Problem 1. This problem considers a piecewise smooth vector field $F$ defined for a simply-connected open set $\Omega_1$ with smooth boundary $\partial \Omega = \partial \Omega_1$ and simply-connected open complement $\Omega_2 = \mathbb{R}^3 \setminus \bar{\Omega}_1$ such that

$$F(x) = \begin{cases} F_1(x) & x \in \Omega_1 \\ F_2(x) & x \in \Omega_2 \end{cases}$$

with smooth $F_1 : \Omega_1 \to \mathbb{R}^3$ and $F_2 : \Omega_2 \to \mathbb{R}^3$.

(a) Defining the distributional divergence of $F$ by

$$\int (\nabla \cdot F) \varphi \, dV = - \int F \cdot \nabla \varphi \, dV$$

for a $C^\infty$ and rapidly decaying scalar test function $\varphi$, show that

$$\nabla \cdot F = \{ \nabla \cdot F \} + \mathbf{n} \cdot [F] \delta (d)$$

where

$$\{ \nabla \cdot F \}(x) = \begin{cases} \nabla \cdot F_1(x) & x \in \Omega_1 \\ \nabla \cdot F_2(x) & x \in \Omega_2 \end{cases}$$

and where $d(x) = \text{dist}(x, \partial \Omega)$, $[F] = F_2 - F_1$ on $\partial \Omega$, and $\mathbf{n}$ is the unit normal on $\partial \Omega$ pointing from $\Omega_1$ into $\Omega_2$.

(a) Defining similarly the distributional curl of $F$ by

$$\int (\nabla \times F) \cdot \varphi \, dV = \int F \cdot (\nabla \times \varphi) \, dV$$

for a $C^\infty$ and rapidly decaying vector test function $\varphi$, show that

$$\nabla \times F = \{ \nabla \times F \} + \mathbf{n} \times [F] \delta (d)$$

where

$$\{ \nabla \times F \}(x) = \begin{cases} \nabla \times F_1(x) & x \in \Omega_1 \\ \nabla \times F_2(x) & x \in \Omega_2 \end{cases}$$

and all other definitions are the same as in part (a).
Problem 2. We study results relevant to the multipole expansion for the vector potential \( \psi(\mathbf{x}) \) of a differentiable, conditionally integrable, solenoidal velocity field \( \mathbf{u}(\mathbf{x}) \) in the domain \( \Omega = \mathbb{R}^3 \setminus B \) outside a smooth, simply-connected body \( B \).

(a) Prove that the integral involving the vorticity \( \mathbf{\omega} = \nabla \times \mathbf{u} \)
\[
\int_{\Omega} \mathbf{x}' \mathbf{\omega}' \, dV' + \int_{\partial B} \mathbf{x}'(\mathbf{n} \times \mathbf{u}') \, dA'
\]
is anti-symmetric in \( i \) and \( j \).

(b) Use the result in (a) to show that the vector impulse defined by
\[
\mathbf{I} = \frac{1}{2} \int_{\Omega} \mathbf{x}' \times \mathbf{\omega}' \, dV' + \frac{1}{2} \int_{\partial B} \mathbf{x}' \times (\mathbf{n} \times \mathbf{u}') \, dA'
\]
satisfies
\[
\mathbf{I} \times \mathbf{x} = \int_{\Omega} (\mathbf{x} \cdot \mathbf{x}') \mathbf{\omega}' \, dV' + \int_{\partial B} (\mathbf{x} \cdot \mathbf{x}') (\mathbf{n} \times \mathbf{u}') \, dA'.
\]

Problem 3. This problem gives a simple derivation of the relation between momentum and impulse, with the same assumptions on velocity \( \mathbf{u}(\mathbf{x}) \) as Problem 2.

(a) Derive the following identity involving the vorticity:
\[
\mathbf{x} \times \mathbf{\omega} = x_i \nabla u_i - (\mathbf{x} \cdot \nabla) \mathbf{u}.
\]

(b) Use the result in part (a) to show that impulse \( \mathbf{I} \) and momentum \( \mathbf{P} = \int_{\Omega} \mathbf{u} \, dV \) are related by
\[
2\mathbf{I} = 2\mathbf{P} + \lim_{R \to \infty} \int_{S_R} \mathbf{x} \times (\mathbf{x} \times \mathbf{u}) \, dA,
\]
where \( S_R \) is the sphere of radius \( R \) centered at the origin.

(c) Using the asymptotic far-field expansion
\[
\mathbf{u}(\mathbf{x}) \sim \frac{-I r^2 + 3(\mathbf{I} \cdot \mathbf{x}) \mathbf{x}}{4\pi r^5}, \quad r \to \infty,
\]
show that
\[
\lim_{R \to \infty} \int_{S_R} \mathbf{x} \times (\mathbf{x} \times \mathbf{u}) \, dA = \frac{2}{3} \mathbf{I}
\]
and conclude from part (b) that \( \mathbf{P} = \frac{2}{3} \mathbf{I} \).
**Problem 4.** With the same notations and assumptions as in the derivation of the Josephson-Anderson relation for external flow around a smooth body $B$, derive the following alternative expressions:

(a) \[ \int_{\Omega} \mathbf{u}_\phi \cdot \mathbf{u} \times \mathbf{\omega} \, dV = \int_{\Omega} \nabla \mathbf{u}_\phi : \mathbf{u}_\phi \mathbf{u}_\phi \, dV \]

(b) \[ \int_{\Omega} \mathbf{u}_\phi \cdot \mathbf{\nu} \nabla \times \mathbf{\omega} \, dV = \int_{\partial B} \mathbf{u}_\phi \cdot \mathbf{\tau}_w \, dA. \]

Carefully justify the neglect of boundary terms in integration by parts.

**Problem 5.** We consider in this problem the general translational motion of a solid body through an incompressible fluid at rest at infinity. The body is represented by the time-dependent set

\[ B(t) = B + \mathbf{X}(t) \]

where $\mathbf{X} : [0, T] \rightarrow \mathbb{R}^3$ is a smooth function with $\mathbf{X}(0) = 0$ and $B$ is a simply-connected open set with a smooth boundary $\partial B$. Set $\mathbf{V}(t) = \dot{\mathbf{X}}(t)$ and $\mathbf{A}(t) = \ddot{\mathbf{X}}(t)$.

(a) The incompressible Navier-Stokes solution $(\bar{\mathbf{u}}(\bar{x}, t), \bar{p}(\bar{x}, t))$ in the space domain $\Omega(t) = \mathbb{R}^3 \setminus B(t)$ for the fluid reference frame satisfies the boundary conditions

\[ \bar{\mathbf{u}} = \mathbf{V}(t) \quad \text{on} \quad \partial B(t); \quad \bar{\mathbf{u}} \rightarrow 0 \quad \text{as} \quad |\bar{x}| \rightarrow \infty. \]

Show that the transformations

\[ \mathbf{u}(\mathbf{x}, t) = \bar{\mathbf{u}}(\mathbf{x} + \mathbf{X}(t), t) - \mathbf{V}(t), \quad p(\mathbf{x}, t) = \bar{p}(\mathbf{x} + \mathbf{X}(t), t) + \mathbf{A}(t) \cdot \mathbf{x} \]

give the solution of the incompressible Navier-Stokes equation in the space domain $\Omega = \mathbb{R}^3 \setminus B$ for the body frame, which satisfies the boundary conditions

\[ \mathbf{u} = 0 \quad \text{on} \quad \partial B; \quad \mathbf{u} \rightarrow -\mathbf{V}(t) \quad \text{as} \quad |\mathbf{x}| \rightarrow \infty. \]

(b) For the same situation as in part (a), consider the potential solution $\bar{\mathbf{u}}_\phi = \nabla \bar{\phi}$ of the incompressible Euler equations in the fluid frame, with $\bar{\phi}(\bar{x}, t)$ solving the Laplace equation $\Delta \bar{\phi} = 0$ in $\Omega(t)$ with boundary conditions

\[ \frac{\partial \bar{\phi}}{\partial n} = \mathbf{V}(t) \cdot \mathbf{n} \quad \text{on} \quad \partial B(t); \quad \bar{\phi} \rightarrow 0 \quad \text{as} \quad |\bar{x}| \rightarrow \infty, \]

and with pressure $\bar{p}_\phi(\bar{x}, t)$ given by the Bernoulli equation

\[ \partial_t \bar{\phi} + \frac{1}{2} |\nabla \bar{\phi}|^2 + \bar{p}_\phi = \bar{c}(t) \]

for some arbitrary function $\bar{c}(t)$. Show that the transformations

\[ \phi(\mathbf{x}, t) = \bar{\phi}(\mathbf{x} + \mathbf{X}(t), t) - \mathbf{V}(t) \cdot \mathbf{x}, \quad p_\phi(\mathbf{x}, t) = \bar{p}_\phi(\mathbf{x} + \mathbf{X}(t), t) + \mathbf{A}(t) \cdot \mathbf{x} \]


give the solution of the incompressible Euler equation in the space domain \( \Omega = \mathbb{R}^3 \setminus B \) for the body frame, where \( p_\phi \) is obtained from the Bernoulli equation and \( \phi \) satisfies the Laplace equation \( \Delta \phi = 0 \) in \( \Omega \) with the boundary conditions

\[
\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \; \partial B; \quad \phi \to -\mathbf{V}(t) \cdot \mathbf{x} \quad \text{as} \; |\mathbf{x}| \to \infty.
\]

Problem 6. In this problem we derive a *generalized d’Alembert theorem* for the arbitrary translational motion of a solid body through an incompressible fluid at rest at infinity, working in the body frame as in part (b) of Problem 5.

(a) Following Lighthill (1979) we define a “pseudo-momentum” of the potential Euler solution by

\[
P_\phi = -\int_{\partial B} \phi \mathbf{n} \, dA.
\]

Explain why the usual momentum \( \int_{\Omega} \mathbf{u}_\phi \, dV \) diverges, but coincides with \( P_\phi \) up to an infinite constant and show that

\[
\frac{dP_\phi}{dt} = \int_{\partial B} \left( p_\phi + \frac{1}{2} \left| \mathbf{u}_\phi \right|^2 \right) \mathbf{n} \, dA.
\]

(b) Prove that \( \nabla \left( \frac{1}{2} \left| \mathbf{u}_\phi \right|^2 \right) = (\mathbf{u}_\phi \cdot \nabla) \mathbf{u}_\phi \) and exploit this relation and the methods used to prove the d’Alembert theorem to show that

\[
\int_{\partial B} \frac{1}{2} \left| \mathbf{u}_\phi \right|^2 \mathbf{n} \, dA = 0.
\]

(c) Conclude from parts (a) and (b) that

\[
\frac{dP_\phi}{dt} = \int_{\partial B} p_\phi \mathbf{n} \, dA = F_\phi
\]

where \( F_\phi \) is the force of the body acting on the fluid. Conclude that the time-average

\[
F_\phi := \lim_{T \to \infty} \frac{1}{T} \int_0^T F_\phi(t) \, dt = 0
\]

whenever \( \mathbf{V}(t) \) and thus \( \mathbf{P}_\phi(t) \) remain bounded in time.