Homework # 3

Problem #1 (a) Changing to integration variable $x = kr$,

$$B_2^{\omega}(r) = 2C\eta^{2/3} \int_{k_f r}^{k_{uv} r} J_0(x)\, \frac{dx}{x}$$

$$\cong 2C\eta^{2/3} \int_{k_f r}^{\infty} J_0(x)\, \frac{dx}{x} \qquad \text{for } k_{uv} r \gg 1$$

Now write

$$\int_{k_f r}^{\infty} J_0(x)\, \frac{dx}{x} = \int_1^{\infty} J_0(x)\, \frac{dx}{x} + \int_{k_f r}^{1} \frac{J_0(x) - 1}{x}\, dx + \int_{k_f r}^{1} \frac{dx}{x}$$

$$= \int_1^{\infty} J_0(x)\, \frac{dx}{x} + \int_0^1 \frac{J_0(x) - 1}{x} - \int_0^{k_f r} \frac{J_0(x) - 1}{x}\, dx - \ln(k_f r)$$

$$= (\ln 2 - \gamma) - O((k_f r)^2) - \ln(k_f r)$$

using first Abramowitz & Stegun, 11.1.20 & the expansion $J_0(x) = 1 + O(x^2)$. Hence,

$$B_2^{\omega}(r) \cong 2C\eta^{2/3} \left[\ln 2 - \gamma - \ln(k_f r) + O((k_f r)^2) \right]$$

(b) Substituting the power series

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}$$

into the first expression in part (a),

$$B_2^{\omega}(r) = 2C\eta^{2/3} \int_{k_f r}^{k_{uv} r} \left(\frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \frac{1}{2^{2n}} x^{2n-1} \right) dx$$

$$= 2C\eta^{2/3} \left[\ln\left(\frac{k_{uv}}{k_f}\right) + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2 2^{2n+1} n} \left((k_{uv} r)^{2n} - (k_f r)^{2n} \right) \right]$$

$$= 2C\eta^{2/3} \left[\ln\left(\frac{k_{uv}}{k_f}\right) - \sum_{n=1}^{\infty} b_n (k_{uv} r)^{2n} \right]$$

with

$$b_n = \frac{(-1)^{n+1}}{(n!)^2 2^{2n+1} n} \left[1 - \left(\frac{k_f}{k_{uv}}\right)^{2n} \right]$$

$$\cong \frac{(-1)^{n+1}}{(n!)^2 2^{2n+1} n}$$

for $k_f \ll k_{uv}$.

(c) It follows from (b) that

$$B_2^\omega(0) = 2C\eta^{2/3} \ln\left(\frac{k_{uv}}{k_f}\right),$$

as is easy to see directly from $B_2^\omega(0) = 2C\eta^{2/3} \int_{k_f r}^{k_{uv} r} \frac{dx}{x}$.

Hence, for $k_f \ll r^{-1} \ll k_{uv}$

$$S_2^\omega(r) = 2\left[B_2^\omega(0) - B_2^\omega(r) \right]$$

$$\cong 4C\eta^{2/3} \left[\ln\left(\frac{k_{uv}}{k_f}\right) + \ln(k_f r) \right.$$

$$\left. - (\ln 2 - \gamma) + O((k_f r)^2) \right]$$

using the result of (a). This gives

$$S_2^\omega(r) \cong 4C\eta^{2/3} \left[\ln(k_{uv} r) - \ln 2 + \gamma + O((k_f r)^2) \right]$$

For $r^{-1} \gg k_{uv}$ it is direct from part (b) that

$$S_2^\omega(r) = 2\left[B_2^\omega(0) - B_2^\omega(r) \right]$$

$$= 4C\eta^{2/3} \sum_{n=1}^{\infty} b_n (k_{uv} r)^{2n} \sim \frac{1}{2} C\eta^{2/3} (k_{uv} r)^2$$

Problem # 2. (a) Using the same change of variables $x = kr$ as in Problem 1

$$B_2^u(r) = 2C\eta^{2/3} r^2 \int_{k_f r}^{k_{uv} r} \frac{J_0(x)}{x^3}\,dx$$

$$\cong 2C\eta^{2/3} r^2 \int_{k_f r}^{\infty} \frac{J_0(x)}{x^3}\,dx \qquad \text{for } k_{uv} r \gg 1$$

Using the hint (we can be proved using integration-by-parts and Bessel function relations)

$$\int_{k_f r}^{\infty} \frac{J_0(x)}{x^3}\,dx = \frac{J_0(k_f r)}{2(k_f r)^2} - \frac{J_1(k_f r)}{4(k_f r)} - \frac{1}{4}\int_{k_f r}^{\infty} \frac{J_0(x)}{x}\,dx$$

From Problem 1(a)

$$\int_{k_f r}^{\infty} \frac{J_0(x)}{x}\,dx = (\ln 2 - \gamma) - \ln(k_f r) + O\left((k_f r)^2\right).$$

Hence,

$$B_2^u(r) \cong C\eta^{2/3} r^2 \left[\frac{J_0(k_f r)}{(k_f r)^2} - \frac{J_1(k_f r)}{2k_f r} - \frac{1}{2}(\ln 2 - \gamma) + \frac{1}{2}\ln(k_f r) + O(k_f^2 r^2) \right]$$

Now use the expansions

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)$$

$$J_1(x) = -J_0'(x)$$

$$= \frac{x}{2} - \frac{x^3}{16} + O(x^5)$$

so that

$$\frac{J_0(x)}{x^2} - \frac{J_1(x)}{2x} = \frac{1}{x^2} - \frac{1}{2} + \frac{3}{64}x^2 + O(x^4).$$

Substituting this into the previous result gives w/ $\beta = \frac{1}{2}(1 + \ln 2 - \gamma)$

$$B_2^u(r) \cong C\eta^{2/3}\left[k_f^{-2} - \beta r^2 + \frac{1}{2}r^2\ln(k_f r) + O(k_f^2 r^4)\right]$$

For $r^{-1} \gg k_{ur} \gg k_f$ on the other hand, we can directly substitute

$$\frac{J_0(x)}{x^3} = \frac{1}{x^3} - \frac{1}{4x} + O(x)$$

to obtain

$$B_2^u(r) \cong 2C\eta^{2/3}r^2 \int_{k_f r}^{k_{ur} r} \left(\frac{1}{x^3} - \frac{1}{4x} + O(x)\right)dx$$

(cont'd)

$\Longrightarrow$

$$B_2^u(r) \cong 2C\eta^{2/3} r^2 \left[-\frac{1}{2x^2} - \frac{1}{4}\ln x + O(x^2) \right]_{x=k_f r}^{x=k_{uv} r}$$

$$\cong C\eta^{2/3} \left[k_f^{-2} - k_{uv}^{-2} - \frac{r^2}{2}\ln\left(\frac{k_{uv}}{k_f}\right) + O(k_{uv}^2 r^4) - O(k_f^2 r^4) \right]$$

$$\cong C\eta^{2/3} \left[k_f^{-2} - \frac{r^2}{2}\ln\left(\frac{k_{uv}}{k_f}\right) + O(k_{uv}^2 r^4) \right]$$

for $k_{uv} \gg k_f$.

(b) It can easily be seen directly that

$$B_2^u(0) = 2C\eta^{2/3} \int_{k_f}^{k_{uv}} \frac{dk}{k^3} = C\eta^{2/3}\left(\frac{1}{k_f^2} - \frac{1}{k_{uv}^2}\right) \cong \frac{C\eta^{2/3}}{k_f^2}.$$

This also follows by the final calculation in part (a). Hence, by the formulas in part (a) and the definition

$$S_2^u(r) = 2\left[B_2^u(0) - B_2^u(r) \right]$$

it follows for $k_f \ll r^{-1} \ll k_{uv}$

$$S_2^u(r) \cong 2C\eta^{2/3} \left[\beta r^2 - \frac{1}{2} r^2 \ln(k_f r) + O(k_f^2 r^4) \right]$$

and for $r^{-1} \gg k_{uv} \gg k_f$

$$S_2^u(r) \cong C\eta^{2/3} r^2 \ln\left(\frac{k_{uv}}{k_f}\right) + O(k_{uv}^2 r^4)$$

(c) Note that if $B_2^u(r) = B_2^{(0)} = (\text{const.})$, then clearly

$$S_2^{2,u}(r) = (6+2-8)\, B_2^{(0)} = 0.$$

Also, if $B_2^u(r) = B_2^{(2)} \cdot r^2$ with $B_2^{(2)} = (\text{const.})$, then

$$S_2^{2,u}(r) = \left[6\cdot 0 + 2\cdot 2^2 - 8\cdot 1^2\right] B_2^{(2)} r^2 = 0$$

as well. Because the relationship of $S_2^{2,u}(r)$ to $B_2^u(r)$ is linear, it follows from the result of part (a) for $k_f \ll r^{-1} \ll k_{ur}$ that

$$S_2^{2,u}(r) = C\eta^{2/3}\left[2\cdot\frac{1}{2}(2r)^2\ln(2k_f r)\right.$$
$$\left. - 8\cdot\frac{1}{2}r^2\ln(k_f r) + O(k_f^2 r^4)\right]$$

$$= 4C\eta^{2/3} r^2\left[\ln(2k_f r) - \ln(k_f r)\right] + O(k_f^2 r^4)$$

$$= 4C\eta^{2/3}(\ln 2)\, r^2 + O(k_f^2 r^4).$$

For $r^{-1} \gg k_{ur} \gg k_f$ we can apply the same argument to the formula $B_2^u(r) \cong C\eta^{2/3}\left[k_f^{-2} - \frac{1}{2}r^2\ln\left(\frac{k_{ur}}{k_f}\right) + O(k_{ur}^2 r^4)\right]$ from (a) and the first two terms cancel, leaving

$$S_2^{3,u}(r) = O(k_{ur}^2 r^4).$$