^2}\int_0^{2\pi} d\theta\; \sin^2(k_0 r\cos\theta).$$

Using $\sin^2\phi = \frac{1-\cos(2\phi)}{2}$ gives likewise

$$E_0 = \frac{\pi\omega_0^2 D^2}{4k_0^2} = \frac{1}{k_0^2}\Omega_0$$

up to corrections of order $O\left(\frac{1}{k_0 D}\right)$.

(b) The solution of the equation

$$(\partial_t + u\cdot\nabla_x)\,\omega = 0$$

can be obtained by the method of characteristics as

$$\omega(x,t) = \omega_0(X_t^{-1}(x))$$

where X_t solves

$$\frac{dX_t}{dt} = u(X_t,t)$$

$$X_0(x) = x.$$

For the stated linear velocity field,

$$X_t(\mathbf{x}) = \begin{pmatrix} e^{\beta(t)} x_1 \\ e^{-\beta(t)} x_2 \end{pmatrix}$$

and thus

$$\begin{aligned} \omega(\mathbf{x},t) &= \omega_0\left(e^{-\beta(t)} x_1,\ e^{+\beta(t)} x_2\right) \\ &= \omega_0 \exp\left(-\frac{e^{2\beta(t)} x_1^2 + e^{-2\beta(t)} x_2^2}{2D^2}\right) \cos(\mathbf{k}(t)\cdot\mathbf{x}) \end{aligned}$$

with

$$\mathbf{k}(t) = \begin{pmatrix} e^{-\beta(t)} k_1 \\ e^{+\beta(t)} k_2 \end{pmatrix}.$$

Since the transformation X_t is volume-preserving

$$\Omega(t) = \frac{1}{2}\int d^2x\ \omega_0^2(X_t^{-1}(\mathbf{x})) = \frac{1}{2}\int d^2x\ \omega_0^2(\mathbf{x}) = \Omega_0 .$$

Note that by the same argument as in part (a), the only derivatives that matter are those applied to the trig functions, to leading order. Hence, at any fixed finite time t

$$\begin{aligned} \mathbf{u}(\mathbf{x},t) = \frac{\mathbf{k}^{\perp}(t)}{k^2(t)}\, \omega_0 \exp\left(-\frac{e^{2\beta(t)} x_1^2 + e^{-2\beta(t)} x_2^2}{2D^2}\right) \\ \times \left[\sin(\mathbf{k}(t)\cdot\mathbf{x}) + O\left(\frac{1}{k_0 D}\right)\right]. \end{aligned}$$

The energy integral at time t can be calculated by using

$$|u(x,t)|^2 = \frac{k_0^2}{k^2(t)} \left| u_0\left(X_t^{-1}(x) \right) \right|^2$$

and the fact that X_t is volume-preserving to obtain

$$E(t) = \frac{1}{2}\int d^2x\, |u(x,t)|^2 = \frac{k_0^2}{k^2(t)} E_0 = \frac{1}{k^2(t)} \Omega_0 .$$

(c) Taking

$$\mathbf{k}(t) = k_0 \begin{pmatrix} e^{-\beta(t)} \cos\varphi \\ e^{+\beta(t)} \sin\varphi \end{pmatrix}$$

and averaging the square magnitude over angle φ gives

$$\langle k^2(t) \rangle = k_0^2 \cdot \frac{1}{2\pi} \int_0^{2\pi} \left[e^{-2\beta(t)} \cos^2\varphi + e^{2\beta(t)} \sin^2\varphi \right] d\varphi$$

$$= k_0^2 \cdot \frac{e^{-2\beta(t)} + e^{2\beta(t)}}{2} = \cosh(2\beta(t))\, k_0^2 .$$

On the other hand, $\langle E(t) \rangle = \langle k^{-2}(t) \rangle \Omega_0$ with

$$\langle k^{-2}(t) \rangle = \frac{1}{k_0^2} \cdot \frac{1}{2\pi} \int_0^{2\pi} \frac{d\varphi}{e^{-2\beta(t)} \cos^2\varphi + e^{2\beta(t)} \sin^2\varphi}$$

$$= \frac{1}{k_0^2} \cdot \frac{1}{2\pi} \int_0^{2\pi} \frac{d\varphi}{\cosh(2\beta(t)) - \sinh(2\beta(t)) \cos(2\varphi)}$$

and this can be evaluated with the standard integral

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{a-b\cos\theta} = \frac{1}{\sqrt{a^2-b^2}}$$

and the hyperbolic function identity $\cosh^2(2\beta)-\sinh^2(2\beta)=1$ to be

$$\langle k^{-2}(t)\rangle = k_0^{-2} \; !$$

It follows that

$$\langle E(t)\rangle = k_0^{-2}\Omega_0 = E_0$$

and the energy of the assemblage of vortex-blobs is conserved, on average.

This result does not apply directly to the 2D inverse cascade range for several reasons. For example,

(i) The strain $\overline{S}_\ell$ acting on small-scale vorticity is not constant in space, but only at a scale 4-8 times larger than the vortex structure itself

(ii) There is no reason that the conditional statistics of the small-scale vorticity given $\overline{S}_\ell$ should be isotropic. In fact, it is known from numerical studies that this is not true, e.g. the strain $S^{[n]}$ at scale $\ell_n = 2^{-n}\ell$ is typically rotated by $\pm 45°$ relative to $\overline{S}_\ell$.

(iii) Kraichnan's result ignores nonlinear self-interactions of the small-scale vorticity $\omega^{[n]}$. It is known that strained elliptical vortices rotate, which will remove atypical configurations where $\nabla\omega^{[n]}$ is reduced in magnitude. See S. Kida, "Motion of an Elliptic Vortex in a Uniform Shear Flow," J. Phys. Soc. Jap. 50 3517-3520 (1981)

Problem 2. The idea is simply to substitude a Taylor-series with reminder for the incremant $\delta u(\mathbf{r})$. It is enough to take

$$\delta u(\mathbf{r}) = \mathbf{D}\cdot\mathbf{r} + \frac{1}{2}(\mathbf{r}\mathbf{r}:\nabla\nabla)u(\mathbf{x}+\theta(\mathbf{x})\mathbf{r})$$

with 2nd-order reminder, because the finiteness of the integrals

$$\int d^2r \, |\partial_{i_1}\partial_{i_2}\cdots\partial_{i_p} G(\mathbf{r})|\cdot|\mathbf{r}|^q , \quad p,q=0,1,2$$

implies that including any reminder term, which will involve 3 factors of $|\mathbf{r}|$, must give a term $O(\ell)$ which vanishes as $\ell\to 0$. Thus, the only non-vanishing contribution as $\ell\to 0$ is obtained from the linear term.

The first integral gives, using $\partial_j^\perp = -\epsilon_{jm}\partial_m$,

$$\frac{1}{\ell^2}\int d^2r \, (\partial_i\partial_j^\perp G)_\ell(\mathbf{r}) \; D_{ik} D_{j\ell} r_k r_\ell$$

$$= -\epsilon_{jm}\int d^2r \, G_\ell(\mathbf{r}) \, D_{ik} D_{j\ell} \, \partial_i\partial_m (r_k r_\ell)$$

$$= -\epsilon_{jm}\int d^2r \, G_\ell(\mathbf{r}) \, D_{ik} D_{j\ell} \, (\underbrace{\cancel{\delta_{ik}}}_{0 \text{ using } \mathrm{tr}(\mathbf{D})=0}\delta_{m\ell} + \delta_{i\ell}\delta_{mk})$$

$$= -\epsilon_{jk}(\mathbf{D}^2)_{jk}\cdot\int d^2r \, G_\ell(\mathbf{r})$$

$$= -\epsilon_{jk}(\mathbf{D}^2)_{jk} \qquad \text{using } \int d^2r \, G_\ell(\mathbf{r}) = 1$$

The second integrals get zero contribution from the linear terms, because $\partial_{r_i} \partial_{r_j}^{\perp}(\mathbf{r}) = 0$.

The third set of integrals gives

$$-\frac{1}{\ell^2} \int d^2r \, (\partial_j^{\perp} G)_\ell(\mathbf{r}) \, D_{ik} r_k \times \int d^2r (\partial_i G)_\ell(\mathbf{r}) D_{j\ell} r_\ell$$

$$= \epsilon_{jm} \int d^2r \, G_\ell(\mathbf{r}) \, D_{ik} \, \cancel{\partial_m (r_k)} \to \delta_{mk}$$

$$\times \int d^2r \, G_\ell(\mathbf{r}) \, D_{j\ell} \, \cancel{\partial_i (r_\ell)} \to \delta_{\ell i}$$

$$= \epsilon_{jk} (\mathbf{D}^2)_{jk} \left[\int d^2r \, G_\ell(\mathbf{r}) \right]^2 = \epsilon_{jk} (\mathbf{D}^2)_{jk} .$$

Hence, the non-vanishing contributions from the first and third term cancel, and the limit is zero pointwise.

Finally, recall from Homework #5, Problem 3(a), that

$$\mathbf{D}^2 = -\det(\mathbf{D}) \, \mathbf{I}$$

from the Cayley-Hamilton Theorem. Hence,

$$\epsilon_{jk} (\mathbf{D}^2)_{jk} = -\det(\mathbf{D}) \epsilon_{jj} = 0$$

by the anti-symmetry of ϵ_{ij}. Thus, each term is in fact separately zero, without the need for cancellation.

Problem 3. The mean value theorem of differential calculus implies that

$$h(\bar{\omega}_\ell(x)) - h(\omega(x)) = h'(\bar{\omega}_\ell^\theta(x)) \cdot (\bar{\omega}_\ell(x) - \omega(x))$$

with

$$\bar{\omega}_\ell^\theta(x) = \theta(x)\bar{\omega}_\ell(x) + (1-\theta(x))\omega(x)$$

for $0 \leq \theta(x) \leq 1$. The triangle inequality implies that

$$\|\bar{\omega}_\ell^\theta\|_p \leq \|\theta \cdot \bar{\omega}_\ell\|_p + \|(1-\theta)\cdot \omega\|_p$$

$$= \|\bar{\omega}_\ell\|_p + \|\omega\|_p .$$

Recall that $\bar{\omega}_\ell = \tilde{G}_\ell * \omega$, where $\tilde{G}(r) = G(-r)$ is the reflection of G and the $*$ indicates convolution. Young's inequality for convolutions gives

$$\|\bar{\omega}_\ell\|_p = \|\tilde{G}_\ell * \omega\|_p \leq \|\tilde{G}_\ell\|_1 \cdot \|\omega\|_p = \|\omega\|_p$$

since $G \geq 0$ and $\int G(r)\,d^d r = 1$ gives $\|\tilde{G}_\ell\|_1 = 1$. Thus,

$$\|\bar{\omega}_\ell^\theta\|_p \leq 2\|\omega\|_p .$$

Now, use the Hölder inequality to get

$$\|h(\bar{\omega}_\ell) - h(\omega)\|_1 \leq \|h'(\bar{\omega}_\ell^\theta)\|_{p/(p-1)} \cdot \|\bar{\omega}_\ell - \omega\|_p$$

for any $p \geq 1$. To complete the argument, we must only derive the general estimate on $\|h'(\omega)\|_{p/(p-1)}$ for $h \in \mathcal{H}_p$.

Note that

$$\| h'(\omega) \|_{p/(p-1)}^{p/(p-1)} = \int_{\mathbb{T}^d} | h'(\omega(x)) |^{\frac{p}{p-1}} d^d x$$

$$= \int_{\mathbb{T}^d \cap \{|\omega| < R\}} | h'(\omega(x)) |^{\frac{p}{p-1}} d^d x$$

$$+ \int_{\mathbb{T}^d \cap \{|\omega| \geq R\}} | h'(\omega(x)) |^{\frac{p}{p-1}} d^d x .$$

The first integral is bounded by $M_h^{\frac{p}{p-1}}$ with

$$M_h = \max_{|\omega| \leq R} | h'(\omega) | ,$$

assuming for convenience that $\int_{\mathbb{T}^d} d^d x = 1$. Using the definition of $\mathcal{H}_p$,

$$| h'(\omega) | \leq C |\omega|^{p-1} \quad \text{for } |\omega| \geq R$$

and thus the second integral is bounded by

$$C^{p/(p-1)} \int_{\mathbb{T}^d} |\omega(x)|^p d^d x = C^{p/(p-1)} \|\omega\|_p^p .$$

Thus,

$$\| h'(\omega) \|_{p/(p-1)}^{p/(p-1)} \leqslant M_h^{p/(p-1)} + C^{p/(p-1)} \|\omega\|_p^p$$

$$\leqslant 2 \max \{ M_h^{p/(p-1)}, C^{p/(p-1)} \|\omega\|_p^p \}$$

and

$$\| h'(\omega) \|_{p/(p-1)} \leqslant 2^{(p-1)/p} \max \{ M_h, C \|\omega\|_p^{p-1} \},$$

which is the desired general estimate.

As a special case

$$\| h'(\bar{\omega}_\ell^\theta) \|_{p/(p-1)} \leqslant 2^{(p-1)/p} \max \{ M_h, C \| \bar{\omega}_\ell^\theta \|_p^{p-1} \}$$

$$\leqslant 2^{(p-1)/p} \max \{ M_h, 2^{p-1} C \|\omega\|_p^{p-1} \}$$

and thus, finally,

$$\| h(\bar{\omega}_\ell) - h(\omega) \|_1 \leqslant 2^{(p-1)/p} \max \{ M_h, 2^{p-1} C \|\omega\|_p^{p-1} \}$$

$$\times \| \bar{\omega}_\ell - \omega \|_p .$$

From this we can infer that

$$h(\bar{\omega}_\ell) \xrightarrow[\ell \to 0]{} h(\omega) \quad \text{strong-}L^1,$$

under the stated assumptions.