# Inertial Momentum Dissipation for Viscosity Solutions of Euler Equations. I. Flow Around a Smooth Body

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**Abstract** We study the local balance of momentum for weak solutions of incompressible Euler equations obtained from the zero-viscosity limit in the presence of solid boundaries, taking as an example flow around a finite, smooth body. We show that both viscous skin friction and wall pressure exist in the inviscid limit as distributions on the body surface. We define a nonlinear spatial flux of momentum toward the wall for the Euler solution, and show that wall friction and pressure are obtained from this momentum flux in the limit of vanishing distance to the wall, for the wall-parallel and wall-normal components, respectively. We show furthermore that the skin friction describing anomalous momentum transfer to the wall will vanish if velocity and pressure are bounded in a neighborhood of the wall and if also the essential supremum of wall-normal velocity within a small distance of the wall vanishes with this distance (a precise form of the non-flow-through condition). In the latter case, all of the limiting drag arises from pressure forces acting on the body and the pressure at the body surface can be obtained as the limit approaching the wall of the interior pressure for the Euler solution. As one application of this result, we show that Lighthill's theory of vorticity generation at the wall is valid for the Euler solutions obtained in the inviscid limit. Further, in a companion work, we show that the Josephson-Anderson relation for the drag, recently derived for strong Navier-Stokes solutions, is valid for weak Euler solutions obtained in their inviscid limit.

#### 1. Introduction

It was proposed by Taylor as early as 1915 [44] that in turbulent fluid flows interacting with a solid boundary there may be a "finite loss of momentum at the walls due to an infinitesimal viscosity", and he suggested also an analogy with weak solutions of the fluid equations describing shocks. The corresponding

phenomenon of "inertial energy dissipation" has been much investigated since Onsager pointed out the criticality of 1/3 Hölder singularity of the velocity field for such dissipation [38]: see [10, 11, 19, 22] for proofs of the necessity of these singularities and [6,31] for proofs that such dissipative solutions exist. This line of investigation has been recently extended to wall-bounded turbulence by Bardos & Titi [2] and by several following works [3,9,16], which all consider the balance of kinetic energy rather than momentum. However, there is a well-developed phenomenology of spatial "momentum cascade" in wall-bounded turbulent flows, closely analogous to the energy cascade through scales in the bulk of the flow away from solid boundaries [32, 45, 49]. As discussed in our previous work [26], the mathematical methods applied to study Onsager's dissipation anomaly due to energy cascade should apply as well to the spatial momentum cascade.

We make such a study here in the context of flow around a finite solid body with smooth surface, which was the subject of the famous paradox of d'Alembert [12, 13]. The type of situation we consider is illustrated in Fig. 1, which shows a finite body B and the exterior flow domain  $\Omega = \mathbb{R}^3 \setminus B$  on which the incompressible Navier-Stokes equation is assumed to be satisfied

$$\partial_t \mathbf{u}^{\nu} + \boldsymbol{\nabla} \cdot (\mathbf{u}^{\nu} \otimes \mathbf{u}^{\nu} + p^{\nu} \mathbf{I}) - \nu \Delta \mathbf{u}^{\nu} = 0, \quad \boldsymbol{\nabla} \cdot \mathbf{u}^{\nu} = 0, \quad \mathbf{x} \in \Omega$$
(1)

subject to the boundary conditions

$$\mathbf{u}^{\nu}|_{\partial B} = \mathbf{0}, \quad \mathbf{u}^{\nu} \underset{|\mathbf{x}| \to \infty}{\sim} \mathbf{V}.$$
 (2)

Here the pressure  $p^{\nu}$  is to obtained from the Poisson equation with Neumann boundary conditions inherited from the previous equations:

$$- \Delta p^{\nu} = \boldsymbol{\nabla} \otimes \boldsymbol{\nabla} : (\mathbf{u}^{\nu} \otimes \mathbf{u}^{\nu}), \ \mathbf{x} \in \Omega; \quad \frac{\partial p^{\nu}}{\partial n} = \nu \mathbf{n} \cdot \Delta \mathbf{u}^{\nu}, \ \mathbf{x} \in \partial \Omega.$$
(3)

where **n** is the normal vector at the boundary  $\partial B$  directed into the domain  $\Omega$ . We shall assume in this work that  $B \subset \mathbb{R}^3$  is closed, bounded, and connected and that the boundary  $\partial B = \partial \Omega$  is a  $C^{\infty}$  manifold embedded in  $\mathbb{R}^3$ . See [42] for a mathematical treatment of Navier-Stokes solutions in such unbounded domains (and even when the solid boundary is non-smooth) and see [43] and references therein for discussion of the closely related problem of the rigid motion of the solid body *B* through an incompressible fluid filling the complement. We consider this particular situation because of a new mathematical approach to the d'Alembert paradox based on a Josephson-Anderson relation inspired by quantum superfluids [24], which will be the subject of a following paper [41] that builds upon our analysis here. However, our results in this paper apply with minor changes to other flows involving solid walls, including interior flows within bounding walls such as Poiseuille flows through pipes and channels.

Our results and analysis here are modelled closely after those of Duchon & Robert [19], who established a kinetic energy balance distributionally in spacetime for weak solutions of incompressible Euler and Navier-Stokes equations. In particular, under suitable assumptions, [19] showed that the (viscous and inertial) dissipation  $\nu |\nabla \mathbf{u}^{\nu}|^2 + D(\mathbf{u}^{\nu})$  for a sequence of Leray solutions with viscosity tending to zero must converge to a positive distribution (Radon measure) which agrees also with the inertial dissipation  $D(\mathbf{u})$  for weak solutions of



**Figure 1.** Flow around a finite body B in an unbounded region  $\Omega$  filled with an incompressible fluid moving at a velocity **V** at far distances.

Euler equations obtained in the inviscid limit. In order to generalize the Duchon-Robert analysis to obtain a momentum balance distributionally in space-time, we have had to make two key modifications. First, we do not treat admissable Leray weak solutions of the Navier-Stokes equations, but instead assume that all Navier-Stokes solutions are strong. The technical reason for this decision is that our argument requires consideration of the global momentum balance of the Navier-Stokes solution, in which spatial integration by parts yields an integral over  $\partial B$  of the viscous Newtonian stress. However, the known regularity of Leray solutions does not suffice to take the trace of the velocity-gradient at the boundary and thus the validity of the global momentum balance, to our knowledge, remains open for Leray solutions. There seems to be no loss of physical significance of our results by assuming strong solutions, however, since there is no empirical evidence for Leray-type singularities in any known fluid flow. The second and related difference is that our argument involves smearing the Navier-Stokes solutions with elements of an enlarged space of test functions, which need not be compactly supported in the open set  $\Omega$  but which may instead be non-vanishing on  $\partial \Omega$  and have there one-sided derivatives of all orders. A convenient definition of this non-standard class of test functions on  $\bar{\Omega} \times (0,T)$ is as restrictions of standard test functions on  $\mathbb{R}^3 \times (0, T)$ :

$$\bar{D}(\bar{\Omega} \times (0,T)) := \left\{ \varphi = \phi |_{\bar{\Omega} \times (0,T)} : \phi \in C_c^{\infty}(\mathbb{R}^3 \times (0,T)), \\ \operatorname{supp}(\phi) \cap (\Omega \times (0,T)) \neq \emptyset \right\}$$
(4)

This class of test functions is employed precisely to obtain crucial surface contributions from the pressure and Newtonian stress after integration by parts. As an aside, we note that for the initial-value problem the space  $\overline{D}(\overline{\Omega} \times [0,T))$ :=  $\{\varphi = \phi |_{\overline{\Omega} \times [0,T)} : \phi \in C_c^{\infty}(\mathbb{R}^3 \times (-T,T)), \operatorname{supp}(\phi) \cap (\Omega \times (0,T)) \neq \emptyset\}$  could be similarly introduced, requiring slight elaboration of the arguments below.

Our first result is that, under stated assumptions, distributional limits exist as viscosity tends to zero both for the normal stress or pressure and for the tangential Newtonian stress on the body surface, when these are considered as distributional sections of the normal and tangent bundles of the surface, respectively. More precisely, since we consider space-time distributions, we define the manifold  $(\partial B)_T := \partial B \times (0,T) \subset \mathbb{R}^3 \times \mathbb{R}$  with the natural product  $C^{\infty}$ structure and with no boundary, or  $\partial(\partial B)_T = \emptyset$ . Recalling that **n** is the normal vector at  $\partial B$  pointing into  $\Omega$ , we define pressure stress acting on the wall by

$$-p_w^{\nu}\mathbf{n} := -p^{\nu}|_{(\partial B)_T}\mathbf{n} \in D'((\partial B)_T, \mathcal{N}(\partial B)_T)$$
(5)

as a distributional section of the normal bundle  $\mathcal{N}(\partial B)_T$  and wall shear stress

$$\boldsymbol{\tau}_{w}^{\nu} = 2\nu \mathbf{S}^{\nu}|_{(\partial B)_{T}} \cdot \mathbf{n} = \nu \frac{\partial \mathbf{u}}{\partial n} \Big|_{(\partial B)_{T}} = \nu \boldsymbol{\omega}^{\nu}|_{(\partial B)_{T}} \times \mathbf{n} \in D'((\partial B)_{T}, \mathcal{T}(\partial B)_{T})$$
(6)

as a distributional section of the tangent bundle  $\mathcal{T}(\partial B)_T$ . Here we have introduced the strain-rate tensor and the vorticity vector

$$S_{ij}^{\nu} = \frac{1}{2} \left( \frac{\partial u_i^{\nu}}{\partial x_j} + \frac{\partial u_j^{\nu}}{\partial x_i} \right), \quad \boldsymbol{\omega}^{\nu} = \boldsymbol{\nabla} \times \mathbf{u}^{\nu}, \tag{7}$$

and note that the second equality in Eq.(6) is a well-known consequence of the stick b.c. on the velocity field [35]. See section 2 for our notations and conventions on differential geometry.

We then prove the following result:

**Theorem 1.** Let  $(\mathbf{u}^{\nu}, p^{\nu})$  be strong solutions of Navier-Stokes equations (1)-(3) on  $\overline{\Omega} \times (0,T)$  for  $\nu > 0$ . Assume that  $(\mathbf{u}^{\nu})_{\nu>0}$  converges strongly to  $\mathbf{u}$  in  $L^{2}((0,T), L^{2}_{loc}(\Omega)):$ 

$$\mathbf{u}^{\nu} \xrightarrow{\nu \to 0} \mathbf{u}. \tag{8}$$

and that  $(p^{\nu})_{\nu>0}$  converges strongly to p in  $L^1((0,T), L^1_{loc}(\Omega))$ :

$$p^{\nu} \xrightarrow{\nu \to 0} p. \tag{9}$$

Further assume that for some  $\epsilon > 0$  arbitrarily small, with  $\Omega_{\epsilon} := \{ \mathbf{x} \in \Omega :$  $dist(\mathbf{x}, \partial B) < \epsilon\},\$ 

$$\mathbf{u}^{\nu}$$
 uniformly bounded in  $L^{2}((0,T), L^{2}(\Omega_{\epsilon}))$  (10)

$$p^{\nu}$$
 uniformly bounded in  $L^1((0,T), L^1(\Omega_{\epsilon})).$  (11)

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Then, the limit  $(\mathbf{u}, p)$  is a weak Euler solution on  $\Omega \times (0, T)$ , and  $\boldsymbol{\tau}_{w}^{\nu}, p_{w}^{\nu} \mathbf{n}$  have limits as surface distributions, i.e.

$$\boldsymbol{\tau}_{w}^{\nu} \xrightarrow{\nu \to 0} \boldsymbol{\tau}_{w} \text{ in } D'((\partial B)_{T}, \mathcal{T}(\partial B)_{T})$$
 (12)

$$p_w^{\nu} \mathbf{n} \xrightarrow{\nu \to 0} p_w \mathbf{n} \text{ in } D'((\partial B)_T, \mathcal{N}(\partial B)_T)$$
(13)

Remark 1. This theorem is analogous to Proposition 4 of Duchon & Robert [19] who proved that the inviscid limit of the local dissipation in Leray solutions, or  $\lim_{\nu\to 0} [\nu |\nabla \mathbf{u}^{\nu}|^2 + D(\mathbf{u}^{\nu})]$ , exists in the sense of space-time distributions, under similar assumptions as ours. The essential identities (83),(100) employed in our proof have been previously exploited to formulate error estimates for drag and lift forces, for the purpose of adaptive mesh refinement in numerical simulation; see [30], Eq.(25). The assumption (8) on strong  $L^2$  convergence of velocities is motivated by results established and reviewed in [18], which provide physically reasonable conditions for such convergence in the case of interior flows in bounded domains. Our assumptions (10)-(11) on boundedness in a small  $\epsilon$ -neighborhood of the boundary are motivated by the similar assumptions in Theorem 1 of [17], but are much weaker and modelled on our hypotheses (8),(9). The latter do not, of course, imply (10)-(11) because the  $L_{\text{loc}}^{\mathrm{p}}(\Omega)$  conditions in (8),(14) imply boundedness of  $L^{p}(U)$ -norms only for  $U \subset \subset \Omega$ .

Remark 2. The assumption (9) on the pressure is much stronger than required. All that is needed is an hypothesis which guarantees that along a suitable subsequence of  $\nu$ ,  $p^{\nu} \rightarrow p \in L^{1}((0,T), L^{1}_{loc}(\Omega))$  distributionally. For example, it would suffice to replace (9) instead with the following:

$$p^{\nu}$$
 is uniformly bounded in  $L^{q}((0,T), L^{q}_{loc}(\Omega))$ , for some  $q > 1$ . (14)

The assumption (14) means more precisely that there exists an increasing sequence of open sets  $\Omega_k \subset \subset \Omega_{k+1}$  with  $\cup_k \Omega_k = \Omega$  such that for each  $k \geq 1$ 

$$\sup_{\nu>0} \|p^{\nu}\|_{L^q((0,T),L^q(\Omega_k))} < \infty.$$
(15)

Thus, by the Banach-Alaoglu theorem applied iteratively in k, we can find for each k a subsequence  $(\nu^{(k)})$  so that  $p^{\nu_j^{(k)}} \rightarrow p$  weakly in  $L^q((0,T), L^q(\Omega_k))$  as  $j \rightarrow \infty$  and such that  $(\nu^{(k+1)})$  is a further subsequence of  $(\nu^{(k)})$ . In that case, it is easy to see that the diagonal subsequence  $\nu_j^* = \nu_j^{(j)}$  has  $\lim_{j\to\infty} p^{\nu_j^*} = p$ weakly in  $L^q((0,T), L^q(\Omega_k))$  for all  $k \ge 1$ , thus also distributionally, and then  $p \in L^q((0,T), L^q_{loc}(\Omega))$ .

Remark 3. The proof of Theorem 1 is based on the concept of an extension operator for smooth test functions on the boundary into the interior flow domain. To prove (12) we must consider test functions  $\boldsymbol{\psi}$  on  $D'((\partial B)_T, \mathcal{T}^*(\partial B)_T)$ , which are smooth sections of the cotangent bundle, and an extension is then a map  $\mathbf{Ext}: \boldsymbol{\psi} \in D((\partial B)_T, \mathcal{T}^*(\partial B)_T) \mapsto \boldsymbol{\varphi} \in \overline{D}(\overline{\Omega} \times (0,T), \mathbb{R}^3)$  which is linear and continuous in the appropriate sense, with the pointwise equality

$$\boldsymbol{\varphi}|_{(\partial B)_T} = (\mathbf{Proj}_{\mathbf{s}} \circ \boldsymbol{\iota}_T)(\boldsymbol{\psi}) \tag{16}$$

where  $\iota_T$  is the natural inclusion map of the tangent bundle into its ambient Euclidean space:

$$\boldsymbol{\iota}_T: \mathcal{T}(\partial B)_T \to (\mathbb{R}^3 \times \mathbb{R}) \times (\mathbb{R}^3 \times \mathbb{R})$$
(17)

and  $\mathbf{Proj}_{s}$  is the projection onto the spatial vector component

$$\mathbf{Proj}_{s}: (\mathbb{R}^{3} \times \mathbb{R}) \times (\mathbb{R}^{3} \times \mathbb{R}) \to \mathbb{R}^{3}$$
(18)

$$((\mathbf{x},t),(\mathbf{u},v)) \mapsto \mathbf{u}.$$
 (19)

We define similarly the projection  $\operatorname{Proj}_{\mathrm{st}}$  onto the space-time component  $(\mathbf{u}, v)$ . See section 2 where we define the set  $\mathcal{E}_{\mathcal{T}}$  of such extensions and prove that it is non-empty, by constructing an explicit example. Likewise, the proof of (13) requires the definition of a set  $\mathcal{E}_{\mathcal{N}}$  consisting of continuous linear extensions  $\operatorname{Ext} : \boldsymbol{\psi} \in D((\partial B)_T, \mathcal{N}^*(\partial B)_T) \mapsto \boldsymbol{\varphi} \in \overline{D}(\overline{\Omega} \times (0, T), \mathbb{R}^3)$  which satisfy the analogous pointwise equality as (16) for smooth sections of the conormal bundle.

The weak Euler solutions obtained in Theorem 1 are "viscosity solutions" resulting from the inviscid limit. Weak solutions are equivalent to "coarse-grained solutions" in the sense of [16], with slight modifications made due to the presence of boundaries. As in [16], we introduce the spatial coarse-graining operation

$$f \in L^1_{\text{loc}}(\Omega) \mapsto \bar{f}_{\ell}(\mathbf{x}) = \int_{\mathbb{R}^3} G_{\ell}(\mathbf{r}) f(\mathbf{x} + \mathbf{r}) V(d\mathbf{r}), \quad \mathbf{x} \in \Omega^{\ell} := \Omega \setminus \Omega_{\ell}$$
(20)

with  $G_{\ell}(\mathbf{r}) \coloneqq \ell^{-3}G(\mathbf{r}/\ell)$  a standard mollifier, assumed supported on the unit ball for simplicity. To take into account the domain boundary, following [2,17] we introduce a smooth window function  $\theta_{h,\ell} : \mathbb{R} \mapsto [0,1]$ , which is non-decreasing, 0 on  $(-\infty, h]$ , and 1 on  $[h + \ell, \infty)$ , with derivative  $\left\| \theta'_{h,\ell} \right\|_{L^{\infty}(\mathbb{R})} \leq C\ell^{-1}$  for some constant *C* independent of *h* and  $\ell$ . We then denote  $\eta_{h,\ell}(\mathbf{x}) := \theta_{h,\ell}(d(\mathbf{x}))$ , where *d* is the distance function

$$d(\mathbf{x}) := \min_{\mathbf{y} \in \partial B} |\mathbf{x} - \mathbf{y}| \tag{21}$$

noting that for  $\mathbf{x} \in \Omega_{\epsilon}$  with sufficiently small  $\epsilon > 0$ ,  $d(\mathbf{x}) = |\mathbf{x} - \pi(\mathbf{x})|$  for a unique choice  $\pi(\mathbf{x}) \in \partial B$  and  $\nabla d(\mathbf{x}) = \mathbf{n}(\pi(\mathbf{x})) := \mathbf{n}(\mathbf{x})$ . See [2,17] and also section 2. If the Navier-Stokes momentum balance equation (1) is both coarse-grained and windowed, then for  $\ell < h$  it yields:

$$\partial_t (\eta_{h,\ell} \bar{\mathbf{u}}_{\ell}^{\nu}) + \boldsymbol{\nabla} \cdot (\eta_{h,\ell} \bar{\mathbf{T}}_{\ell}^{\nu} + \eta_{h,\ell} \bar{p}_{\ell}^{\nu} \mathbf{I}) = \boldsymbol{\nabla} \eta_{h,\ell} \cdot \bar{\mathbf{T}}_{\ell}^{\nu} + \bar{p}_{\ell}^{\nu} \boldsymbol{\nabla} \eta_{h,\ell} + \nu \eta_{h,\ell} \triangle \bar{\mathbf{u}}_{\ell}^{\nu} \quad (22)$$

where we have introduced the *advective stress tensor*  $\bar{\mathbf{T}}_{\ell}^{\nu} = \overline{\mathbf{u}^{\nu} \otimes \mathbf{u}^{\nu}}$ . The following result describes the inviscid limit:

**Proposition 1.** Assume conditions (8)-(14) as in Theorem 1. Then as  $\nu \to 0$ , the coarse-grained momentum equation (22) converges pointwise for  $\mathbf{x} \in \Omega$  and distributionally for  $t \in [0, T]$  to the following equation,

$$\partial_t(\eta_{h,\ell}\bar{\mathbf{u}}_\ell) + \boldsymbol{\nabla} \cdot (\eta_{h,\ell}\bar{\mathbf{T}}_\ell + \eta_{h,\ell}\bar{p}_\ell \mathbf{I}) = \boldsymbol{\nabla}\eta_{h,\ell} \cdot \bar{\mathbf{T}}_\ell + \bar{p}_\ell \boldsymbol{\nabla}\eta_{h,\ell}.$$
(23)

with  $\bar{\mathbf{T}}_{\ell} = \overline{\mathbf{u} \otimes \mathbf{u}}$  for the limiting Euler solution  $(\mathbf{u}, p)$  in Theorem 1. The set of equations (23) for all  $h > \ell > 0$  are equivalent to the standard weak formulation of the momentum balance for incompressible Euler equations.

The proof of this proposition is straightforward and left to the reader. For the final statement, see [16], Section 2. The importance of the proposition is that it identifies *nonlinear spatial flux of momentum* toward the wall at distance h as

$$-\left(\boldsymbol{\nabla}\eta_{h,\ell}\cdot\bar{\mathbf{T}}_{\ell}+\bar{p}_{\ell}\boldsymbol{\nabla}\eta_{h,\ell}\right)\in D'((0,T),C_c^{\infty}(\Omega)),\tag{24}$$

where recall that  $\nabla \eta_{h,\ell} = \eta'_{h,\ell}(d(\mathbf{x}))\mathbf{n}(\pi(\mathbf{x}))$ , when h is sufficiently small.

Our next main theorem states that this spatial flux of momentum (both its components wall-parallel and wall-normal) matches onto the corresponding components of the limiting wall stress which were established in Theorem 1. Since those inviscid limits were defined as sectional distributions of the tangent and normal bundles, we must identify momentum flux (24) with similar sectional distributions. To accomplish this, we use the idea of extensions in Theorem 1 to define e.g.  $\mathbf{Ext}^*(\nabla \eta_{h,\ell} \cdot \bar{\mathbf{T}}_{\ell} + \bar{p}_{\ell} \nabla \eta_{h,\ell}) \in D'((\partial B)_T, \mathcal{T}(\partial B)_T)$  with  $\mathbf{Ext} \in \mathcal{E}_T$  as

$$\langle \mathbf{Ext}^* ( \mathbf{
abla} \eta_{h,\ell} \cdot \bar{\mathbf{T}}_\ell + ar{p}_\ell \mathbf{
abla} \eta_{h,\ell} ), oldsymbol{\psi} 
angle = \langle \mathbf{
abla} \eta_{h,\ell} \cdot \bar{\mathbf{T}}_\ell + ar{p}_\ell \mathbf{
abla} \eta_{h,\ell}, \mathbf{Ext}(oldsymbol{\psi}) 
angle$$

for all  $\psi \in D((\partial B)_T, \mathcal{T}^*(\partial B)_T)$ . The righthand side is meaningful and defines a sectional distribution of the tangent bundle because of regularity (24) and linearity and continuity of  $\mathbf{Ext} \in \mathcal{E}_T$ . Likewise, with  $\mathbf{Ext} \in \mathcal{E}_N$  one can define  $\operatorname{Ext}^*(\nabla \eta_{h,\ell} \cdot \overline{\mathbf{T}}_{\ell} + \overline{p}_{\ell} \nabla \eta_{h,\ell}) \in D'((\partial B)_T, \mathcal{N}(\partial B)_T).$  For details, see Section 2.

We then have:

**Theorem 2.** Under the assumptions (8)-(11) of Theorem 1, then for  $0 < \ell < h$ and for all  $Ext \in \mathcal{E}_T$ 

$$-\lim_{h,\ell\to 0} \boldsymbol{Ext}^*(\boldsymbol{\nabla}\eta_{h,\ell}\cdot\bar{\mathbf{T}}_{\ell}+\bar{p}_{\ell}\boldsymbol{\nabla}\eta_{h,\ell})=\boldsymbol{\tau}_w \text{ in } D'((\partial B)_T,\mathcal{T}(\partial B)_T)$$
(25)

Likewise, for  $0 < \ell < h$  and for all  $Ext \in \mathcal{E}_N$ 

$$-\lim_{h,\ell\to 0} \boldsymbol{Ext}^*(\boldsymbol{\nabla}\eta_{h,\ell}\cdot\bar{\mathbf{T}}_{\ell}+\bar{p}_{\ell}\boldsymbol{\nabla}\eta_{h,\ell})=-p_w\mathbf{n} \text{ in } D'((\partial B)_T,\mathcal{N}(\partial B)_T).$$
(26)

*Remark* 4. This result is analogous to the second part of Proposition 4 of Duchon & Robert [19], stating not only that  $D(\mathbf{u}) = \lim_{\nu \to 0} [\nu |\nabla \mathbf{u}^{\nu}|^2 + D(\mathbf{u}^{\nu})]$  exists but also that it coincides with the "inertial energy dissipation" of [19], Proposition 2, which defines it as a distributional limit of energy flux to vanishingly small length scales,  $D(\mathbf{u}) = \lim_{\ell \to 0} D_{\ell}(\mathbf{u})$ . In fact, our proof of Theorem 2 is a direct adaptation of the proof in [19].

*Remark 5.* It is not geometrically natural that pressure stress should contribute to the cascade of wall-parallel momentum, as it apparently does in (25). In fact, as previously noted,  $\nabla \eta_{h,\ell} = \theta'_{h,\ell} \mathbf{n}$  for sufficiently small h, and the term  $\bar{p}_{\ell} \nabla \eta_{h,\ell}$ should give vanishing contribution in the tangent bundle. This can be shown if we define a class of *natural extensions*  $\mathcal{E}_{\mathcal{T}}$  which consists of those  $\mathbf{Ext} \in \mathcal{E}_{\mathcal{T}}$  such that  $\forall \boldsymbol{\psi} \in D((\partial B)_T, \mathcal{T}^*(\partial B)_T), \boldsymbol{\varphi} = \mathbf{Ext}(\boldsymbol{\psi})$  satisfies

$$\|\boldsymbol{\varphi} \cdot \mathbf{n}\|_{L^{\infty}((\Omega_{h+\ell} \setminus \Omega_h) \times (0,T))} \le C\ell \tag{27}$$

(possibly with  $\ell/h$  bounded from below) for constant C independent of  $h, \ell$ . We show in Section 2 that  $\tilde{\mathcal{E}}_{\mathcal{T}} \neq \emptyset$  by explicit construction. We then obtain from the preceding theorem the following simple corollary:

**Corollary 1.** For  $\mathbf{Ext} \in \tilde{\mathcal{E}}_{\mathcal{T}}$ , then under the assumption (11) of Theorem 1,  $\lim_{h,\ell\to 0} \mathbf{Ext}^*(\bar{p}_\ell \nabla \eta_{h,\ell}) = 0$ . Thus, under all of the assumptions (8)-(11) of Theorem 1,

$$-\lim_{h,\ell\to 0} \boldsymbol{Ext}^*(\boldsymbol{\nabla}\eta_{h,\ell}\cdot\bar{\mathbf{T}}_\ell) = \boldsymbol{\tau}_w \text{ in } D'((\partial B)_T, \mathcal{T}(\partial B)_T)$$
(28)

for any  $Ext \in \tilde{\mathcal{E}}_{\mathcal{T}}$ .

Finally, we establish sufficient conditions for vanishing cascade of momentum to the wall via spatial advection:

**Proposition 2.** Assume that  $\mathbf{u} \in L^2((0,T), L^2_{loc}(\Omega))$  so that  $\mathbf{T}_{\ell} = \overline{\mathbf{u} \otimes \mathbf{u}}$  is well-defined. Assume further for some  $\epsilon > 0$  the boundedness property in the vicinity of the wall

$$\mathbf{u} \in L^2((0,T), L^\infty(\Omega_\epsilon)) \tag{29}$$

and the no-flow-through condition at the boundary in the sense

$$\lim_{\delta \to 0} \|\mathbf{n} \cdot \mathbf{u}\|_{L^2((0,T), L^\infty(\Omega_\delta))} = 0.$$
(30)

Then, for all  $\mathbf{Ext} \in \mathcal{E}_{\mathcal{T}}$ ,

$$\lim_{h,\ell\to 0} \operatorname{\textit{Ext}}^*(\nabla\eta_{h,\ell}\cdot\bar{\mathbf{T}}_\ell) = \mathbf{0} \text{ in } D'((\partial B)_T, \mathcal{T}(\partial B)_T)$$
(31)

and for all  $Ext \in \mathcal{E}_N$ ,

$$\lim_{h,\ell\to 0} \operatorname{Ext}^*(\nabla\eta_{h,\ell}\cdot\bar{\mathbf{T}}_\ell) = \mathbf{0}, \text{ in } D'((\partial B)_T, \mathcal{N}(\partial B)_T).$$
(32)

Remark 6. This result can be regarded as an analogue of Duchon & Robert, [19] Proposition 3, which showed that  $\lim_{\ell \to 0} D_{\ell}(\mathbf{u}) = 0$  when the velocity field satisfies a regularity condition slightly stronger than  $\mathbf{u} \in L^3((0,T), B_3^{1/3,\infty}(\Omega))$ . Our assumption (30) can be regarded as a corresponding assumption on continuity of the normal velocity at the wall, the importance of which has been recognized in prior work: see Remark 3.2 in [2], assumption 1, Eq.(4.3b) of Theorem 4.1 in [3], and assumption (12) of Theorem 1 in [17]. Our near-wall boundedness assumption (29) is likewise motivated by assumption (11) of Theorem 1 in [17], but requiring only  $L^2$  rather than  $L^3$  sense in time.

Combining Proposition 2 with Theorems 1 & 2, and Corollary 1 yields our main result:

**Theorem 3.** Make all of the assumptions (8)-(11) of Theorem 1, and assume further that the limiting weak Euler solution  $(\mathbf{u}, p)$  in that theorem satisfies the near-wall boundedness (29) and no-flow-though condition (30) in Proposition 2. Then, for all  $\mathbf{Ext} \in \tilde{\mathcal{E}}_{\tau}$ ,

$$-\lim_{h,\ell\to 0} \boldsymbol{E}\boldsymbol{x}\boldsymbol{t}^*(\boldsymbol{\nabla}\eta_{h,\ell}\cdot\bar{\mathbf{T}}_\ell) = \boldsymbol{\tau}_w = \boldsymbol{0} \text{ in } D'((\partial B)_T, \mathcal{T}(\partial B)_T)$$
(33)

and for all  $Ext \in \mathcal{E}_N$ ,

$$-\lim_{h,\ell\to 0} \operatorname{Ext}^*(\bar{p}_{\ell} \nabla \eta_{h,\ell}) = -p_w \mathbf{n}, \text{ in } D'((\partial B)_T, \mathcal{N}(\partial B)_T).$$
(34)

where the distributions  $\boldsymbol{\tau}_w \in D'((\partial B)_T, \mathcal{T}(\partial B)_T)$ ,  $p_w \mathbf{n} \in D'((\partial B)_T, \mathcal{N}(\partial B)_T)$ are those obtained in Theorem 1. Remark 7. The result (33) implies that Taylor's conservation anomaly for tangential momentum, under the stated hypotheses, can be only a "weak anomaly". Here we employ the terminology from [5] (also [26]) according to which  $\tau_w^{\nu}$  is "weakly anomalous" if it vanishes as  $Re \to \infty$ , but more slowly than it does for laminar flow where  $\tau_w^{\nu} \propto 1/Re$ . Such a weak anomaly for tangential momentum conservation would imply that all drag in the inviscid limit arises from the "form drag" due to pressure stress (34) acting in the direction of the external flow **V**.

There is a great deal of empirical evidence from experiments and numerical simulations which supports this picture. For example, in the experimental study [1] for high-Reynolds flow around a smooth sphere,  $\tau_w^{\nu} \propto Re^{-1/2}$  in the front of the sphere, consistent with the boundary-layer theory of Prandtl [20,35,39], and vanishes a bit slower in the turbulent wake region after flow separation behind the sphere (see [1],Fig.7(a)). The form drag from pressure stress thus becomes becomes dominant for very large Reynolds numbers (see [1], Fig.10). For flow through a straight, smooth-walled pipe, as reviewed in [26], geometry does not permit wall pressure stress to act parallel to the mean flow direction and drag vanishes as  $Re \to \infty$ . If, instead, the pipe walls are mathematically smooth but "hydraullically rough", then form drag is again geometrically possible and it becomes dominant over the contribution from  $\tau_w^{\nu}$  in the large-Re limit; e.g. see [7], Fig.10. For related evidence in many other flows, see [8, 24].

The only possible exception of which we are aware comes from a 2D numerical simulation of a vortex quadrupole impinging on a flat wall [37]. Evidence was presented in [37], Figure 12, that the maximum vorticity at the wall in that flow scales  $\sim Re$ , which would imply  $\tau_w \neq \mathbf{0}$  at least at one point. It is possible that our strong non-flow-through assumption (30) is invalid in this flow, since reference [37] reports "a blow-up of the wall-normal velocity associated with an abrupt acceleration of fluid particles away from the wall," corresponding to explosive boundary-layer separation. Another possible reconciliation of our Theorem 3 with the numerical observations of [37] is that the nonzero  $\tau_w$  values reported may occur at only a zero-measure set of points of  $\partial \Omega$ , so that still  $\tau_w = \mathbf{0}$  in the sense of distributions and  $\lim_{\delta \to 0} \text{ess.sup}_{\mathbf{x} \in \Omega_{\delta}} |\mathbf{u}(\mathbf{x})| = 0$ .

Remark 8. On the other hand, the assumptions (29), (30) invoked in Theorem 3 imply the strong-weak uniqueness property for the resulting viscosity solutions of Euler equations, e.g. see [48]. (We thank T. Drivas for insisting on this fact.) This result is immediate when the flow domain  $\Omega$  is a bounded open set with  $C^{\infty}$ boundary  $\partial \Omega$  and if there is an incompressible Euler solution  $\mathbf{U} \in C^{\infty}(\Omega \times [0,T))$ which satisfies  $\mathbf{U} \cdot \mathbf{n} = 0$  everywhere on the boundary. In that case, we may consider  $\mathbf{U}$  as an extension  $\varphi$  into  $\Omega$  of a smooth section of the surface cotangent bundle and from the proof of Theorem 3 we obtain that the limiting viscosity solution  $\mathbf{u}$  must satisfy for a.e.  $\tau \in (0,T)$ 

$$\int_{\Omega} \left[ \mathbf{u}(\cdot, \tau) \cdot \mathbf{U}(\cdot, \tau) - \mathbf{u}_0 \cdot \mathbf{U}_0 \right] dV = \int_0^{\tau} \int_{\Omega} \left[ \partial_t \mathbf{U} \cdot \mathbf{u} + \nabla \mathbf{U} : \mathbf{u} \otimes \mathbf{u} \right] dV dt.$$
(35)

Strong-weak uniqueness for the admissable weak solution **u** then follows by a remark of E. Feireisl recorded in [48], section 5. This argument may not apply if  $\mathbf{U} \cdot \mathbf{n} \neq 0$  on part of the boundary (as for open flows through pipes), since the above equation then gets a surface contribution from the pressure p of the weak solution. This argument also does not apply for flow around a smooth finite

body *B* as discussed in the present paper, because the smooth Euler solution **U** will not generally be compactly supported in  $\Omega$  and cannot be regarded as a smooth extension. However, we shall see in our companion paper [40] that strong-weak uniqueness nevertheless holds by a relative energy argument when **U** is the potential Euler solution of d'Alembert and when assumptions (29), (30) of Theorem 3 hold  $L^3$ -in-time. In particular, if initial data  $\mathbf{u}_0^r$  for the Navier-Stokes solution converge to  $\mathbf{U}_0$  strong in  $L^2(\Omega)$  (allowing a vanishing boundary layer to enforce stick conditions at the surface), then the limiting weak Euler solution **u** must coincide with **U**, unless the conditions (29), (30) are violated. It should be emphasized that, in fact, it is the consequence  $\boldsymbol{\tau}_w = \mathbf{0}$  of Theorem 3 which implies strong-weak uniqueness for viscosity weak solutions, even if  $\boldsymbol{\tau}_w = \mathbf{0}$  follows from assumptions weaker than (29), (30). Since  $\boldsymbol{\tau}_w^{\nu} = \nu \boldsymbol{\omega}_w^{\nu} \times \mathbf{n}$ , a thin enough boundary layer in the initial data  $\mathbf{u}_0^r$  may correspond to  $\boldsymbol{\tau}_w^{\nu} \sim O(1)$  in the surface vortex sheet and subsequent explosive separation of such a boundary layer may violate our assumptions (29), (30) at early times.

*Remark 9.* The result (34) of Theorem 3 is a statement that pressure is continuous at the wall in the inviscid limit, in the sense that the limit of zero distance to the wall and the limit of infinite Reynolds-number commute with each other. Such continuity helps to justify one of the fundamental assumptions in the theory of Prandtl [20, 35, 39], which posited that pressure would be continuous across thin viscous boundary layers at solid walls.

This result has further important implications for turbulence modelling, because it suggests that the asymptotic drag arising from pressure forces might be calculated from Euler solutions in the fluid interior which arise from the infinite-Re limit [18], without the need to resolve small viscous lengths at the wall. To obtain the pressure field p from the Euler solution velocity field **u** involves the solution of a Poisson equation analogous to Eq.(3), and this requires suitable boundary conditions on the pressure. For smooth Euler solutions, the following Neumann problem is generally solved:

$$- \Delta p = \boldsymbol{\nabla} \otimes \boldsymbol{\nabla} : (\mathbf{u} \otimes \mathbf{u}), \ \mathbf{x} \in \Omega; \quad \frac{\partial p}{\partial n} = (\mathbf{u} \otimes \mathbf{u}) : \boldsymbol{\nabla} \mathbf{n}, \ \mathbf{x} \in \partial \Omega,$$
(36)

where the latter condition arises from the normal component of the Euler equation at the wall, assuming  $\mathbf{u} \cdot \mathbf{n} = 0$ . Recently, in an interesting work [14] (following [4]) it has been shown, assuming a weak Euler solution in a bounded domain  $\Omega$  with velocity  $\mathbf{u} \in C^{\alpha}(\Omega)$ ,  $\alpha \in (0, 1)$  and  $\mathbf{u} \cdot \mathbf{n} = 0$  on  $\partial \Omega$ , that the pressure pmust satisfy the Neumann problem (36) in the weak form

$$\int_{\Omega} [p \triangle \varphi + \mathbf{u} \otimes \mathbf{u} : (\nabla \otimes \nabla) \varphi] \, dV = \int_{\partial \Omega} p \frac{\partial \varphi}{\partial n} \, dA, \quad \forall \varphi \in C^2(\bar{\Omega}) \tag{37}$$

and, furthermore, that there is a unique weak solution of (37) with zero spacemean which is at least  $C^{\alpha}$  up to the boundary. This result offers hope that the drag on the body in the infinite Reynolds limit can be computed entirely from the limiting weak Euler solution.

Remark 10. The methods of this paper can be applied to another fundamental cascade process in wall-bounded turbulence, which is the "inverse cascade" of vorticity away from the wall; e.g. see [23, 24]. This topic will be discussed in

detail in another work [25]. Here we just note a key result for inviscid-limit Euler solutions which follows directly from the considerations in the present paper: with the assumptions of Theorem 3, then for all  $\mathbf{Ext} \in \mathcal{E}_{\mathcal{T}}$ ,

$$\lim_{h,\ell\to 0} \mathbf{Ext}^* \left[ \nabla \eta_{\ell,h} \times \partial_t \bar{\mathbf{u}}_\ell + \nabla \eta_{h,\ell} \times (\nabla \cdot \bar{\mathbf{T}}_\ell) \right] = -(\mathbf{n} \times \nabla) p_w.$$
(38)

The quantity on the righthand side of this equation is the Lighthill vorticity source [24, 35, 36], which describes the rate of generation of tangential vorticity due to pressure gradients at the body surface. The term involving  $\bar{\mathbf{T}}_{\ell}$  on the lefthand side represents a spatial flux of vorticity away from the solid surface; see [25]. One might naively expect the Lighthill source to be in balance with this vorticity flux into the flow interior  $\Omega$ . However, the time-derivative term has also a simple physical interpretation, representing the rate of change of a tangential vortex sheet of strength  $\mathbf{n} \times \mathbf{u}$  at the body surface  $\partial \Omega$  [25]. The meaning of (38) is thus that vorticity generated at the surface by pressure gradients is either cascaded into the flow interior or else accumulates in the surface vortex sheet.

It is worth sketching here briefly the derivation of this important result. For any  $\boldsymbol{\psi} \in D((\partial B)_T, \mathcal{T}^*(\partial B)_T)$ , let  $\boldsymbol{\varphi} = \mathbf{Ext}(\boldsymbol{\psi})$ . Then it is not hard to show that  $((\mathbf{n} \times \boldsymbol{\nabla}) \cdot \boldsymbol{\psi})\mathbf{n} \in D((\partial B)_T, \mathcal{N}^*(\partial B)_T)$  and that  $(\mathbf{n} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\varphi}))\mathbf{n} \in \overline{D}(\overline{\Omega} \times (0,T), \mathbb{R}^3)$  extends this test section into the interior. Since

$$-\langle (\mathbf{n} \times \boldsymbol{\nabla}) p_w, \boldsymbol{\psi} \rangle = \langle p_w \mathbf{n}, ((\mathbf{n} \times \boldsymbol{\nabla}) \cdot \boldsymbol{\psi}) \mathbf{n} \rangle$$
(39)

we obtain from (34) in Theorem 3 that

$$-\langle (\mathbf{n} \times \boldsymbol{\nabla}) p_w, \boldsymbol{\psi} \rangle = \lim_{h, \ell \to 0} \int_0^T \int_{\Omega} (\boldsymbol{\nabla} \times \boldsymbol{\varphi}) \cdot \boldsymbol{\nabla} \eta_{h,\ell} \, \bar{p}_\ell \, dV \, dt.$$
(40)

On the other hand,

$$\int_{0}^{T} \int_{\Omega} (\boldsymbol{\nabla} \times \boldsymbol{\varphi}) \cdot \boldsymbol{\nabla} \eta_{h,\ell} \, \bar{p}_{\ell} \, dV \, dt = -\int_{0}^{T} \int_{\Omega} \boldsymbol{\varphi} \cdot \boldsymbol{\nabla} \eta_{h,\ell} \times \boldsymbol{\nabla} \bar{p}_{\ell} \, dV \, dt$$
$$= \int_{0}^{T} \int_{\Omega} \left[ -(\partial_{t} \boldsymbol{\varphi}) \cdot \boldsymbol{\nabla} \eta_{h,\ell} \times \bar{\mathbf{u}}_{\ell} + \boldsymbol{\varphi} \cdot \boldsymbol{\nabla} \eta_{h,\ell} \times (\boldsymbol{\nabla} \bar{\mathbf{T}}_{\ell}) \right] \, dV \, dt \tag{41}$$

where in the final line we used the coarse-grained momentum balance (23). Combining the two results (40),(41) yields exactly (38), thus showing that the Lighthill theory of vorticity generation is valid even in the infinite Reynolds-number limit. The inviscid nature of vorticity production by tangential pressure gradients was already emphasized by Morton [36].

Remark 11. A further application of the results of this work is given in the companion paper [41], where the infinite-Reynolds limit will be established for the Josephson-Anderson relation, which precisely relates vorticity flux from the body to drag [24]. That relation decomposes the velocity into a contribution  $\mathbf{u}_{\phi} = \nabla \phi$  from the smooth, potential Euler solution studied by d'Alembert [12,12] and the complementary contribution  $\mathbf{u}_{\omega}^{\nu} = \mathbf{u}^{\nu} - \mathbf{u}_{\phi}$  which represents the rotational fluid motions. Most importantly, this field satisfies an equation for conservation of "rotational momentum"

$$\partial_{t}\mathbf{u}_{\omega}^{\nu} + \nabla \cdot (\mathbf{u}_{\omega}^{\nu} \otimes \mathbf{u}_{\omega}^{\nu} + \mathbf{u}_{\omega}^{\nu} \otimes \mathbf{u}_{\phi} + \mathbf{u}_{\phi} \otimes \mathbf{u}_{\omega}^{\nu} + p_{\omega}^{\nu}\mathbf{I}) - \nu \bigtriangleup \mathbf{u}_{\omega}^{\nu} = \mathbf{0}, \quad \nabla \cdot \mathbf{u}_{\omega}^{\nu} = 0, \quad \mathbf{x} \in \Omega$$
(42)

subject to the boundary conditions

$$\mathbf{u}_{\omega}^{\nu}|_{\partial B} = -\mathbf{u}_{\phi}|_{\partial B}, \quad \mathbf{u}_{\omega}^{\nu} \underset{|\mathbf{x}| \to \infty}{\sim} \mathbf{0}.$$

$$\tag{43}$$

and with the pressure  $p_{\omega}^{\nu}$  determined by the incompressibility constraint. Of course, Eqs.(42),(43) are equivalent to the incompressible Navier-Stokes equations in their standard representation, Eqs.(1),(2). Because the equations (42) are conservation-type, they have a weak formulation and therefore all of the results of the present work are valid also for Eqs.(42),(43) and, in particular, the Theorems 1-3. Note in this context that the weak Euler solutions obtained in the inviscid limit satisfy in distributional sense the equations

$$\partial_t \mathbf{u}_\omega + \boldsymbol{\nabla} \cdot (\mathbf{u}_\omega \otimes \mathbf{u}_\omega + \mathbf{u}_\omega \otimes \mathbf{u}_\phi + \mathbf{u}_\phi \otimes \mathbf{u}_\omega + p_\omega \mathbf{I}) = \mathbf{0}, \quad \boldsymbol{\nabla} \cdot \mathbf{u}_\omega = 0. \quad \mathbf{x} \in \Omega$$
(44)

The resulting weak solutions  $\mathbf{u} = \mathbf{u}_{\phi} + \mathbf{u}_{\omega}$  of incompressible Euler equations in their standard form differ from the potential solution  $\mathbf{u}_{\phi}$  of d'Alembert, with non-vanishing vorticity corresponding to the rotational flow  $\mathbf{u}_{\omega}$  in the turbulent wake behind the solid body.

### 2. Preliminaries

In this section, we summarize our notations and conventions on differential geometry and introduce the concept of extensions that we employ in our proofs.

2.1. Manifolds and Vector Bundles Associated to a Smooth Body. We consider a body B that is a connected, compact domain in  $\mathbb{R}^3$ , with  $\Omega = \mathbb{R}^3 \setminus B$  also connected, and with common  $C^{\infty}$  boundary  $\partial B = \partial \Omega$ . The boundary  $\partial B$  is then a connected compact  $C^{\infty}$  hypersurface in  $\mathbb{R}^3$ , which is thus a level set of a  $C^{\infty}$  function  $f : B \to [0, \infty)$ . That is,  $\partial B = f^{-1}(0)$  and  $\nabla f(\mathbf{x}) \neq \mathbf{0}$  for all  $\mathbf{x} \in \partial B$ . By the Regular Level Set Theorem ([33], Corollary 5.14) the tangent space at any  $\mathbf{x} \in \partial B$  is given by

$$\mathcal{T}_{\mathbf{x}}\partial B = \ker(\nabla f(\mathbf{x})) = (\nabla f(\mathbf{x})\mathbb{R})^{\perp}.$$
(45)

Furthermore, the vector field

$$\mathbf{n}(\mathbf{x}) = \frac{\nabla f(\mathbf{x})}{|\nabla f(\mathbf{x})|} \tag{46}$$

defines a unit normal vector of  $\partial B$ , and **n** is also smooth on  $\partial B$  by definition. See chapter 5 in [33] for more details on submanifolds with a boundary.

Since  $\partial \Omega$  is a compact  $C^{\infty}$  submanifold of  $\Omega$ , there exists  $\eta(\Omega) > 0$  such that  $\Omega_{\epsilon}$  for any  $\epsilon < \eta(\Omega)$  is a neighborhood of  $\partial B \subset \Omega$  with the unique nearest point property: for any  $\mathbf{x} \in \overline{\Omega}_{\epsilon}$ , there exists a unique point  $\pi(\mathbf{x}) \in \partial B$  such that  $\operatorname{dist}(\mathbf{x}, \partial B) = |\mathbf{x} - \pi(\mathbf{x})|$ . The map  $\pi : \overline{\Omega}_{\epsilon} \to \partial B$  is called the projection onto  $\partial B$ . One can show this projection map  $\pi$  is  $C^{\infty}$  using the Tubular Neighborhood Theorem. See chapter 6 in [33], and [27,34] for more details. Thus the distance function  $d: \overline{\Omega}_{\epsilon} \to \mathbb{R}$  is a smooth function in  $C^{\infty}(\overline{\Omega}_{\epsilon})$ , and

$$d(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, \partial B) = |\mathbf{x} - \pi(\mathbf{x})|, \quad \nabla d(\mathbf{x}) = \mathbf{n}(\pi(\mathbf{x}))$$
(47)

The latter result follows by using appropriate local coordinates: see [29]. p.9. Finally, we observe that  $\partial B$  is naturally Riemannian, with metric induced by the embedding in Euclidean space.

We need to consider also additional manifolds associated with B. The first is the space-time manifold  $(\partial B)_T := \partial B \times (0,T)$  with the product differentiable structure, so that  $\partial(\partial B)_T = \emptyset$ . Since  $(\partial B)_T$  is a closed smooth hypersurface in  $\mathbb{R}^3 \times \mathbb{R}$ , it is orientable and Riemannian. We consider also the associated *tangent bundle*  $\mathcal{T}(\partial B)_T$  ([15], 16.15.4; [46], section 15.6; [33], Proposition 3.18). As  $(\partial B)_T$  is an embedded submanifold of  $\mathbb{R}^3 \times \mathbb{R}$ ,  $\mathcal{T}(\partial B)_T \subset (\mathbb{R}^3 \times \mathbb{R}) \times (\mathbb{R}^3 \times \mathbb{R})$ We can describe the tangent space  $T_{(\mathbf{x},t)}(\partial B)_T \cong \mathcal{T}_{\mathbf{x}}\partial B \times T_t(0,T)$  embedded in  $\mathbb{R}^3 \times \mathbb{R}$ . We use  $\boldsymbol{\iota}_T$  to denote the natural inclusion map of the tangent bundle into its ambient Euclidean space:

$$\boldsymbol{\iota}_T: \mathcal{T}(\partial B)_T \to (\mathbb{R}^3 \times \mathbb{R}) \times (\mathbb{R}^3 \times \mathbb{R}).$$
(48)

Finally, we need the normal bundle  $\mathcal{N}(\partial B)_T$  ([46], section 15.6; [33], Proposition 13.21), and we can take the normal space  $N_{(\mathbf{x},t)}(\partial B)_T \cong \mathcal{N}_{\mathbf{x}}\partial B \times \{0\}$  embedded in  $\mathbb{R}^3 \times \mathbb{R}$ . We use  $\boldsymbol{\iota}_N$  to denote the natural inclusion map of the normal bundle into its ambient Euclidean space:

$$\boldsymbol{\iota}_N: \mathcal{N}(\partial B)_T \to (\mathbb{R}^3 \times \mathbb{R}) \times (\mathbb{R}^3 \times \mathbb{R}).$$
(49)

Because  $(\partial B)_T$  is orientable, the normal bundle  $\mathcal{N}(\partial B)_T$  is trivial ([33],Exercise 15.8) and every smooth section  $\sigma : (\partial B)_T \to \mathcal{N}(\partial B)_T$  can be identified with the map  $(\mathbf{x}, t) \mapsto (\mathbf{x}, t, \sigma(\mathbf{x}, t)\mathbf{n}(\mathbf{x}), 0)$  for a smooth function  $\sigma : (\partial B)_T \to \mathbb{R}$ .

2.2. Distributions on Manifolds. The results on distributions that we require in this paper follow as a special case of general theory for a  $C^{\infty}$  manifold Xof dimension n, and a rank k vector bundle  $(E, \Pi, X)$  of X. Let  $\bigcup_{i \in I} (V_i, \Phi_i)$ ,  $V_i \subset X, \Phi_i : \Pi^{-1}(V_i) \to \mathbb{R}^n \times \mathbb{R}^k$  be a smooth structure of E, and  $\bigcup_{i \in I} (V_i, \phi_i)$ ,  $\phi_i : V_i \to \mathbb{R}^n$  be a corresponding smooth structure on X with  $\Pi_1 \Phi_i = \phi_i \Pi$ . Here  $\Pi_1$  projects onto the first factor of  $\mathbb{R}^n \times \mathbb{R}^k$  and  $\Pi_2$  onto the second. We shall denote by D(X, E) the space of smooth sections with compact support, which is a Fréchet space with the seminorms defined by

$$p_{s,m,i}(\psi) := \sum_{j=1}^{k} \tilde{p}_{s,m,i}((\Pi_2 \Phi_i)^j \circ \psi|_{V_i} \circ \phi_i^{-1})$$
(50)

where  $\psi \in D(X, E)$  and the  $\tilde{p}_{s,m,i}$ 's are a countable and separating basis of seminorms on  $C^{\infty}(\phi_i(V_i))$  defined by

$$\tilde{p}_{s,m,i}(f) = \sup_{x \in K_m^{(i)}, |\alpha| \le s} |D^{\alpha} f(x)|$$
(51)

for  $f \in C^{\infty}(\phi_i(V_i))$ . Here, *m* is the index of a fundamental increasing sequence  $(K_m^{(i)})$  of compact subsets of  $\phi_i(V_i)$ . For further details, see [15], Chapter XVII. Then, one can define the space of *distributional sections* by

$$D'(X,E) := D(X,E^* \otimes \Lambda^n(X))'$$
(52)

Here,  $E^*$  is the dual bundle of E and  $\hat{A}^n(X)$  denotes the bundle of densities on X. For these standard notions, see e.g. [28, 47]. One can embed D(X, E) into D'(X, E) by

$$D(X,E) \hookrightarrow D'(X,E) : \psi \mapsto T_{\psi}, \ \langle T_{\psi}, f \rangle := \int_X \operatorname{trace}(\psi \otimes f)$$
(53)

where  $\operatorname{trace}(\psi \otimes f) \in L^1_{\operatorname{loc}}(X, \hat{A}^n(X))$  defines an integrable Radon measure on X, for any  $\psi \in D(X, E)$  and  $f \in D(X, E^* \otimes \hat{A}^n(X))$ . We now specialize these results for general vector bundles to the cases of interest.

Let  $D((\partial B)_T; \mathcal{T}^*(\partial B)_T)$  denote the space of smooth sections with compact support of the cotangent bundle  $\mathcal{T}^*(\partial B)_T$ . Note that the tangent spaces are finite-dimensional at each  $(\mathbf{x}, t) \in (\partial B)_T$  and thus  $\mathcal{T}^*(\partial B)_T \simeq \mathcal{T}(\partial B)_T$  as a bundle isomorphism. For  $\psi \in D((\partial B)_T; \mathcal{T}^*(\partial B)_T)$  and  $(\mathbf{x}, t) \in V_i \subset (\partial B)_T$ 

$$\boldsymbol{\iota}_T(\boldsymbol{\psi}(\mathbf{x},t)) = (\mathbf{x},t,\mathbf{u},v), \text{ with } \mathbf{u} \in \mathcal{T}^*_{\mathbf{x}} \partial B \subset \mathbb{R}^3, \ v \in \mathcal{T}_t(0,T) = \mathbb{R}$$
(54)

By Prop.16.36 in [33],  $\hat{A}^3((\partial B)_T)$  is a smooth line bundle of  $(\partial B)_T$  and as a consequence of 15.29 in [33], this density bundle is trivialized by the Riemannian volume form. Thus, we may identify

$$D((\partial B)_T; \mathcal{T}^*(\partial B)_T) \longleftrightarrow D((\partial B)_T, \mathcal{T}^*(\partial B)_T \otimes \hat{A}^3((\partial B)_T))$$
(55)

$$\chi \longleftrightarrow \chi \, dS \, dt \tag{56}$$

where dS is the volume form of  $\partial B$  (surface area). In that case, by the general definition (52) applied to the tangent bundle

$$D'((\partial B)_T, \mathcal{T}(\partial B)_T) = D((\partial B)_T, \mathcal{T}^*(\partial B)_T)',$$
(57)

and we may embed

$$D((\partial B)_T; \mathcal{T}(\partial B)_T) \hookrightarrow D'((\partial B)_T, \mathcal{T}(\partial B)_T)$$
(58)

$$\chi \mapsto T_{\chi}, \quad \langle T_{\chi}, \psi \rangle = \int_{(\partial B)_T} \langle \psi, \chi \rangle \, dS \, dt$$
 (59)

for all  $\psi \in D((\partial B)_T; \mathcal{T}^*(\partial B)_T)$  and  $\chi \in D((\partial B)_T; \mathcal{T}(\partial B)_T)$ .

Likewise,  $D((\partial B)_T; \mathcal{N}^*(\partial B)_T)$  denotes the space of smooth sections with compact support of the conormal bundle  $\mathcal{N}^*(\partial B)_T \simeq \mathcal{N}(\partial B)_T$ , so that for  $\psi \in D((\partial B)_T; \mathcal{N}^*(\partial B)_T)$  and  $(\mathbf{x}, t) \in V_i \subset (\partial B)_T$ ,

$$\boldsymbol{\iota}_N(\boldsymbol{\psi}(\mathbf{x},t)) = (\mathbf{x},t,\mathbf{u},0), \text{ with } \mathbf{u} \in \mathcal{N}^*_{\mathbf{x}} \partial B = \{\mathbf{n}(\mathbf{x})\mathbb{R}\}.$$
 (60)

Similarly as before, we may identify

$$D((\partial B)_T; \mathcal{N}^*(\partial B)_T) \longleftrightarrow D((\partial B)_T, \mathcal{N}^*(\partial B)_T \otimes \hat{A}^3((\partial B)_T))$$
(61)  
$$\chi \longleftrightarrow \chi \, dS \, dt$$
(62)

In that case, by the general definition (52) applied to the normal bundle

$$D'((\partial B)_T, \mathcal{N}(\partial B)_T) = D((\partial B)_T, \mathcal{N}^*(\partial B)_T)', \tag{63}$$

and we may embed

$$D((\partial B)_T; \mathcal{N}(\partial B)_T) \hookrightarrow D'((\partial B)_T, \mathcal{N}(\partial B)_T)$$
(64)

$$\boldsymbol{\chi} \mapsto T_{\boldsymbol{\chi}}, \quad \langle T_{\boldsymbol{\chi}}, \boldsymbol{\psi} \rangle = \int_{(\partial B)_T} \langle \boldsymbol{\psi}, \boldsymbol{\chi} \rangle \, dS \, dt$$
 (65)

for all  $\psi \in D((\partial B)_T; \mathcal{N}^*(\partial B)_T)$  and  $\chi \in D((\partial B)_T; \mathcal{N}(\partial B)_T)$ .

2.3. Extensions. The notion of an extension operator allows us to identify functions in the interior domain  $\Omega \times (0,T)$  with sectional distributions of  $\mathcal{T}(\partial B)_T$ and of  $\mathcal{N}(\partial B)_T$ . Beginning with the tangent bundle, we define  $\mathcal{E}_{\mathcal{T}}$  as the set of all linear operators

$$\mathbf{Ext}: \boldsymbol{\psi} \in D((\partial B)_T, \mathcal{T}^*(\partial B)_T) \mapsto \boldsymbol{\varphi} \in \bar{D}(\bar{\Omega} \times (0, T), \mathbb{R}^3 \times \mathbb{R})$$
(66)

satisfying pointwise equality (16) and continuous in the sense that for all multiindices  $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  with  $|\alpha| \leq N, \forall (\mathbf{x}, t) \in \overline{\Omega} \times (0, T)$  and  $\forall m > 0$ 

$$|D^{\alpha}\varphi(\mathbf{x},t)| = |D^{\alpha}\mathbf{Ext}(\psi)(\mathbf{x},t)| \lesssim \sup_{i \in I} p_{N,m,i}(\psi)$$
(67)

where,  $\leq$  denotes inequality with constant prefactor depending on the domain  $(\partial B)_T$  and the extension operator **Ext**. Note that for  $(\mathbf{x},t) \in \partial \Omega \times (0,T)$ , the derivatives  $D^{\alpha}$  with non-vanishing spatial indices  $\alpha_i, i = 1, 2, 3$  are one-sided derivatives, which according to definition (4) may be calculated as  $D^{\alpha}\varphi = D^{\alpha}\phi|_{\bar{\Omega}\times(0,T)}$  for  $\phi \in C_c^{\infty}(\mathbb{R}^3 \times (0,T),\mathbb{R}^3)$ , independent of the choice of  $\phi$ . Furthermore, if  $\boldsymbol{\psi} \in D((\partial B)_T, \mathcal{T}^*(\partial B)_T)$  is a space-like section of  $\mathcal{T}^*(\partial B)_T$ , so that

$$\boldsymbol{\iota}_T(\boldsymbol{\psi}(\mathbf{x},t)) = (\mathbf{x},t,\mathbf{u},0), \text{ with } \mathbf{u} \in \mathcal{T}^*_{\mathbf{x}} \partial B \subset \mathbb{R}^3$$
(68)

for all  $(\mathbf{x}, t) \in (\partial B)_T$ , then we may require that

$$\mathbf{Ext}: \boldsymbol{\psi} \in D((\partial B)_T, \mathcal{T}^*(\partial B)_T) \mapsto \boldsymbol{\varphi} \in \bar{D}(\bar{\Omega} \times (0, T), \mathbb{R}^3 \times \{0\})$$
(69)

We show that the set  $\mathcal{E}_T$  is non-empty, by constructing such an extension operator explicitly. We define  $\mathbf{Ext}_T^0 \in \mathcal{E}_T$  as a map

$$\mathbf{Ext}_T^0: \boldsymbol{\psi} \in D((\partial B)_T, \mathcal{T}^*(\partial B)_T) \mapsto \boldsymbol{\varphi} \in \bar{D}(\bar{\Omega} \times (0, T), \mathbb{R}^3 \times \mathbb{R})$$
(70)

by the explicit formula

$$\boldsymbol{\varphi}(\mathbf{x},t) = \begin{cases} \exp\left(-\frac{d(\mathbf{x})}{\epsilon - d(\mathbf{x})}\right) (\mathbf{Proj}_{st} \circ \boldsymbol{\iota}_T \circ \boldsymbol{\psi})(\pi(\mathbf{x}), t), & d(\mathbf{x}) < \epsilon \\ 0 & d(\mathbf{x}) \ge \epsilon \end{cases}$$
(71)

for any  $\epsilon < \eta(\Omega)$ . Then  $\mathbf{Ext}_T^0$  is clearly linear by the linearity of  $\iota_T$  and satisfies  $\varphi|_{\partial B} = (\mathbf{Proj}_{st} \circ \iota_T)(\psi)$ .  $\varphi$  is smooth by the smoothness of distance function d and projection  $\pi$  in  $\overline{\Omega}_{\epsilon}$ . One can easily obtain the bound (67) for  $\mathbf{Ext}_T^0$  by product rule and chain rule in calculus. Thus,  $\mathbf{Ext}_T^0$  is continuous. In particular, for a space-like section  $\psi \in D((\partial B)_T, \mathcal{T}^*(\partial B)_T)$ , the condition (69) holds, so that we may take  $\varphi \in \overline{D}(\overline{\Omega} \times (0,T), \mathbb{R}^3)$  with

$$\varphi|_{\partial B} = (\mathbf{Proj}_s \circ \boldsymbol{\iota}_T)(\boldsymbol{\psi}), \quad \varphi(\mathbf{x}, t) \perp \mathbf{n}(\pi(\mathbf{x})) \text{ in } \Omega_{\eta(\Omega)}.$$
 (72)

As a consequence of the second property,  $\mathbf{Ext}_T^0$  satisfies also the natural condition (27) and  $\mathbf{Ext}_T^0 \in \widetilde{\mathcal{E}}_T \neq \emptyset$ .

Similarly, we can define a set  $\mathcal{E}_{\mathcal{N}}$ , consisting of maps

$$\mathbf{Ext}: \boldsymbol{\psi} \in D((\partial B)_T, \mathcal{N}^*(\partial B)_T) \mapsto \boldsymbol{\varphi} \in \overline{D}(\overline{\Omega} \times (0, T), \mathbb{R}^3), \tag{73}$$

which are linear, continuous and satisfy

$$\boldsymbol{\varphi}|_{\partial B} = (\mathbf{Proj}_s \circ \boldsymbol{\iota}_N)(\boldsymbol{\psi}). \tag{74}$$

The set  $\mathcal{E}_{\mathcal{N}}$  is non-empty, because  $\mathbf{Ext}_{\mathcal{N}}^{0}$ , defined for  $\epsilon < \eta(\Omega)$  by

$$\boldsymbol{\varphi}(\mathbf{x},t) = \begin{cases} \exp\left(-\frac{d(\mathbf{x})}{\epsilon - d(\mathbf{x})}\right) (\mathbf{Proj}_s \circ \boldsymbol{\iota}_N \circ \boldsymbol{\psi})(\pi(\mathbf{x}),t), & d(\mathbf{x}) < \epsilon \\ 0 & d(\mathbf{x}) \ge \epsilon \end{cases}$$
(75)

for any  $\psi \in D((\partial B)_T, \mathcal{N}^*(\partial B)_T)$ , provides an explicit example which satisfies also the condition

$$\varphi(\mathbf{x},t) \parallel \mathbf{n}(\pi(\mathbf{x})) \text{ in } \Omega_{\eta(\Omega)}.$$
 (76)

One can use extension operators to identify  $\mathbf{F} \in D'((0,T), C^{\infty}(\bar{\Omega}, \mathbb{R}^3))$  with sectional distributions of the tangent and normal bundles. For example, for some  $\mathbf{Ext} \in \mathcal{E}_{\mathcal{T}}$  we define

$$\mathbf{Ext}^*: D'((0,T), C^{\infty}(\bar{\Omega}, \mathbb{R}^3)) \to D'((\partial B)_T, \mathcal{T}(\partial B)_T) \\
\mathbf{F} \mapsto \mathbf{Ext}^*(\mathbf{F})$$
(77)

as follows:

$$\langle \mathbf{Ext}^*(\mathbf{F}), \boldsymbol{\psi} \rangle := \langle \mathbf{F}, \mathbf{Ext}(\boldsymbol{\psi}) \rangle$$
 (78)

for all  $\psi \in D((\partial B)_T, \mathcal{T}^*(\partial B)_T)$ . Linearity and continuity properties of  $\mathbf{Ext}^*(\mathbf{F})$ follow from those of  $\mathbf{Ext} \in \mathcal{E}_{\mathcal{T}^*}$ , so that  $\mathbf{Ext}^*(\mathbf{F}) \in D'((\partial B)_T, \mathcal{T}(\partial B)_T)$ . Note that this identification depends on the choice of the extension operator  $\mathbf{Ext}$ . Likewise, we can define

$$\mathbf{Ext}^*: D'((0,T), C^{\infty}(\bar{\Omega}, \mathbb{R}^3)) \to D'((\partial B)_T, \mathcal{N}(\partial B)_T)$$
(79)

for each  $\mathbf{Ext} \in \mathcal{E}_{\mathcal{N}}$ , in exactly the same manner.

## 3. Proof of Theorem 1

The proof will proceed in steps. First, note that  $\boldsymbol{\tau}_{w}^{\nu} = \nu \boldsymbol{\omega}^{\nu}|_{(\partial B)_{T}} \times \mathbf{n}$  can be embedded into  $D((\partial B)_{T}, \mathcal{T}(\partial B)_{T})$  by

$$\boldsymbol{\tau}_{w}^{\nu} \mapsto \left( (\mathbf{x}, t) \mapsto (\mathbf{x}, t, \boldsymbol{\tau}_{w}^{\nu}(\mathbf{x}, t), 0) \right)$$
(80)

which can be further embedded into  $D'((\partial B)_T, \mathcal{T}(\partial B)_T)$  by (58). For the rest of this article, we abuse the notation  $\boldsymbol{\tau}_w^{\nu}$  to mean both vector fields on  $(\partial B)_T$  and smooth sections (80) in  $D((\partial B)_T, \mathcal{T}(\partial B)_T)$ , according to the context. We then show that  $\langle \boldsymbol{\tau}_w^{\nu}, \boldsymbol{\psi} \rangle$  for any  $\boldsymbol{\psi} \in D((\partial B)_T, \mathcal{T}^*(\partial B)_T)$  converges to a quantity denoted  $\langle \boldsymbol{\tau}_w, \boldsymbol{\psi} \rangle$ . Finally, we prove that  $\boldsymbol{\tau}_w$  is a continuous linear functional on  $D((\partial B)_T, \mathcal{T}^*(\partial B)_T)$ , thus obtaining the convergence (12) in the sense of distributional sections of the tangent bundle.

Similarly, wall pressure stress  $p_w^{\nu} \mathbf{n}$  is embedded into  $D((\partial B)_T, \mathcal{N}(\partial B)_T)$  by

$$p_w^{\nu} \mathbf{n} \mapsto \left( (\mathbf{x}, t) \mapsto (\mathbf{x}, t, p_w^{\nu}(\mathbf{x}, t) \mathbf{n}(\mathbf{x}), 0) \right)$$
(81)

which can be further embedded into  $D'((\partial B)_T, \mathcal{N}(\partial B)_T)$  by (65). An analogous argument shows that  $\langle p_w^{\nu} \mathbf{n}, \psi \rangle \rightarrow \langle p_w \mathbf{n}, \psi \rangle$  for all  $\psi \in D((\partial B)_T, \mathcal{N}^*(\partial B)_T)$ , with a suitable element  $p_w \mathbf{n} \in D'((\partial B)_T, \mathcal{N}(\partial B)_T)$ .

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3.1. Convergence of skin friction  $\tau_w^{\nu}$  to  $\tau_w$ . Consider an arbitrary extension operator  $\mathbf{Ext} \in \mathcal{E}_{\mathcal{T}}$ , and a smooth section  $\psi \in D((\partial B)_T, \mathcal{T}^*(\partial B)_T)$ . Let  $\varphi =$  $\mathbf{Ext}(\boldsymbol{\psi})$  so that  $\boldsymbol{\varphi} \in \overline{D}(\overline{\Omega} \times (0,T), \mathbb{R}^3)$  and  $\boldsymbol{\varphi} \cdot \mathbf{n} = 0$  on  $(\partial B)_T$ . Integrating the Navier-Stokes equations (1) against  $\varphi$  yields

$$-\int_{0}^{T}\int_{\Omega}\partial_{t}\boldsymbol{\varphi}\cdot\mathbf{u}^{\nu}+\boldsymbol{\nabla}\boldsymbol{\varphi}:\left[\mathbf{u}^{\nu}\otimes\mathbf{u}^{\nu}+p^{\nu}\mathbf{I}\right]dVdt$$
  
$$-\int_{0}^{T}\int_{\Omega}\nu\Delta\boldsymbol{\varphi}\cdot\mathbf{u}^{\nu}dVdt=-\int_{0}^{T}\int_{\partial\Omega}\nu\frac{\partial\mathbf{u}^{\nu}}{\partial n}\cdot\boldsymbol{\varphi}|_{\partial\Omega}\,dS\,dt.$$
(82)

As a useful shorthand, we write this as

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$$-\langle\!\langle \mathbf{u}^{\nu}, \partial_t \boldsymbol{\varphi} \rangle\!\rangle - \langle\!\langle \mathbf{u}^{\nu} \otimes \mathbf{u}^{\nu} : \boldsymbol{\nabla} \boldsymbol{\varphi} \rangle\!\rangle - \langle\!\langle p^{\nu}, \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} \rangle\!\rangle - \langle\!\langle \nu \mathbf{u}^{\nu}, \Delta \boldsymbol{\varphi} \rangle\!\rangle = -\langle \boldsymbol{\tau}_w^{\nu}, \boldsymbol{\psi} \rangle \tag{83}$$

where  $\langle\!\langle,\rangle\!\rangle$  denotes the integration over space-time domain  $\Omega \times (0,T)$  and

$$\langle \boldsymbol{\tau}_{w}^{\nu}, \boldsymbol{\psi} \rangle = \int_{0}^{T} \int_{\partial \Omega} \langle \boldsymbol{\psi}, \boldsymbol{\tau}_{w}^{\nu} \rangle \, dS \, dt = \int_{0}^{T} \int_{\partial \Omega} \nu \frac{\partial \mathbf{u}}{\partial n} \cdot \boldsymbol{\varphi}|_{\partial \Omega} \, dS \, dt. \tag{84}$$

By Cauchy-Schwartz,

$$|\langle\!\langle \nu \mathbf{u}^{\nu}, \triangle \boldsymbol{\varphi} \rangle\!\rangle| \leq \nu \sqrt{\int_0^T \int_{\Omega} |\triangle \boldsymbol{\varphi}|^2 \, dV dt} \sqrt{\int_0^T \int_{\mathrm{supp}(\boldsymbol{\varphi})} |\mathbf{u}^{\nu}|^2 \, dV dt} \qquad (85)$$
  
  $\to 0, \text{ as } \nu \to 0, \qquad (86)$ 

$$\rightarrow 0, \text{ as } \nu \rightarrow 0,$$
 (8)

as a consequence of the assumptions (8),(10) on velocity  $\mathbf{u}^{\nu}$ .

The convergence of the rest of the lefthand side of (83) as  $\nu \to 0$  follows from the following elementary lemma:

**Lemma 1.** If  $f^{\nu}$  converges weakly to f in  $L^{p}((0,T), L^{p}_{loc}(\Omega)), 1 \leq p < \infty$ , and if  $f^{\nu}$  is uniformly bounded in  $L^{p}((0,T), L^{p}(\Omega_{\epsilon}))$  for sufficiently small  $\epsilon > 0$ , then  $f \in L^p((0,T), L^p(\Omega_{\epsilon}))$ , and for  $\varphi \in \overline{D}(\overline{\Omega} \times (0,T))$ , we have the following limit

$$\lim_{\nu \to 0} \int_0^T \int_\Omega \varphi f^{\nu} \, dV dt = \int_0^T \int_\Omega \varphi f \, dV dt \tag{87}$$

*Proof.* Let  $M_{\epsilon} = \sup_{\nu>0} \|f^{\nu}\|_{L^{p}((0,T),L^{p}(\Omega_{\epsilon}))} < \infty$ . Then let  $\epsilon_{n} = 2^{-n}\epsilon$  for  $n \geq 0$ and  $\Gamma_n = \Omega_{\epsilon_n} \setminus \Omega_{\epsilon_{n+1}}$ . Then  $\Omega_{\epsilon} = \bigcup_{n=0}^{\infty} \Gamma_n$  and the union is a disjoint union. With weak lower-semicontinuity of the  $L^p$ -norm and Fatou's lemma, we have

$$\int_{(0,T)\times\Omega_{\epsilon}} |f|^{p} \, dV \, dt = \sum_{n=0}^{\infty} \int_{(0,T)\times\Gamma_{n}} |f|^{p} \, dV \, dt \le \sum_{n=0}^{\infty} \liminf_{\nu\to0} \int_{(0,T)\times\Gamma_{n}} |f^{\nu}|^{p} \, dV \, dt$$
(88)

$$\leq \liminf_{\nu \to 0} \sum_{n=0}^{\infty} \int_{(0,T) \times \Gamma_n} |f^{\nu}|^p \, dV \, dt = \liminf_{\nu \to 0} \int_{(0,T) \times \Omega_{\epsilon}} |f^{\nu}|^p \, dV \, dt \leq M_{\epsilon}^p < \infty$$
(89)

Thus, we obtain that  $f \in L^p((0,T), L^p(\Omega_{\epsilon}))$ .

Furthermore, for any  $0<\delta<\epsilon$  we obtain by Hölder inequality and the uniform  $L^p$  bound on  $f^\nu$  that for  $\frac1p+\frac1{p'}=1$ 

$$\sup_{\nu>0} \left| \int_{(0,T)\times\Omega_{\delta}} \varphi f^{\nu} dV dt \right| \le \|\varphi\|_{L^{p'}((0,T)\times\Omega_{\delta})} M^{p}_{\epsilon}$$
(90)

with an identical bound for the limit function f. As a consequence

$$\left| \int_{(0,T)\times\Omega} \varphi f^{\nu} dV dt - \int_{(0,T)\times\Omega} \varphi f dV dt \right| \leq 2 \left\| \varphi \right\|_{L^{p'}((0,T)\times\Omega_{\delta})} M_{\epsilon}^{p} + \left| \int_{(0,T)\times\Omega^{\delta}} \varphi f^{\nu} dV dt - \int_{(0,T)\times\Omega^{\delta}} \varphi f dV dt \right|$$
(91)

where  $\Omega^{\delta} := \Omega \setminus \Omega_{\delta}$ . Using convergence of  $f^{\nu}$  to f weakly in  $L^{p}((0,T), L^{p}_{\text{loc}}(\Omega))$ and  $\lim_{\delta \to 0} \|\varphi\|_{L^{p'}((0,T) \times \Omega_{\delta})} = 0$ , we conclude.  $\Box$ 

Conditions (8),(14) imply that, at least along a subsequence,  $\mathbf{u}^{\nu}$ ,  $\mathbf{u}^{\nu} \otimes \mathbf{u}^{\nu}$ ,  $p^{\nu}$  have local weak convergence to  $\mathbf{u}$ ,  $\mathbf{u} \otimes \mathbf{u}$ , p respectively. Then by Lemma 1,

$$\mathbf{u} \in L^2((0,T), L^2(\Omega_{\epsilon})), \quad p \in L^1((0,T), L^1(\Omega_{\epsilon}))$$
(92)

and as  $\nu \to 0$ , the left hand side of (83) converges to

$$-\langle\!\langle \mathbf{u}, \partial_t \boldsymbol{\varphi} \rangle\!\rangle - \langle\!\langle \mathbf{u} \otimes \mathbf{u} : \boldsymbol{\nabla} \boldsymbol{\varphi} \rangle\!\rangle - \langle\!\langle p, \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} \rangle\!\rangle := \langle \boldsymbol{\tau}_w, \boldsymbol{\psi} \rangle.$$
(93)

As  $\psi$  was arbitrary, , we conclude that

$$\lim_{\nu \to 0} \langle \boldsymbol{\tau}_{w}^{\nu}, \boldsymbol{\psi} \rangle = \langle \boldsymbol{\tau}_{w}, \boldsymbol{\psi} \rangle, \ \forall \boldsymbol{\psi} \in D((\partial B)_{T}, \mathcal{T}^{*}(\partial B)_{T})$$
(94)

3.2.  $\tau_w$  is a distributional section. Linearity of  $\tau_w$  follows easily from the linearity of **Ext** and the definition (93). Then, it suffices to prove the continuity of  $\tau_w$ . Let K be a compact subset of  $(\partial B)_T$ . Then there exists a finite set Jsuch that  $K \subset \bigcup_{i \in J} V_i$ , where  $\bigcup (V_i, \phi_i)$  is a smooth structure of  $(\partial B)_T$ . Furthermore, there exists some  $m_0 > 0$  such that for each  $i \in J$ ,  $\phi_i(K \cap V_i) \subset K_{m_0}^{(i)}$ for a compact set  $K_{m_0}^{(i)}$  in the fundamental sequence of  $\phi_i(V_i)$ . Therefore, for all  $\psi \in D((\partial B)_T, \mathcal{T}^*(\partial B)_T)$  supported on K and for all  $m \geq m_0$ 

$$\langle\!\langle \mathbf{u}, \partial_t \boldsymbol{\varphi} \rangle\!\rangle \lesssim \|\mathbf{u}\|_{L^2(\operatorname{supp}(\boldsymbol{\varphi}))} \sup_{i \in I} p_{1,m,i}(\boldsymbol{\psi})$$
 (95)

$$\langle\!\langle \mathbf{u} \otimes \mathbf{u} : \boldsymbol{\nabla} \boldsymbol{\varphi} \rangle\!\rangle \lesssim \|\mathbf{u}\|_{L^2(\mathrm{supp}(\boldsymbol{\varphi}))}^2 \sup_{i \in I} p_{1,m,i}(\boldsymbol{\psi})$$
 (96)

where  $\varphi = \text{Ext}(\psi)$ , so that  $\text{supp}(\varphi)$  is a compact subset of  $\overline{\Omega} \times (0,T)$  by definition (4). Here,  $\leq$  denotes inequality up to a constant prefactor, depending on K, **Ext**. Note that **u** is bounded in  $L^2(\text{supp}(\varphi))$ , as a result of interior boundedness (8) and near-boundary boundedness (92) in  $L^2$ . Similarly, for all

 $\boldsymbol{\psi} \in D((\partial B)_T, \mathcal{T}^*(\partial B)_T)$  supported on K and for all  $m \geq m_0, p$  is bounded in  $L^1(\operatorname{supp}(\boldsymbol{\varphi}))$  and

$$\langle\!\langle p, \nabla \cdot \varphi \rangle\!\rangle \lesssim \|p\|_{L^1(\mathrm{supp}(\varphi))} \sup_{i \in I} p_{1,m,i}(\psi).$$
 (97)

In conclusion,  $\boldsymbol{\tau}_w$  is continuous and  $\boldsymbol{\tau}_w$  is thus a well defined distribution in  $D'((\partial B)_T), \mathcal{T}(\partial B)_T)$  for each  $\mathbf{Ext} \in \mathcal{E}_T$ .

Note that  $\tau_w$  is independent of  $\mathbf{Ext} \in \mathcal{E}_{\mathcal{T}}$ . Indeed, by combining the result in this section with that in 3.1, we see for each  $\mathbf{Ext} \in \mathcal{E}_{\mathcal{T}}$  that  $\lim_{\nu \to 0} \tau_w^{\nu} = \tau_w$  in the standard topology of  $D'((\partial B)_T), \mathcal{T}(\partial B)_T)$ . Since such limits are unique,  $\tau_w$  is independent of the choice of extension operator and depends only upon the subsequence  $\nu_k \to 0$  chosen to obtain the limiting weak Euler solution  $(\mathbf{u}, p)$ .

3.3. Pressure stress  $p_w \mathbf{n}$ . Consider now instead an arbitrary extension operator  $\mathbf{Ext} \in \mathcal{E}_{\mathcal{N}}$  and a smooth section  $\boldsymbol{\psi} \in D((\partial B)_T, \mathcal{N}^*(\partial B)_T)$ . Let  $\boldsymbol{\varphi} = \mathbf{Ext}(\boldsymbol{\psi})$  so that  $\boldsymbol{\varphi} \in \overline{D}(\overline{\Omega} \times (0,T), \mathbb{R}^3)$  and  $\boldsymbol{\varphi} \parallel \mathbf{n}$  on  $(\partial B)_T$ . Integrating the Navier-Stokes equations (1) against  $\boldsymbol{\varphi}$  yields

$$-\int_{0}^{T}\int_{\Omega}\left[\partial_{t}\boldsymbol{\varphi}\cdot\mathbf{u}^{\nu}+\boldsymbol{\nabla}\boldsymbol{\varphi}:\left(\mathbf{u}^{\nu}\otimes\mathbf{u}^{\nu}+p^{\nu}\mathbf{I}\right)\right]dVdt$$

$$+\nu\int_{0}^{T}\int_{\Omega}(\Delta\boldsymbol{\varphi})\cdot\mathbf{u}^{\nu}dVdt=\int_{0}^{T}\int_{\partial\Omega}p^{\nu}|_{(\partial B)_{T}}\mathbf{n}\cdot\boldsymbol{\varphi}|_{(\partial B)_{T}}dSdt$$
(98)

On the other hand,

$$\langle p_w^{\nu} \mathbf{n}, \boldsymbol{\psi} \rangle = \int_0^T \int_{\partial B} \langle \boldsymbol{\psi}, p_w^{\nu} \mathbf{n} \rangle \, dS \, dt = \int_0^T \int_{\partial B} p^{\nu}|_{(\partial B)_T} \mathbf{n} \cdot \boldsymbol{\varphi}|_{(\partial B)_T} \, dS \, dt.$$
(99)

In shorthand,

$$-\langle\!\langle \mathbf{u}^{\nu}, \partial_t \boldsymbol{\varphi} \rangle\!\rangle - \langle\!\langle \mathbf{u}^{\nu} \otimes \mathbf{u}^{\nu} : \boldsymbol{\nabla} \boldsymbol{\varphi} \rangle\!\rangle - \langle\!\langle p^{\nu}, \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} \rangle\!\rangle + \langle\!\langle \nu \mathbf{u}^{\nu}, \Delta \boldsymbol{\varphi} \rangle\!\rangle = \langle p_w^{\nu} \mathbf{n}, \boldsymbol{\psi} \rangle$$
(100)

By an analogous argument as that used to prove convergence of  $\boldsymbol{\tau}_{w}^{\nu}$  to  $\boldsymbol{\tau}_{w}$ , it follows that (100) in the limit  $\nu \to 0$  yields for all  $\boldsymbol{\psi} \in D((\partial B)_T, \mathcal{N}^*(\partial B)_T)$ 

$$\langle p_w^{\nu} \mathbf{n}, \boldsymbol{\psi} \rangle \xrightarrow{\nu \to 0} \langle p_w \mathbf{n}, \boldsymbol{\psi} \rangle := -\langle \langle \mathbf{u}, \partial_t \boldsymbol{\varphi} \rangle \rangle - \langle \langle \mathbf{u} \otimes \mathbf{u} : \boldsymbol{\nabla} \boldsymbol{\varphi} \rangle \rangle - \langle \langle p, \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} \rangle \rangle \quad (101)$$

and  $p_w \mathbf{n} \in D'((\partial B)_T), \mathcal{N}(\partial B)_T)$ , independent of the extension  $\mathbf{Ext} \in \mathcal{E}_{\mathcal{N}}$ .

# 4. Proof of Theorem 2

We give here a detailed proof only of the result (25) on the convergence in the space of distributional sections of the tangent bundle. The statement (26) on convergence in the space of distributional sections of the normal bundle is proved by a very similar argument, which is left to the reader.

# 4.1. Proof of a lemma. We first prove:

**Lemma 2.** Let K be a compact subset of  $\overline{\Omega}$ . Then for any  $f \in L^p((0,T), L^p_{loc}(\Omega))$  $\cap L^p((0,T), L^p(\Omega_{\epsilon})), \text{ with } 1 \leq p < \infty \text{ and } \epsilon > 0, \text{ we have for } 0 < \ell < h,$ 

$$\eta_{h,\ell} \bar{f}_{\ell} \xrightarrow{h,\ell \to 0}{L^p((0,T),L^p(K))} f$$
(102)

*Proof.* Let  $\delta > 0$  be a sufficiently small number with  $\ell < h < \delta < \frac{\epsilon}{2}$ . It is wellknown for  $f \in L^p_{\text{loc}}(\Omega)$  that  $\bar{f}_{\ell} \to f$  in  $L^p_{\text{loc}}(\Omega)$  (e.g. see Appendix C.5 of [21]). In particular, for a.e.  $t \in (0,T)$ ,

$$\lim_{h,\ell\to 0} \left\| \eta_{h,\ell} \bar{f}_{\ell} - f \right\|_{L^p(\Omega^{\delta} \cap K)} = 0$$
(103)

and

$$\left\|\eta_{h,\ell}\bar{f}_{\ell} - f\right\|_{L^{p}(\Omega^{\delta}\cap K)} \le 2\left\|f\right\|_{L^{p}(K_{\epsilon}\cap\Omega)}$$
(104)

for  $K_{\epsilon} = {\mathbf{x} \in \mathbb{R}^3 : \exists \mathbf{y} \in K, |\mathbf{x} - \mathbf{y}| < \epsilon}$ . On the other hand, for a.e.  $t \in (0, T)$ ,

$$\left\|\eta_{h,\ell}\bar{f}_{\ell} - f\right\|_{L^{p}(\Omega_{\delta}\cap K)} \leq \left\|\eta_{h,\ell}\bar{f}_{\ell} - f\right\|_{L^{p}(\Omega_{\delta})} \tag{105}$$

$$= \left\| \eta_{h,\ell} \bar{f}_{\ell} - f \right\|_{L^p(\Omega_{\delta} \setminus \Omega_h)} + \left\| f \right\|_{L^p(\Omega_h)}$$
(106)

$$\leq 2 \|f\|_{L^{p}(\Omega_{2\delta})} + \|f\|_{L^{p}(\Omega_{h})}$$
(107)

$$\leq 3 \|f\|_{L^{p}(\Omega_{2\delta})} \leq 3 \|f\|_{L^{p}(\Omega_{\epsilon})}$$
(108)

Then by combining (103),(108)

$$\limsup_{h,\ell\to 0} \left\| \eta_{h,\ell} \bar{f}_{\ell} - f \right\|_{L^p(K)} \le 3 \left\| f \right\|_{L^p(\Omega_{2\delta})} \xrightarrow{\delta \to 0} 0 \tag{109}$$

where the latter follows by dominated convergence theorem. Since (109) is true for a.e.  $t \in (0, T)$ , one obtains (102), or convergence in  $L^p((0, T), L^p(K))$ . 

4.2. Proof of Theorem 2. Take any extension operator  $\mathbf{Ext} \in \mathcal{E}_{\mathcal{T}}$  and smooth section  $\boldsymbol{\psi} \in D((\partial B)_T, \mathcal{T}^*(\partial B)_T)$ . Let  $\boldsymbol{\varphi} = \mathbf{Ext}(\boldsymbol{\psi})$  so that  $\boldsymbol{\varphi} \in \overline{D}(\overline{\Omega} \times (0,T), \mathbb{R}^3)$ and  $\boldsymbol{\varphi} \cdot \mathbf{n} = 0$  on  $(\partial B)_T$ . Integrating the coarse-grained Euler equations (23) against  $\varphi$  yields

$$\int_{0}^{T} \int_{\Omega} \partial_{t} \boldsymbol{\varphi} \cdot (\eta_{h,\ell} \bar{\mathbf{u}}_{\ell}) \, dV \, dt + \int_{0}^{T} \int_{\Omega} \boldsymbol{\nabla} \boldsymbol{\varphi} : \eta_{h,\ell} (\bar{\mathbf{T}}_{\ell} + \bar{p}_{\ell} \mathbf{I}) \, dV \, dt$$
(110)

$$= -\int_{0}^{T} \int_{\Omega} \boldsymbol{\varphi} \cdot \left( \boldsymbol{\nabla} \eta_{h,\ell} \cdot \bar{\mathbf{T}}_{\ell} + \bar{p}_{\ell} \boldsymbol{\nabla} \eta_{h,\ell} \right) dV dt$$
(111)

As  $\varphi \in \overline{D}(\overline{\Omega} \times (0,T), \mathbb{R}^3)$ , there exists a compact set  $K \subset \overline{\Omega}$  such that

$$\operatorname{supp}(\boldsymbol{\varphi}) \subset K \times (0, T) \subset \overline{\Omega} \times (0, T)$$
(112)

By Lemma 2, as  $h, \ell \to 0$ ,

$$\eta_{h,\ell} \bar{\mathbf{u}}_{\ell} \to \mathbf{u} \text{ in } L^2((0,T), L^2(K))$$
(113)

$$\eta_{h,\ell} \bar{\mathbf{T}}_{\ell} \to \mathbf{T} \text{ in } L^1((0,T), L^1(K))$$
(114)

$$\eta_{h,\ell} \bar{p}_{\ell} \to p \text{ in } L^1((0,T), L^1(K))$$
 (115)

Then, as  $h, \ell \to 0$ , (110) converges to

$$\int_{0}^{T} \int_{\Omega} \partial_{t} \boldsymbol{\varphi} \cdot \mathbf{u} \, dV \, dt + \int_{0}^{T} \int_{\Omega} \boldsymbol{\nabla} \boldsymbol{\varphi} : [\mathbf{T} + p\mathbf{I}] \, dV \, dt \tag{116}$$

Thus we obtain from (110)-(111) that

$$-\lim_{h,\ell\to 0} \int_0^T \int_\Omega \boldsymbol{\varphi} \cdot (\boldsymbol{\nabla}\eta_{h,\ell} \cdot \bar{\mathbf{T}}_\ell + \bar{p}_\ell \boldsymbol{\nabla}\eta_{h,\ell}) \, dV \, dt \tag{117}$$

$$= \int_{0}^{T} \int_{\Omega} \partial_{t} \boldsymbol{\varphi} \cdot \mathbf{u} \, dV \, dt + \int_{0}^{T} \int_{\Omega} \boldsymbol{\nabla} \boldsymbol{\varphi} : [\mathbf{T} + p\mathbf{I}] \, dV \, dt \tag{118}$$

Then by comparison with  $\langle \boldsymbol{\tau}_w, \boldsymbol{\psi} \rangle$  defined by (93), we obtain that

$$-\lim_{h,\ell\to 0} \int_0^T \int_{\Omega} \boldsymbol{\varphi} \cdot (\boldsymbol{\nabla}\eta_{h,\ell} \cdot \bar{\mathbf{T}}_{\ell} + \bar{p}_{\ell} \boldsymbol{\nabla}\eta_{h,\ell}) \, dV \, dt = \langle \boldsymbol{\tau}_w, \boldsymbol{\psi} \rangle \tag{119}$$

In other words, for any  $\mathbf{Ext} \in \mathcal{E}_{\mathcal{T}}$ 

$$-\lim_{h,\ell\to 0} \mathbf{Ext}^* (\nabla \eta_{h,\ell} \cdot \bar{\mathbf{T}}_{\ell} + \bar{p}_{\ell} \nabla \eta_{h,\ell}) = \boldsymbol{\tau}_w \text{ in } D'((\partial B)_T, \mathcal{T}(\partial B)_T)$$
(120)

4.3. Proof of Corollary 1. For any  $\mathbf{Ext} \in \tilde{\mathcal{E}}_{\mathcal{T}}$  as in (27),  $\forall \psi \in D((\partial B)_T, \mathcal{T}^*(\partial B)_T)$ ,  $\varphi = \mathbf{Ext}(\psi)$ 

$$\langle \mathbf{Ext}^*(\bar{p}_{\ell} \nabla \eta_{h,\ell}), \psi \rangle = \int_0^T \int_{\Omega} \boldsymbol{\varphi} \cdot (\bar{p}_{\ell} \nabla \eta_{h,\ell}) \, dV \, dt$$

$$= \int_0^T \int_{\Omega_{h+\ell} \setminus \Omega_h} \theta'_{h,\ell}(d(\mathbf{x})) \bar{p}_{\ell}(\mathbf{x},t) \boldsymbol{\varphi}(\mathbf{x},t) \cdot \mathbf{n}(\pi(\mathbf{x})) \, dV \, dt$$

$$(122)$$

$$|\langle \mathbf{Ext}^*(\bar{p}_{\ell} \nabla \eta_{h,\ell}), \psi \rangle| \leq \left\| \theta_{h,\ell}' \varphi \cdot \mathbf{n} \right\|_{L^{\infty}((\Omega_{h+\ell} \setminus \Omega_h) \times (0,T))} \left( \int_0^T \int_{\Omega_{h+\ell} \setminus \Omega_h} |\bar{p}_{\ell}| \, dV \, dt \right)$$
(123)

$$\leq \frac{C}{\ell} \| \boldsymbol{\varphi} \cdot \mathbf{n} \|_{L^{\infty}((\Omega_{h+\ell} \setminus \Omega_h) \times (0,T))} \| p \|_{L^1((\Omega_{h+2\ell} \setminus \Omega_{h-\ell}) \times (0,T))}$$
(124)

$$\leq C' \|p\|_{L^1((\Omega_{3h}) \times (0,T))} \xrightarrow{h,\ell \to 0} 0 \tag{125}$$

by (27) and dominated convergence. By comparison with (25) we obtain that

$$-\lim_{h,\ell\to 0} \mathbf{Ext}^* (\nabla \eta_{h,\ell} \cdot \bar{\mathbf{T}}_\ell) = \boldsymbol{\tau}_w \text{ in } D'((\partial B)_T, \mathcal{T}(\partial B)_T)$$
(126)

## 5. Proof of Theorem 3

Theorem 3 follows from Proposition 2, in conjunction with Theorem 1 & 2 and Corollary 1. We thus prove Proposition 2 in this section. We follow the idea in [17] by bounding the following term directly

$$\nabla \eta_{h,\ell} \cdot \bar{\mathbf{T}}_{\ell}(\mathbf{x},t) = \theta_{h,\ell}'(d(\mathbf{x})) \mathbf{n}(\pi(\mathbf{x})) \cdot \overline{\mathbf{u} \otimes \mathbf{u}}_{\ell}(\mathbf{x},t)$$
(127)

which is supported in  $\Omega_{h+\ell} \setminus \Omega_h \subset \Omega_{3h} \subset \Omega_{\epsilon}$ . We write,  $\forall \mathbf{x} \in \Omega_{h+\ell} \setminus \Omega_h$ , a.e.  $t \in (0,T)$ ,

$$\mathbf{n}(\pi(\mathbf{x})) \cdot \overline{\mathbf{u} \otimes \mathbf{u}}_{\ell}(\mathbf{x}, t) = \int_{\mathbb{R}^{3}} G_{\ell}(\mathbf{r}) [\mathbf{n}(\pi(\mathbf{x})) - \mathbf{n}(\pi(\mathbf{x} + \mathbf{r}))] \cdot \mathbf{u} \otimes \mathbf{u}(\mathbf{x} + \mathbf{r}, t) V(d\mathbf{r}) + \int_{\mathbb{R}^{3}} G_{\ell}(\mathbf{r}) \mathbf{n}(\pi(\mathbf{x} + \mathbf{r})) \cdot \mathbf{u} \otimes \mathbf{u}(\mathbf{x} + \mathbf{r}, t) V(d\mathbf{r})$$
(128)

Since  $\mathbf{n} \circ \pi$  is smooth in  $\overline{\Omega_{\epsilon}}$ ,  $\forall \delta > 0$ ,  $\exists \rho = \rho(\delta) > 0$  s.t.

$$\mathbf{n}(\pi(\mathbf{x})) - \mathbf{n}(\pi(\mathbf{x} + \mathbf{r}))| \le \delta$$
(129)

for all  $\mathbf{x} \in \Omega_{h+\ell} \backslash \Omega_h$  and  $|r| < \ell < \rho$ . Then it follows that

$$|\mathbf{n}(\pi(\mathbf{x})) \cdot \overline{\mathbf{u} \otimes \mathbf{u}}_{\ell}(\mathbf{x}, t)| \leq \left(\delta \|\mathbf{u}(t)\|_{L^{\infty}(\Omega_{\epsilon})} + \|\mathbf{n} \cdot \mathbf{u}(t)\|_{L^{\infty}(\Omega_{\epsilon})}\right) \|\mathbf{u}(t)\|_{L^{\infty}(\Omega_{\epsilon})}$$
(130)

Using these bounds above, together with the fact that  $\left\| \theta'_{h,\ell}(d(\mathbf{x})) \right\|_{L^{\infty}} \leq \frac{C}{\ell}$  and  $|\Omega_{h+\ell} \backslash \Omega_h| \leq C'\ell$ , we obtain that for  $\psi \in D((\partial B)_T, \mathcal{T}^*(\partial B)_T)$ ,  $\mathbf{Ext} \in \mathcal{E}_T$ 

$$\langle \mathbf{Ext}^*(\boldsymbol{\nabla}\eta_{h,\ell}\cdot\bar{\mathbf{T}}_\ell),\boldsymbol{\psi}\rangle = \int_0^T \int_{\Omega} \boldsymbol{\varphi} \cdot (\boldsymbol{\nabla}\eta_{h,\ell}\cdot\bar{\mathbf{T}}_\ell) \, dV \, dt \tag{131}$$

$$\left| \left\langle \mathbf{Ext}^{*}(\boldsymbol{\nabla}\eta_{h,\ell}) \cdot \bar{\mathbf{T}}_{\ell}, \boldsymbol{\psi} \right\rangle \right| \leq \left\| \boldsymbol{\varphi} \right\|_{L^{\infty}((0,T) \times \Omega)} \int_{0}^{T} \int_{\Omega_{h+\ell} \setminus \Omega_{h}} \left| \boldsymbol{\nabla}\eta_{h,\ell} \cdot \bar{\mathbf{T}}_{\ell} \right| dV dt$$
(132)

$$\lesssim \sup_{i \in I} p_{N,m,i}(\boldsymbol{\psi}) \times \left[ \delta \| \mathbf{u} \|_{L^2((0,T),L^\infty(\Omega_\epsilon))}^2 \right]$$
(133)

$$+ \left\| \mathbf{n} \cdot \mathbf{u} \right\|_{L^{2}((0,T),L^{\infty}(\Omega_{\epsilon}))} \left\| \mathbf{u} \right\|_{L^{2}((0,T),L^{\infty}(\Omega_{\epsilon}))} \right\|$$
(134)

where  $\varphi = \text{Ext}(\psi)$ . Thus, by the assumptions on the near wall uniform boundedness of **u** (29) and the continuity of wall normal velocity (30), the first result (31) in Proposition 1 follows:

$$\lim_{h,\ell\to 0} \mathbf{Ext}^* (\nabla \eta_{h,\ell} \cdot \bar{\mathbf{T}}_\ell) = 0 \text{ in } D'((\partial B)_T, \mathcal{T}(\partial B)_T)$$
(135)

It is easy to see that the argument above applies also for all  $\mathbf{Ext} \in \mathcal{E}_{\mathcal{N}}$  and  $\psi \in D((\partial B)_T, \mathcal{N}^*(\partial B)_T)$ . Thus, the result (32) in Proposition 1 also follows:

$$\lim_{h,\ell\to 0} \mathbf{Ext}^*(\boldsymbol{\nabla}\eta_{h,\ell}\cdot\bar{\mathbf{T}}_\ell) = 0 \text{ in } D'((\partial B)_T, \mathcal{N}(\partial B)_T)$$
(136)

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