Homework No.2, 550.794, Due March 6, 2023.

Problem 1. Suppose that a smooth curve C is the boundary $\partial\Omega$ of a flow domain $\Omega \in \mathbb{R}^2$ where $\mathbf{u} : \Omega \to \mathbb{R}^2$ is a stationary solution of the Navier-Stokes equation with stick b.c. If $\partial/\partial n$ is the derivative along the direction of $\mathbf{n}$, the unit normal at the wall pointing into the fluid, and if $\partial/\partial s$ is the derivative in arclength s along C, then show that the tangential velocity u_t and wall-normal velocity u_n on C satisfy

$$(i) \quad \nu \tfrac{\partial^2 u_t}{\partial n^2} = \pm \kappa \tau_w + \tfrac{\partial p}{\partial s}$$

$$(ii) \quad \nu \tfrac{\partial^2 u_n}{\partial n^2} = \tfrac{\partial p}{\partial n}$$

where κ is the curvature of the boundary and $+$ sign holds at points where the center of curvature lies in the fluid and the $-$ sign holds otherwise.

Hint: Let $\mathbf{t}$ be the unit tangent vector along C in the direction of increasing s and then write the gradient as $\boldsymbol{\nabla} = \mathbf{t}\frac{\partial}{\partial s} + \mathbf{n}\frac{\partial}{\partial n}$ and use the well-known relation $\partial \mathbf{t}/\partial s = \pm\kappa\mathbf{n}$.

Problem 2. Let $x(\xi, \eta, t)$, $y(\xi, \eta, t)$ be the Lagrangian position variables defined by

$$\dot{x} = u(x, y, t)$$
$$\dot{y} = v(x, y, t)$$

and $x(\xi, \eta, 0) = \xi$, $y(\xi, \eta, t) = \eta$.

(a) Show by direct calculation that

$$\frac{d}{dt}(x_\xi y_\eta - x_\eta y_\xi) = (u_x + v_y)(x_\xi y_\eta - x_\eta y_\xi)$$

and thus incompressibility holds if and only if $x_\xi y_\eta - x_\eta y_\xi \equiv 1$. Assume true hereafter.

(b) Derive the following relations

$$\xi_x = y_\eta, \quad \xi_y = -x_\eta, \quad \eta_x = -y_\xi, \quad \eta_y = x_\xi.$$

(c) Show that Prandtl's equation for the velocity u in Lagrangian formulation becomes

$$\ddot{x} = -p_x(x, t) + (x_\eta \partial_\xi - x_\xi \partial_\eta)^2 \dot{x}.$$

Problem 3. In this problem we consider an incompressible velocity field $\mathbf{u}$ satisfying stick b.c. at a smooth 2D surface $S \subset \mathbb{R}^3$ with unit normal $\mathbf{n}$ pointing into the fluid:

(a) Defining the surface strain field by $\boldsymbol{\epsilon}_w = \frac{\partial \mathbf{u}}{\partial n}\big|_S$, show that $\mathbf{n} \cdot \boldsymbol{\epsilon}_w = 0$.

(b) Defining the viscous skin friction by $\boldsymbol{\tau}_w = 2\nu \mathbf{S} \cdot \mathbf{n}|_S$, where $S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ is the strain rate tensor, show that

$$\boldsymbol{\tau}_w = \nu \boldsymbol{\epsilon}_w = \nu \boldsymbol{\omega}_w \times \mathbf{n}$$

where $\boldsymbol{\omega}_w = \boldsymbol{\nabla} \times \mathbf{u}|_S$ is the wall vorticity.

(c) Defining the Weingarten tensor by $\mathbf{K} = -\boldsymbol{\nabla} \mathbf{n}|_S$, show that $n_i K_{ij} = K_{ij} n_j = 0$.

(d) Defining $\Delta = -\left.\frac{\partial^2 u_n}{\partial n^2}\right|_S$ and using parts (b) and (c), show that

$$\Delta = \boldsymbol{\nabla}_s \boldsymbol{\cdot} \boldsymbol{\epsilon}_w = (\mathbf{n} \times \boldsymbol{\nabla}_s) \boldsymbol{\cdot} \boldsymbol{\omega}_w$$

where $\boldsymbol{\nabla}_s$ is the surface gradient.

Problem 4. The index of a critical point $\mathbf{x}_* = (x_*, y_*)$ for a 2D velocity field $\mathbf{u} = (u(x, y), v(x, y))$ is defined by taking any simple closed loop C encircling that critical point and no other, and then defining the index as the winding number of the map $S^1 \to S^1$ that takes $\mathbf{x} \in C \simeq S^1 \mapsto \hat{\mathbf{u}}(\mathbf{x}) := \mathbf{u}(\mathbf{x})/|\mathbf{u}(\mathbf{x})| \in S^1$. More concretely, the index is the integer number of times that the unit vector $\hat{\mathbf{u}}(\mathbf{x})$ rotates around S^1 as the point $\mathbf{x}$ moves once around $C \simeq S^1$ in the positive (counterclockwise) direction.

(a) Show that the vector field $(u, v) = (x, -y)$ has the critical point $(x_*, y_*) = (0, 0)$ corresponding to a saddle point with index=-1.

(b) Show that the vector field $(u, v) = (-x, -2y)$ has the critical point $(x_*, y_*) = (0, 0)$ corresponding to a stable node with index=+1.

(c) Show that the vector field $(u, v) = (-2x - 2y, x)$ has the critical point $(x_*, y_*) = (0, 0)$ corresponding to a stable spiral with index=+1.

To calculate the index, you may use a qualitative geometric argument.

Remark: It can be shown that the index of each critical point is independent of the specific curve C (because it is an integer-valued function continuous under changes of C—try it!). Also the above vector fields are "normal forms" of general saddles, stable nodes and stable spirals, so that the results for the index of these particular types of critical points hold in general.

Problem 5. (a) Show that any twice-continuously differentiable, incompressible velocity field (u, v) which is defined in the upper 2D half-plane for (x, y) with $y \geq 0$ and which also vanishes at $y = 0$ can be written exactly as

$$u(x, y, t) = \bar{u}_y(x, y, t)y, \quad v(x, y, t) = \frac{1}{2}\bar{\bar{v}}_{yy}(x, y, t)y^2$$

with

$$\bar{u}_y(x, y, t) := \int_0^1 u_y(x, sy, t)\, ds, \quad \bar{\bar{v}}_{yy}(x, y, t) := \int_0^1 \int_0^1 v_{yy}(x, rsy, t)\, 2r dr\, ds.$$

(b) Show that incompressibility implies that

$$\bar{u}_{y,x} + \bar{\bar{v}}_{yy} + \frac{1}{2} y \bar{\bar{v}}_{yy,y} = 0.$$

(c) Assuming that there exists a material line given by the graph $x = \gamma + yF(y, t)$, derive the equation

$$F_t = \bar{u}_y(\gamma + yF, y, t) - \frac{1}{2} y \bar{\bar{v}}_{yy}(\gamma + yF, y, t)(F + yF_y)$$

for the function $F(y, t)$.

Problem 6. The 2D velocity field of a point vortex located at the origin $(x, y) = 0$ with circulation κ is given by

$$u = \frac{\kappa y}{r^2}, \quad v = -\frac{\kappa x}{r^2}, \quad r^2 = x^2 + y^2.$$

(a) If there is pair of vortices, one with circulation κ initially located at $(x, y) = (0, a)$ and the other with circulation $-\kappa$ initially located at $(x, y) = (0, -a)$, then explain why the vortex pair move together in the x-direction with velocity u_{vort} and calculate this velocity.

(b) Show that $v = 0$ at $y = 0$ for the resultant velocity field of the vortex pair in (a). Write down an explicit formula for the velocity field $(u(x, y, t), v(x, y, t))$ of the pair in the upper half-plane, which thus represents the Euler solution for a *single* vortex of circulation κ starting at $(x, y) = (0, a)$ with the no-flow-through condition at $y = 0$.

(c) Now add a constant velocity field U in the x-direction. What is now the velocity u_{vort} of the vortex pair? Write down an explicit formula for the velocity field $(u(x, y, t), v(x, y, t))$ in the upper half-plane due both to the point vortex and to the uniform external flow U.

(d) Using the ratio $\alpha = u_{vort}/U$ to eliminate u_{vort}, write down the streamwise velocity u/U on the boundary $y = 0$ for the flow in (c) and show that it has a maximum magnitude at the instantaneous x-location of the vortex for all $\alpha < \frac{3}{4}$.

Problem 7. Suppose that $(\mathbf{U}, P)$ is a steady potential-flow solution of the 3D Euler equations, so that $\mathbf{U} = \boldsymbol{\nabla}\phi$ and P is given by the Bernoulli equation $P + \frac{1}{2}U^2 = \text{const.}$

(a) Show that $(\mathbf{U}\cdot\boldsymbol{\nabla})\mathbf{U} = \boldsymbol{\nabla}(\frac{1}{2}U^2)$.

(b) Represent the potential flow by $\mathbf{U} = U\mathbf{t}_s$, where $\mathbf{t}_s$ is the local tangent vector to its streamlines. Show that

$$-\boldsymbol{\nabla}P = \partial_s(\frac{1}{2}U^2)\mathbf{t}_s + U^2\kappa_s\mathbf{n}_s,$$

where s is arclength along the streamlines of $\mathbf{U}$, κ_s is the local curvature of the streamline, and $\mathbf{n}_s$ is the normal vector to the streamline directed toward its local center of curvature.

(c) Explain how, acting across a thin boundary layer at the wall, the Lighthill source $-\mathbf{n}\times\boldsymbol{\nabla}P$ due to the pressure-gradient of a curved streamline of $\mathbf{U}$ may create both spanwise and streamwise vorticity.