Problem 1. Following the hint, we write
\[ \nabla = t \frac{\partial}{\partial s} + n \frac{\partial}{\partial n} \]
near the boundary \( C = \partial \Omega \). Note first that
\[ \nabla u \bigg|_C = t \frac{\partial u}{\partial s} + n \frac{\partial u}{\partial n} \bigg|_C = n \nabla u \]
since \( u \equiv 0 \) on \( C \) and thus \( \partial u / \partial s \equiv 0 \) as well. For the Laplacian \( \Delta u \) we can likewise calculate
\[
\Delta u = \left( t \frac{\partial}{\partial s} + n \frac{\partial}{\partial n} \right) \cdot \left( t \frac{\partial u}{\partial s} + n \frac{\partial u}{\partial n} \right)
\]
\[ = \left( \frac{\partial^2 u}{\partial s^2} + t \cdot \frac{\partial t}{\partial s} \frac{\partial u}{\partial s} + n \cdot \frac{\partial t}{\partial n} \frac{\partial u}{\partial n} \right)
\]
\[ + \left( t \cdot \frac{\partial n}{\partial s} \frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial n^2} \right) \]
where the first parenthetical expression arises from the divergence of \( t \frac{\partial u}{\partial s} \) and the second expression arises from \( n \frac{\partial u}{\partial n} \).

We have made use of \( t \cdot n = 0 \) to eliminate some terms.
If we now consider the restriction to the wall, then
\[ \Delta u \bigg|_C = t \cdot \frac{\partial n}{\partial s} \frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial n^2} \bigg|_C \]
since again \( \partial u / \partial s = \partial^2 u / \partial s^2 \equiv 0 \) on \( C \).
Recalling also the relation
\[ t \cdot \frac{\partial n}{\partial s} = -n \cdot \frac{\partial t}{\partial s} \]

obtained by differentiating \( t \cdot n = 0 \), we then use the result from the hint
\[ \frac{\partial t}{\partial s} = \pm k n \]

which follows from the Frenet-Serre equations and our definition of \( n \) as pointing inward to \( S \). Thus,
\[ \Delta u \bigg|_C = \mp k \left( \frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial n^2} \right) \bigg|_C, \]

and using the Navier-Stokes equation on the boundary
\[ \nu \Delta u - \nabla p = 0 \text{ on } C, \]

we obtain
\[ \nu \frac{\partial^2 u}{\partial n^2} = \pm k T_w + \nabla p \text{ on } C. \]

We have set \( T_w = \nu \partial w \). Now taking the tangential component
\[ \nu \frac{\partial^2 u_t}{\partial n^2} = \pm k T_w + \frac{\partial p}{\partial s} \]

whereas the normal component gives
\[ \nu \frac{\partial^2 u_n}{\partial n^2} = \frac{\partial p}{\partial n}, \]

since \( n \cdot T_w = 0 \).
Problem 2. (a) From the defining equations
\[ x = u, \quad y = v \]
we obtain
\[ \dot{x}_\alpha = u_x x_\alpha + u_y y_\alpha, \quad \alpha = \xi, \eta \]
\[ \dot{y}_\alpha = v_x x_\alpha + v_y y_\alpha, \quad \alpha = \xi, \eta \]
by taking derivatives of both sides with respect to either \( \xi \) or \( \eta \). In that case,
\[
\frac{d}{dt} (x_\xi y_\eta - x_\eta y_\xi)
= (u_x x_\xi + u_y y_\xi) y_\eta + x_\xi (v_x x_\eta + v_y y_\eta)
- (u_x x_\eta + u_y y_\eta) y_\xi - x_\eta (v_x x_\xi + v_y y_\xi)
= (u_x + v_y) (x_\xi y_\eta - x_\eta y_\xi)
\]
Note that at the initial time \( x_\xi = y_\eta = 1 \), \( x_\eta = y_\xi = 0 \) so that
\[ x_\xi y_\eta - x_\eta y_\xi \bigg|_{t=0} = 1. \]
Since this linear equation for \( (x_\xi y_\eta - x_\eta y_\xi) \) has a unique solution, it follows that
\[ u_x + v_y \equiv 0 \quad \text{iff} \quad x_\xi y_\eta - x_\eta y_\xi \equiv 1. \]
(b) Note that

\[
\begin{bmatrix}
\xi_x & \xi_y \\
\eta_x & \eta_y
\end{bmatrix}
= \begin{bmatrix}
x_\xi & x_\eta \\
y_\xi & y_\eta
\end{bmatrix}^{-1}
\]

since \((\xi(x,y), \eta(x,y))\) is the inverse map to \((x(\xi,\eta), y(\xi,\eta))\).

However, because

\[
\begin{vmatrix}
x_\xi & x_\eta \\
y_\xi & y_\eta
\end{vmatrix}
= x_\xi y_\eta - x_\eta y_\xi = 1
\]

\[
\begin{bmatrix}
x_\xi & x_\eta \\
y_\xi & y_\eta
\end{bmatrix}
= \begin{bmatrix}
y_\eta & -x_\eta \\
-y_\xi & x_\xi
\end{bmatrix}
\]

and thus we read off

\[
\xi_x = y_\eta, \quad \xi_y = -x_\eta,
\]

\[
\eta_x = -y_\xi, \quad \eta_y = x_\xi.
\]

(c) We note that by the chain rule and the equation \(\dot{x} = u\)

\[
\dot{x} = \frac{d}{dt} u(x,y,t)
\]

\[
= u_t + \dot{x}u_x + \dot{y}u_y
\]

\[
= u_t + uu_x + vv_y
\]

Next, we use the x-momentum equation of the Prandtl theory to infer that
\[ \dot{x} = -p_x + \partial_y \gamma \]

We next use

\[
\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta}
\]

\[ = -x \eta \frac{\partial}{\partial \xi} + x \xi \frac{\partial}{\partial \eta} \]

by exploiting the relations from part (b). However

\[ u = \dot{x} \]

and

\[
\frac{\partial^2}{\partial y^2} = \left( x_\eta \frac{\partial}{\partial \xi} - x_\xi \frac{\partial}{\partial \eta} \right)^2
\]

so that we obtain the final result that

\[ \ddot{x} = -p_x(x) + (x_\eta \frac{\partial}{\partial \xi} - x_\xi \frac{\partial}{\partial \eta})^2 \dot{x}. \]

This is a second-order in time, nonlinear PDE for the Lagrangian flow map \( x(\xi, \eta) \). The other component \( y(\xi, \eta) \) can be obtained from the relation

\[ x \xi y \eta - x_\eta y \eta \xi = 1. \]
Problem 3. (a) First we note that from \( \mathbf{n} = \nabla d(x) \)

\[
\frac{\partial \mathbf{n}}{\partial n} = (\mathbf{n} \cdot \nabla) \mathbf{n} = \mathbf{n} \cdot (\nabla \nabla d) \\
= n_i \nabla n_i \quad \text{ (summing over repeated i)} \\
= \nabla \left( \frac{1}{2} \mathbf{n} \cdot \mathbf{n} \right) \\
= 0
\]

since \( \mathbf{n} \cdot \mathbf{n} = 1 \). Then using the above

\[
\mathbf{n} \cdot \mathbf{e}_w = \mathbf{n} \cdot \left. \frac{\partial \mathbf{u}}{\partial n} \right|_s = \left. \frac{\partial}{\partial n} (\mathbf{n} \cdot \mathbf{u}) \right|_s = 0
\]

since by incompressibility and stick b.c.

\[
\frac{\partial \mathbf{u}_n}{\partial n} = -\nabla_s \cdot \mathbf{u}_s \equiv 0 \quad \text{on } S,
\]

(b) We begin with

\[
\tau_w = 2 \nu S \cdot \mathbf{n} = \nu \left( \frac{\partial \mathbf{u}}{\partial n} + \nabla \mathbf{u} \cdot \mathbf{n} \right) \bigg|_S \\
= \nu \frac{\partial \mathbf{u}}{\partial n} + \nu \nabla (\mathbf{u} \cdot \mathbf{n}) \bigg|_S
\]

since \( \mathbf{u} \cdot \nabla \mathbf{n} \equiv 0 \) on \( S \). However, by part (a)

\[
\frac{\partial}{\partial n} (\mathbf{u} \cdot \mathbf{n}) = \frac{\partial \mathbf{u}_n}{\partial n} = 0
\]

and

\[
\nabla_s \mathbf{u}_n \equiv 0 \quad \text{on } S
\]

because of the stick b.c. Thus, we see that
\[ \mathbf{T}_w = \nu \frac{\partial \mathbf{u}}{\partial n} = \nu \mathbf{e}_w \text{ on } S. \]

For the second relation we perform a similar calculation

\[
(w \times n)_i = \epsilon_{ijk} w_j n_k
\]

\[
= \epsilon_{ijk} \epsilon_{jlm} \partial_l u_m n_k
\]

\[
= (\delta_{im} \delta_{lk} - \delta_{il} \delta_{km}) \partial_l u_m n_k
\]

\[
= (n_k \partial_k) u_i - (\partial_i u_k) n_k
\]

\[
= \frac{\partial u_i}{\partial n} - 0 \quad \text{since } (\partial_i u_k) n_k = \partial_i (u_k n_k) = 0 \text{ as above}
\]

or

\[
w \times n = \frac{\partial \mathbf{u}}{\partial n} = \mathbf{e}_w.
\]

Thus,

\[ \mathbf{T}_w = \nu \mathbf{e}_w = \nu w \times n. \]

(c) We note that

\[ n_i K_{ij} = -\frac{\partial n_j}{\partial n} = 0 \]

from part (a), while

\[ K_{ij} n_j = -(\partial_i n_j) n_j = -\partial_i (\frac{1}{2} n_j n_j) = 0 \]

since \( n_j n_j = 1 \).
Incompressibility implies that
\[ \frac{\partial u_s}{\partial n} + \nabla_s \cdot \mathbf{u}_s = 0 \]
Taking a second derivative \( \partial / \partial n \) then gives
\[ -\frac{\partial^2 u_s}{\partial n^2} = \nabla_s \cdot \frac{\partial u_s}{\partial n} = \nabla_s \cdot \mathbf{e}_w \text{ on } S. \]
Next we use \( \mathbf{e}_w = \mathbf{w}_w \times \mathbf{n} \) from part (b) and a vector calculus identity to obtain
\[ \nabla_s \cdot \mathbf{e}_w = \nabla_s \cdot (\mathbf{w}_w \times \mathbf{n}) \]
\[ = (\nabla_s \times \mathbf{w}_w) \cdot \mathbf{n} - \mathbf{w}_w \cdot (\nabla \times \mathbf{n}) \]
Note that
\[ \mathbf{w}_w \cdot (\nabla \times \mathbf{n}) = \varepsilon_{ijk} \mathbf{w}_i \delta_{jnk} = -\varepsilon_{ijk} \mathbf{w}_i \mathbf{K}_{jk}. \]
However, \( \mathbf{w} \) is tangent to \( S \) and can thus be written as \( \mathbf{w} = \alpha \mathbf{e} + \beta \mathbf{f} \) where \( \mathbf{e}, \mathbf{f} \) is a basis for the tangent space. Likewise, from part (c) we see that
\[ \nabla \mathbf{m} = \gamma \mathbf{e} \mathbf{e} + \delta \mathbf{e} \mathbf{f} + \epsilon \mathbf{f} \mathbf{e} + \theta \mathbf{f} \mathbf{f} \]
so that \( \varepsilon_{ijk} \mathbf{w}_i \delta_{jnk} = 0. \) It follows that
\[ \nabla_s \cdot \mathbf{e}_w = (\nabla_s \times \mathbf{w}_w) \cdot \mathbf{n} = (\mathbf{n} \times \nabla_s) \cdot \mathbf{w}_w. \]
\( \Box \)
Problem 4. (a) The linearization at the fixed point $A = \nabla u(x^* y^*)$ is

$$
A = \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
$$

which has $\det A = -1 < 0$ and is thus a saddle point. In fact, the $y$-axis is the stable manifold and the $x$-axis is the unstable manifold. Note also that

$$
\frac{d}{dt} (xy) = \dot{x} y + x \dot{y} = x \cdot y + x \cdot (-y) = 0
$$

so that phase orbits are given by hyperbolae with $xy = \text{const.}$ The phase portrait looks like:

On a circle around the critical point at the origin we indicate the unit vectors $\hat{u}$ at eight points numbered counterclockwise. These are the vectors
Clearly, these vectors rotate once clockwise so that

\[ \text{Index} = -1. \]

(b) The linearization at the fixed point is

\[ A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \]

which has \( \det A = 2 > 0, \text{ tr} A = -3 < 0, \text{ Dis} A = 9 - 8 = 1 > 0 \)
so that the critical point is a stable node. Note that

\[
\frac{d}{dt} \left( \frac{y}{x^2} \right) = \frac{y}{x^3} \left( \dot{y} x - 2y \dot{x} \right)
\]

\[
= \frac{y}{x^3} \left( -2y \dot{x} + 2y \dot{x} \right) = 0
\]

so that phase orbits are parabolas \( y = Cx^2, \ C = \text{const.} \)

The phase portrait is:

![Phase portrait diagram](image-url)
The vector plot is now

so that \( \mathbf{u} \) rotates **counterclockwise** and thus

\[
\text{Index} = +1
\]

(c) The linearization at the fixed point is

\[
A = \begin{bmatrix}
-2 & -2 \\
1 & 0
\end{bmatrix}
\]

so that \( \det A = 2 > 0 \), \( \text{tr} A = -2 < 0 \), \( \text{dis} A = 4 - 8 = -4 < 0 \) and thus the critical point is a **stable spiral**. It is easiest here to consider a square box around the critical point.
Which yields the vector plot

so that again $\mathbf{u}$ rotates once counterclockwise and thus

$$\text{Index} = +1.$$  

Note that the continuity of the index under changes of the contour $C$ can be inferred from the integral expression

$$\text{Index}(f, x_*) = \frac{1}{2\pi i} \oint_C \frac{f(x) \, df}{\|f\|^2}.$$  

Since the index is continuous in $C$ but also integer-valued, it does not change under deformations of $C$, as long as these do not hit the critical point $x_*$. Thus, the value is independent of the particular contour $C$ selected to encircle the critical point.
Problem 5. (a) Note by change of variable $s \rightarrow y = sy$ that

$$\bar{u}_y(x, y, t) := \int_0^1 u_y(x, sy, t) \, ds$$

$$= \frac{1}{y} \int_0^y u_y(x, y, t) \, dy$$

$$= \frac{1}{y} \left[ u(x, y, t) - u(x, 0, t) \right]$$

$$= \frac{1}{y} u(x, y, t) \quad \text{since } u(x, 0, t) = 0$$

and thus

$$u(x, y, t) = \bar{u}_y(x, y, t) \cdot y.$$  

Similarly, we change the variable $s \rightarrow y = ry$ to obtain

$$\int_0^1 u_{yy}(x, rsy, t) \, r \, ds$$

$$= \frac{1}{y} \int_0^y u_y(x, ry, t) \, dy$$

$$= \frac{1}{y} \left[ u_y(x, ry, t) - u_y(x, 0, t) \right]$$

$$= \frac{1}{y} u_y(x, ry, t)$$

Since $u_y(x, 0, t) = -u_x(x, 0, t) = 0$ by incompressibility and stick b.c.
In that case, using the definition of $\overline{v_{yy}}(x,y,t)$

\[
\int_0^1 \int_0^1 v_{yy}(x,rsy,t)2r \, dr \, ds
\]

\[
= \frac{2}{y} \int_0^1 v_y(x,ry,t) \, dr \quad \quad r \to y = ry
\]

\[
= \frac{2}{y^2} \int_0^y v_y(x,y,t) \, dy
\]

\[
= \frac{2}{y^2} \left[ v(x,y,t) - v(x,0,t) \right]
\]

\[
= \frac{2}{y^2} v(x,y,t)
\]

and thus

\[
v(x,y,t) = \frac{1}{2} \overline{v_{yy}}(x,y,t) y^2.
\]

(b) By incompressibility

\[
0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x} (\overline{uy} \, y) + \frac{\partial}{\partial y} \left( \frac{1}{2} \overline{v_{yy}} y^2 \right)
\]

\[
= y \cdot \overline{uy} + \frac{1}{2} y^2 \cdot \overline{v_{yy}} + y \cdot \overline{v_{yy}}
\]

\[
\Rightarrow \quad \overline{uy} + \overline{v_{yy}} + \frac{1}{2} \overline{v_{yy}} y = 0.
\]
(c) Finally, $x = \gamma + y F(y, t)$ is a material curve if and only if each point of it evolves according to the equations

$$\dot{x} = u = \bar{u}_y y$$
$$\dot{y} = v = \frac{1}{2} \bar{v}_{yy} y^2$$

Substituting into the first equation yields by chain rule

$$y F_t(y, t) + \dot{y} F(y, t) + y \dot{y} F_y(y, t)$$

$$= \bar{u}_y (x, y, t) y$$

$$= \bar{u}_y (\gamma + y F, y, t) y$$

so that

$$F_t(y, t) = \bar{u}_y (\gamma + y F, y, t) - \frac{\dot{y}}{y} (F + y F_y).$$

Now substituting from the second equation for $\dot{y}$ gives

$$F_t(y, t) = \bar{u}_y (\gamma + y F, y, t)$$

$$- \frac{1}{2} y \bar{v}_{yy} (\gamma + y F, y, t) (F + y F_y)$$

QED
Problem 6. (a) The vortex at \((x,y)=(0,a)\) has velocity

\[ u_+ = \frac{-k(y-a)}{x^2+(y-a)^2}, \quad v_+ = \frac{-kx}{x^2+(y-a)^2} \]

while the vortex at \((x,y)=(0,-a)\) has velocity

\[ u_- = \frac{-k(y+a)}{x^2+(y+a)^2}, \quad v_- = \frac{kx}{x^2+(y+a)^2}. \]

The vortex at \((x,y)=(0,a)\) feels the velocity of the other vortex, which is

\[ u_-(0,a) = \frac{-k}{2a}, \quad v_-(0,a) = 0. \]

On the other hand, the vortex at \((x,y)=(0,-a)\) feels the velocity of the first vortex, which is

\[ u_+(0,-a) = \frac{-k}{2a}, \quad v_+(0,-a) = 0. \]

Thus, the pair of vortices are mutually advected with the velocities

\[ U_{\text{vort}} = \frac{-k}{2a}, \quad V_{\text{vort}} = 0 \]

and move parallel to the \(x\)-axis together.
(b) When the point vortices are located at \((x, y) = (uv_{0} + t, \pm a)\), then the resultant velocity at all other points is given by

\[
U = \frac{k(y-a)}{(x-uv_{0}+t)^2 + (y-a)^2} - \frac{k(y+a)}{(x-uv_{0}+t)^2 + (y+a)^2}
\]

\[
V = \frac{-k(x-uv_{0}t)}{(x-uv_{0}+t)^2 + (y-a)^2} + \frac{k(x-uv_{0}t)}{(x-uv_{0}+t)^2 + (y+a)^2}
\]

In particular, at \(y=0\)

\[
V(x, 0, t) = \frac{-k(x-uv_{0}t)}{(x-uv_{0}+t)^2 + a^2} + \frac{k(x-uv_{0}t)}{(x-uv_{0}+t)^2 + a^2}
\]

\[= 0.\]

Thus, we see that the velocity in the upper half-plane satisfies the no-flow-through condition at \(y=0\).

The vortex in the lower half-plane is the so-called image vortex required to enforce that condition for the flow in the upper half-plane.

(c) Since vortices are material points in 2D, adding the uniform velocity \(U\) changes the velocity of both vortices to

\[u_{\text{vort}} = U - \frac{k}{2a}\]
Using this new definition of $U_{vort}$, we then obtain

$$U = U + \frac{k(y-a)}{(x-U_{vort}+t)^2+(y-a)^2} - \frac{k(y+a)}{(x-U_{vort}+t)^2+(y+a)^2}$$

$$V = -\frac{k(x-U_{vort}+t)}{(x-U_{vort}+t)^2+(y-a)^2} + \frac{k(x-U_{vort}+t)}{(x-U_{vort}+t)^2+(y+a)^2}.$$ 

(d) The streamwise velocity on the boundary $y=0$ now becomes

$$U = U - \frac{2ka}{(x-U_{vort}+t)^2+a^2}$$

Note that

$$\alpha = \frac{U_{vort}}{U} = 1 - \frac{k}{2aU}$$

$$\implies k = 2a U (1-\alpha)$$

Thus,

$$\frac{U_{vort}}{U} = 1 - \frac{4a^2 (1-\alpha)}{(x-U_{vort}+t)^2+a^2} \quad \text{and} \quad U_{vort} = \alpha U.$$ 

As long as $\alpha < \frac{3}{4}$, then $a^2 (1-\alpha) > 0$ and we see that the most negative value is achieved for $x = U_{vort} t$ where

$$\frac{|u|_{\text{max}}}{U} = 4(1-\alpha) - 1 > 0.$$
Problem 7. (a) Noting that $U^2 = U \cdot U$, we see that

$$\nabla \left( \frac{1}{2} U^2 \right) = U_j \nabla U_j \quad \text{(sum over repeated } i \text{)}$$

However,

$$\partial_i U_j = \partial_i (\partial_j \phi) = \partial_j (\partial_i \phi) = \partial_j U_i$$

is symmetric in $i, j$ (i.e. there is no vorticity for potential flow!). Thus

$$U_j \partial_i U_j = U_j \partial_j U_i = (U \cdot \nabla) U_i$$

and

$$\nabla \left( \frac{1}{2} U^2 \right) = (U \cdot \nabla) U.$$

(b) From the Bernoulli equation we obtain

$$- \nabla P = \nabla \left( \frac{1}{2} U^2 \right) = (U \cdot \nabla) U$$

by part (a). This, of course, is just the stationary Euler equation! Now substituting $U = Ut_s$ and noting that $t_s \nabla = \frac{\partial}{\partial s}$ gives

$$- \nabla P = U \frac{\partial}{\partial s} (U t_s)$$

$$= U \frac{\partial U}{\partial s} t_s + U^2 \frac{\partial t_s}{\partial s}$$
Now, using the Frenet-Serret equation
\[ \frac{\partial t_s}{\partial s} = k_s n_s \]
and \( U \frac{\partial U}{\partial s} = \frac{\partial}{\partial s} \left( \frac{1}{2} U^2 \right) \), we get finally that
\[ -\nabla P = \frac{\partial}{\partial s} \left( \frac{1}{2} U^2 \right)t_s + U^2 k_s n_s. \]

(c) Assuming continuity of the pressure gradient across the thin boundary layer, we calculate the Lighthill source at the wall surface as
\[ \sigma = -n \times \nabla P = \frac{\partial}{\partial s} \left( \frac{1}{2} U^2 \right)n \times t_s + U^2 k_s n \times n_s. \]
Since \( t_s \) is, by definition, the streamwise direction, then \( n \times t_s \) is the spanwise direction and the term \( \frac{\partial}{\partial s} \left( \frac{1}{2} U^2 \right) \) creates spanwise vorticity. On the other hand, if the streamlines bend in a direction perpendicular to both \( n \) and \( t_s \), e.g. to pass around an obstacle in the flow, then
\[ \sigma \cdot t_s = U^2 k_s (n \times n_s) \cdot t_s \neq 0 \]
and the term \( \alpha U^2 k_s \) can thus generate streamwise vorticity.