Problem 1. This problem discusses the heat-flow regularization of quantities in the upper half-space \( \mathbb{R}^3_+ = \{ (x, y, z) : y > 0 \} \).

(a) Show for any locally integrable function \( f \) defined on \( \mathbb{R}^3_+ \) that

\[
\overline{f}^{D/N}_\ell (x, y, z) := \frac{1}{(2\pi \ell^2)^{3/2}} \int_{\mathbb{R}^3_+} \left[ e^{-\frac{(y-y')^2}{4\ell^2}} \mp e^{-\frac{(y+y')^2}{4\ell^2}} \right] e^{-\frac{(x-x')^2+(z-z')^2}{2\ell^2}} f(x', y', z') \, dx' dy' dz'
\]

solve the heat flow equation \( \frac{\partial}{\partial \ell^2} \overline{f}_\ell = \frac{1}{2} \triangle \overline{f}_\ell \) with, respectively, the Dirichlet b.c

\[
\overline{f}^D_\ell (x, 0, z) = 0.
\]

and the Neumann b.c

\[
\partial_y \overline{f}^N_\ell (x, 0, z) = 0.
\]

(b) Use part (a) to show that

\[
\begin{align*}
(i) & \quad \overline{(\partial_y f)}^D_\ell = \partial_y \overline{f}^N_\ell \\
(ii) & \quad \overline{(\partial_y f)}^N_\ell = \partial_y \overline{f}^D_\ell \quad \text{if } f(x, 0, z) = 0 \\
(iii) & \quad \overline{\partial_y f}^{D/N}_\ell := \overline{(\partial_y f)}^{D/N}_\ell - \partial_y \overline{f}^{D/N}_\ell = \pm \partial_y \overline{f}^W_\ell
\end{align*}
\]

where the third result for case \( N \) also requires \( f(x, 0, z) = 0 \) and we have made the definition \( \overline{f}^W_\ell := \overline{f}_\ell - \overline{f}^D_\ell \) so that

\[
\overline{f}^W_\ell (x, y, z) = \frac{2}{(2\pi \ell^2)^{3/2}} \int_{\mathbb{R}^3_+} e^{-\frac{(y+y')^2}{4\ell^2}} e^{-\frac{(x-x')^2+(z-z')^2}{2\ell^2}} f(x', y', z') \, dx' dy' dz'.
\]

Show that \( \overline{f}^W_\ell (x, y, z) \) is negligible for \( y \gg \ell \).

(c) Use part (b) to derive the coarse-grained Navier-Stokes equations in the half-space

\[
\partial_t \overline{\mathbf{u}}^D_\ell + \nabla \cdot \left[ (\overline{uu})^D_\ell + \overline{p}^D_\ell I \right] = \nu \triangle \overline{\mathbf{u}}^D_\ell - \overline{f}^D_\ell, \quad \nabla \cdot \overline{\mathbf{u}}^D_\ell = \sigma^D_\ell
\]

with

\[
\overline{f}^D_\ell := \partial_y \overline{(\nu \mathbf{u})}^W_\ell + \overline{p}^W_\ell \, \mathbf{y}, \quad \sigma^D_\ell := -\partial_y \overline{p}^W_\ell.
\]

Explain why

\[
\int_{\mathbb{R}^3_+} \sigma^D_\ell \, dx \, dy \, dz = 0.
\]
Problem 2. This problem discusses the elliptic regularization of Germano (1986) and Bose & Moin (2014) defined by the solution \( \mathbf{f}_\ell \) of the equation

\[
\mathbf{f}_\ell - \frac{\partial}{\partial x_k} \left( \ell^2(x) \frac{\partial \mathbf{f}_\ell}{\partial x_k} \right) = f, \quad x \in \Omega.
\]

(a) When \( \Omega = \mathbb{R}^3 \) and \( \ell^2 \) is constant, then \( \mathbf{f}_\ell = G_\ell \ast f \) for \( G_\ell(r) = \ell^{-3} G(r/\ell) \) and

\[
G(\rho) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{-i \kappa \cdot \rho} \frac{1}{1 + \kappa^2} d^3 \kappa.
\]

Show that \( G(\rho) = e^{-\rho}/4\pi \rho \). Hint: Evaluate the above Fourier integral in spherical coordinates using calculus of residues.

(b) If \( \ell^2(x) \) is a smooth function vanishing at \( \partial \Omega \), then show that

\[
\int_\Omega \mathbf{f}_\ell dV = \int_\Omega f dV.
\]

for any \( f \) which is integrable and spatially differentiable.

(c) If \( \ell^2(x) \) is a smooth function vanishing at \( \partial \Omega \), then show that

\[
\mathbf{f}_\ell - \frac{\partial \ell^2}{\partial n} \frac{\partial \mathbf{f}_\ell}{\partial n} = f, \quad x \in \partial \Omega.
\]

For a velocity field satisfying \( \mathbf{u} = 0 \) on \( \partial \Omega \), show that

\[
\mathbf{u}_\ell = 0 \text{ on } \partial \Omega, \text{ if } \frac{\partial \ell^2}{\partial n} \bigg|_{\partial \Omega} = 0,
\]

\[
\mathbf{u}_\ell = \Delta_w \frac{\partial \mathbf{u}_\ell}{\partial n} \text{ on } \partial \Omega, \text{ if } \Delta_w = \frac{\partial \ell^2}{\partial n} \bigg|_{\partial \Omega} > 0.
\]

(d) Show that the derivative-commutator for this filter is given in closed form by

\[
\left[ \frac{\partial f}{\partial x_i} \right]_\ell = - \left( \frac{\partial}{\partial x_k} \left( \ell^2 \frac{\partial \mathbf{f}_\ell}{\partial x_k} \right) \right)_\ell
\]

(e) Use part (d) to show that \( \nabla \cdot \mathbf{u}_\ell = \sigma_\ell \) with

\[
\sigma_\ell = \left( \frac{\partial}{\partial x_k} \left( \nabla \ell^2 \cdot \frac{\partial \mathbf{u}_\ell}{\partial x_k} \right) \right)_\ell
\]

and therefore when \( \nabla \ell^2 = \Delta_w \mathbf{n} \) on \( \partial \Omega \)

\[
\int_\Omega \sigma_\ell dV = \int_{\partial \Omega} \Delta_w \frac{\partial \mathbf{u}_\ell}{\partial n} dA.
\]
Problem 3. This problem discusses the regularization using filtering & windowing of Bardos & Titi (2018), defined by
\[
\tilde{f}_{h,\ell}(x) = \theta_{h,\ell}(d(x))\overline{f}_\ell(x), \quad x \in \Omega, \ h > \ell > 0,
\]
for \(\theta_{h,\ell}\) the smoothed step-function discussed in the course notes and \(d(x) = \text{dist}(x, \partial \Omega)\).

(a) Derive the coarse-grained Navier-Stokes equations
\[
\partial_t \tilde{u}_{h,\ell} + \nabla \cdot \left[\overline{uu}_{h,\ell} + \overline{p}_h I - \nu \overline{\nabla u}_h\right] = -f_{h,\ell}, \quad \nabla \cdot \tilde{u}_{h,\ell} = \sigma_{h,\ell}
\]
with
\[
f_{h,\ell} := -\theta'_{h,\ell}(d(x))n(x) \cdot (\overline{uu}_\ell + \overline{p}_\ell I - \nu \overline{\nabla u}_\ell), \quad \sigma_{h,\ell} := \theta'_{h,\ell}(d(x))n(x) \cdot \overline{u}_\ell.
\]
(b) Explain why \(\int \sigma_{h,\ell} dV = 0\) and derive the Poisson equation
\[
-\Delta \tilde{p}_{h,\ell} = \partial_t \sigma_{h,\ell} + \nabla \nabla \cdot (\overline{uu}_{h,\ell} - \nu \overline{\nabla u}_{h,\ell}) + \nabla \cdot f_{h,\ell}
\]
to determine the coarse-grained pressure with zero Dirichlet boundary conditions.
(c) Explain carefully why the equations in (a) with \(\nu = 0\) for all \(h > \ell > 0\) are equivalent to the standard weak formulation of the incompressible Euler equations in the flow domain \(\Omega\).

Problem 4. (a) If \((u', p')\) is a smooth solution of incompressible Navier-Stokes equations with viscosity \(\nu\) and if \(\varphi \in D((0, T) \times \Omega, \mathbb{R}^3)\), then derive the relation
\[
\int_0^T dt \int_{\partial \Omega} dA \left[-p'(n \cdot \varphi) + \tau^\nu_w \cdot \varphi \right] = \int_0^T dt \int_{\Omega} dV \left[(\partial_t - \nu \Delta) \varphi \cdot u'^\nu + \nabla \varphi : uu'^\nu + (\nabla \cdot \varphi)p'^\nu\right].
\]
If one assumes strong convergence
\[
\begin{align*}
    u' &\to u \text{ in } L^2((0, T), L^2_{\text{loc}}(\Omega)), \\
p' &\to p \text{ in } L^1((0, T), L^1_{\text{loc}}(\Omega))
\end{align*}
\]
as \(\nu \to 0\), then prove that
\[
\int_0^T dt \int_{\Omega} dV \left[(\partial_t - \nu \Delta) \varphi \cdot u'^\nu + \nabla \varphi : uu'^\nu + (\nabla \cdot \varphi)p'^\nu\right] \to \int_0^T dt \int_{\Omega} dV \left[\partial_t \varphi \cdot u + \nabla \varphi : uu + (\nabla \cdot \varphi)p\right]
\]

(b) If \((u, p)\) is a solution of the coarse-grained Euler equations in the sense of the equations in Problem 3(a) with \(\nu = 0\) for all \(h > \ell > 0\) and if \(\varphi \in D((0, T) \times \Omega, \mathbb{R}^3)\), then derive the relation

\[-\int_0^T dt \int_{\Omega} dV \nabla \eta_{h, \ell} \cdot (T_{\ell} + p_{\ell} I) \cdot \varphi = \int_0^T dt \int_{\Omega} dV \eta_{h, \ell} [ (\partial_t \varphi \cdot \overline{u}_\ell + \nabla \varphi \cdot \overline{T}_\ell + (\nabla \cdot \varphi) \overline{p}_\ell ].\]

for \(\eta_{h, \ell}(x) := \theta_{h, \ell}(d(x))\) and \(T := uu\). If \(u \in L^2((0, T), L^2_{\text{loc}}(\Omega))\), \(p \in L^1((0, T), L^1_{\text{loc}}(\Omega))\), then prove that

\[\int_0^T dt \int_{\Omega} dV \eta_{h, \ell} [ (\partial_t \varphi \cdot \overline{u}_\ell + \nabla \varphi \cdot \overline{T}_\ell + (\nabla \cdot \varphi) \overline{p}_\ell ] \to \int_0^T dt \int_{\Omega} dV [ (\partial_t \varphi \cdot u + \nabla \varphi \cdot uu + (\nabla \cdot \varphi) p] \]

as \(h, \ell \to 0\).

**Problem 5.** In this problem we derive the boundary vorticity flux relation for a smooth solution of incompressible Euler equation, in the form

\[-n \times \nabla p = D_t \gamma - n((\gamma \cdot \nabla) u) \cdot n\]

which is invoked in the force field method of Prandtl (1918). Here \(\gamma = n \times u\) is the strength of the boundary vorticity sheet; we assume that the Euler solution satisfies the no-penetration condition \(u \cdot n = 0\) so that \(u\) is tangent to the boundary \(\partial \Omega\). Our starting point is the alternative vorticity flux relation

\[-n \times \nabla p = D_t \gamma - (u \cdot n) n \times u\]

derived in the course notes.

(a) Show that the Weingarten matrix \(\nabla n\) is symmetric and use this result to derive

\[(u \cdot \nabla) n \times u = -(u \times \nabla S) n \cdot u = (u \times \nabla S) u \cdot n\]

where \(\nabla S\) is the surface gradient operator, i.e. the component of the gradient tangent to the boundary surface \(\partial \Omega\).

(b) Show that

\[u \times \nabla = n(\gamma \cdot \nabla) - \gamma (n \cdot \nabla)\]

and use this relation in the result of part (a) to show that

\[(u \cdot \nabla) n \times u = n((\gamma \cdot \nabla) u) \cdot n,\]

completing the derivation.