Homework #5

Problem 1. (a) The "nonlinear model" for the subscale stress is in 2D

$$\tau_{ij} = \frac{1}{2} C \ell^2 \bar{u}_{i,k} \bar{u}_{j,k}$$

using the simplified notations $\tau = \tau_\ell$, $\bar{u} = \bar{u}_\ell$ and $\bar{u}_{i,k} = \frac{\partial \bar{u}_i}{\partial x_k}$.

By incompressibility $\bar{u}_{j,jk} = 0$ and thus

$$\partial_j \tau_{ij} = \frac{1}{2} C \ell^2 \bar{u}_{i,jk} \bar{u}_{j,k} .$$

Furthermore,

$$\partial_i \left(\frac{1}{2} \tau_{jj} \right) = \partial_i \left(\frac{C\ell^2}{4} \bar{u}_{j,k} \bar{u}_{j,k} \right) = \frac{C\ell^2}{2} \bar{u}_{j,ik} \bar{u}_{j,k}$$

Hence,

$$\sigma_i^\perp = \partial_j \tau_{ij} - \frac{1}{2} \partial_i \tau_{jj} = \frac{1}{2} C \ell^2 (\bar{u}_{i,jk} - \bar{u}_{j,ik}) \bar{u}_{j,k}$$

(b) Note that in 2D the anti-symmetric part of the velocity-gradient is

$$\bar{\Omega}_{ij} \equiv \frac{1}{2} (\bar{u}_{i,j} - \bar{u}_{j,i}) = - \frac{1}{2} \omega \epsilon_{ij} .$$

Hence, from (a),

$$\sigma_i^\perp = - \frac{1}{2} C \ell^2 \epsilon_{ij} \, \omega_{,k} \bar{u}_{j,k}$$

and since $\sigma^\perp = - \epsilon \sigma$, this implies that

$$\sigma_i = \frac{1}{2} C \ell^2 \bar{u}_{i,k} \, \omega_{,k} .$$

Problem 2. (a) In general

$$(\bar{S}_\ell)^2_{rms} = \frac{1}{2}\int_0^{2\pi/\ell} k^2 E(k)\, dk .$$

For the given spectrum this becomes an elementary integral

$$(\bar{S}_\ell)^2_{rms} = \frac{1}{2} C \eta^{2/3} \int_{k_f}^{2\pi/\ell} \frac{dk}{k \ln^p (k/k_f)}$$

$$= \frac{1}{2} C \eta^{2/3} \int_0^{\ln(2\pi/k_f \ell)} \frac{dx}{x^p} \quad , \; x = \ln(k/k_f)$$

$$= \frac{1}{2(1-p)} C \eta^{2/3} \ln^{1-p}(2\pi/k_f \ell)$$

$$\Longrightarrow \quad (\bar{S}_\ell)_{rms} = C' \eta^{1/3} \ln^{\frac{1-p}{2}} (2\pi/k_f \ell)$$

with $C' = \left(\frac{C}{2(1-p)}\right)^{1/2}$.

(b) In general

$$(\nabla \bar{\omega}_\ell)^2_{rms} = \int_0^{2\pi/\ell} k^4 E(k)\, dk$$

and for the given spectrum this becomes

$$(\nabla \bar{\omega}_\ell)^2_{rms} = C\eta^{2/3} \int_{k_f}^{2\pi/\ell} \frac{k\,dk}{\ln^p (k/k_f)}$$

$$= C\eta^{2/3} k_f^2 \int_0^{\ln(\pi/k_f\ell)} \frac{e^{2x}\,dx}{x^p} \qquad x = \ln(k/k_f)$$

$$= \frac{C\eta^{2/3} k_f^2 e^{2A}}{2A^p} \int_0^1 \frac{e^{2A(u-1)}}{u^p} 2A\,du \qquad x = Au, \quad A = \ln\left(\frac{2\pi}{k_f \ell}\right)$$

However, it is a simple standard argument that

$$\lim_{A \to \infty} \int_0^1 2A\, e^{2A(u-1)} \varphi(u)\,du = \varphi(1)$$

for any function φ continuous on $(0,1]$ and integrable there. Applying this result for $\varphi(u) = \frac{1}{u^p}$, one gets that

$$(\nabla \bar{\omega}_\ell)^2_{rms} \sim \frac{C\eta^{2/3} k_f^2 e^{2A}}{2A^p} \qquad A \gg 1$$

$$\sim \frac{C\eta^{2/3}}{2} \left(\frac{2\pi}{\ell}\right)^2 \frac{1}{\ln^p (2\pi/k_f \ell)} \qquad \frac{2\pi}{\ell} \gg k_f$$

and

$$(\nabla \bar{\omega}_\ell)_{rms} \sim C'' \eta^{1/3} \frac{1}{\ell \ln^{p/2}(2\pi/k_f \ell)} \qquad \frac{2\pi}{\ell} \gg k_f$$

with $C'' = 2\pi \left(\frac{C}{2}\right)^{1/2}$.

(c) From parts (a) and (b)

$$Z_\ell \sim (\text{const.})\, \eta \left[\ln(2\pi / k_f \ell)\right]^{\frac{1-3p}{2}}$$

for $k_f \ll \frac{2\pi}{\ell} < k_{ur}$. This becomes independent of ℓ precisely when $p = \frac{1}{3}$.

Problem 3. (a) The characteristic equation for any 2×2 matrix $\mathbf{D}$ has the form

$$\lambda^2 - \operatorname{tr}(\mathbf{D})\lambda + \det(\mathbf{D}) = 0.$$

Since $\operatorname{tr}(\mathbf{D}) = 0$ by incompressibility, this becomes

$$\lambda^2 = -\det(\mathbf{D}).$$

Hence, by the Cayley-Hamilton Theorem,

$$\mathbf{D}^2 = -\det(\mathbf{D})\, \mathbf{I}$$

and taking the trace gives

$$\operatorname{tr}(\mathbf{D}^2) = -2\det(\mathbf{D}).$$

Of course, this result is also easy to prove by a direct calculation using the general form

$$\mathbf{D} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

(b) Decomposing $\mathbf{D}$ into its antisymmetric and symmetric parts, $\mathbf{\Omega}$ and $\mathbf{S}$, one has

$$\mathbf{D} = \mathbf{\Omega} + \mathbf{S} = -\frac{1}{2}\omega\,\boldsymbol{\epsilon} + \mathbf{S} .$$

Then

$$\mathrm{tr}(\mathbf{D}^2) = -\frac{1}{2}\omega^2 + S^2$$

using $\boldsymbol{\epsilon}^2 = -\mathbf{I}$ and $\mathrm{tr}(\boldsymbol{\epsilon}\mathbf{S}) = 0$, since $\boldsymbol{\epsilon}$ is antisymmetric and $\mathbf{S}$ symmetric. Together with part (a) this gives

$$\det(\mathbf{D}) = \frac{1}{2}\left(\frac{1}{2}\omega^2 - S^2\right) \qquad (*)$$

Since the eigenvalues of $\mathbf{D}$ are generally of the form $\lambda, -\lambda$ by the traceless condition, $\det(\mathbf{D}) = -\lambda^2$ and is thus positive when $\lambda = i\beta$ is pure imaginary and is negative when $\lambda = \gamma$ is a (positive) real number. From $(*)$ we see that

$$\frac{1}{2}\omega^2 > S^2 \implies \lambda = i\beta \text{ pure imaginary}$$

$$\frac{1}{2}\omega^2 < S^2 \implies \lambda = \gamma \text{ (positive) real}$$

(c) Taking the divergence of the Navier-Stokes equation

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\Delta\mathbf{u}$$

using $\nabla\cdot\mathbf{u} = 0$ gives the standard pressure Poisson equation

$$-\Delta p = u_{i,j}\, u_{j,i} = \mathrm{tr}(\mathbf{D}^2) .$$

Problem 4. (a) By the condition of symmetry ($S_{12} = S_{21}$) and tracelessness ($S_{11} = -S_{22}$) we can write

$$\mathbf{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{12} & -S_{11} \end{pmatrix}.$$

Now represent (S_{11}, S_{12}) in standard polar form as

$$(S_{11}, S_{12}) = \sigma(\cos(2\theta), \sin(2\theta))$$

using $\phi = 2\theta$ as the polar angle and $\sigma = \sqrt{S_{11}^2 + S_{12}^2} = \sqrt{S_{21}^2 + S_{22}^2}$.

Then,

$$\mathbf{S} = \sigma \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}.$$

Using the trigonometric identities

$$\cos(2\theta)\cos\theta + \sin(2\theta)\sin\theta = \cos(2\theta - \theta) = \cos\theta$$

and

$$\sin(2\theta)\cos\theta - \cos(2\theta)\sin\theta = \sin(2\theta - \theta) = \sin\theta$$

it then easy to check that

$$\mathbf{e}_+ = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \mathbf{e}_- = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

are eigenvectors with eigenvalues $+\sigma, -\sigma$, respectively. Note that $\mathbf{e}_+, \mathbf{e}_-$ are the rotations of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, resp. by angle θ.

(b) The matrix scalar product

$$\mathbf{S}_1 : \mathbf{S}_2 = \sum_{ij} (\mathbf{S}_1)_{ij} (\mathbf{S}_2)_{ij}$$

$$= 2\sigma_1\sigma_2 \left[\cos(2\theta_1)\cos(2\theta_2) + \sin(2\theta_1)\sin(2\theta_2) \right]$$

$$= 2\sigma_1\sigma_2 \cos(2(\theta_1 - \theta_2))$$

(c) By direct matrix multiplication

$$\tilde{\mathbf{S}} = \mathbf{S}\boldsymbol{\epsilon} = \sigma \begin{pmatrix} -\sin(2\theta) & \cos(2\theta) \\ \cos(2\theta) & \sin(2\theta) \end{pmatrix} = -\boldsymbol{\epsilon}\mathbf{S}$$

(d) Immediately by its definition

$$\tilde{\mathbf{S}} : \mathbf{S} = \sum_{ij} \tilde{S}_{ij} S_{ij} = \sum_{ijk} S_{ik} \epsilon_{kj} \cdot S_{ij}$$

$$= \sum_{jk} \left(\sum_i S_{ik} S_{ij} \right) \epsilon_{kj} = 0$$

since the first factor is symmetric in jk and the second antisymmetric. Furthermore, we see that the expression for $\mathbf{S}$ in part (a) becomes the expression for $\tilde{\mathbf{S}}$ in part (c) by the transformation $\theta \to \theta + \frac{\pi}{4}$, using

$$\cos\left(2\theta + \frac{\pi}{2}\right) = -\sin(2\theta), \quad \sin\left(2\theta + \frac{\pi}{2}\right) = \cos(2\theta).$$

Thus, the eigenframe of $\tilde{\mathbf{S}}$ is rotated by $+45°$ relative to the eigenframe of $\mathbf{S}$.