# Homework #1 – Solutions

**Problem 1** (a) We use the fact that

$$\ell_{mf} \sim \nu / c_s,$$

since the sound speed $c_s \sim v_{th}$, the r.m.s. thermal velocity. In that case,

$$\ell_{mf} / \delta_\nu \sim \frac{\nu / c_s}{\nu / u_\tau} = \frac{u_\tau}{c_s} = \frac{U}{c_s} \cdot \frac{u_\tau}{U} \sim Ma\sqrt{f}$$

since $f = \tau_\nu / \frac{1}{2}U^2 \sim (u_\tau/U)^2$.

(b) Ensemble-averaging Navier's law gives

$$\delta \bar{u}_s = \ell_{mf} \frac{\partial \bar{u}}{\partial y} \sim \frac{\nu}{c_s} \frac{\partial \bar{u}}{\partial y}$$

Using $\nu \frac{\partial \bar{u}}{\partial y} = u_\tau^2$ thus gives

$$\frac{\delta \bar{u}_s}{u_\tau} = \frac{u_\tau}{c_s} \sim Ma\sqrt{f}$$

exactly as before.

Problem 2. (a) The Leray formulation of the NS equation in terms of $\mathbf{u}$ alone

$$\frac{d\mathbf{u}}{dt} + B(\mathbf{u},\mathbf{u}) + \nu A\mathbf{u} = 0 \qquad \text{in } \Omega \times [0,T]$$

is equivalent to a velocity-pressure formulation

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nu\Delta\mathbf{u} + \nabla p = 0$$

where the pressure $p$ is the solution of the Poisson equation with Neumann b.c.

$$\begin{cases} -\Delta p = \dfrac{\partial u_i}{\partial x_j}\dfrac{\partial u_j}{\partial x_i} & \text{in } \Omega \\ \dfrac{\partial p}{\partial n} = \hat{n}\cdot\nu\Delta\mathbf{u}. & \text{in } \partial\Omega \end{cases}$$

This follows by the definition of the Helmholtz–Leray projection $P$ which gives

$$B(\mathbf{u},\mathbf{u}) = P(\mathbf{u}\cdot\nabla)\mathbf{u}, \quad A\mathbf{u} = P\Delta\mathbf{u}$$

in the space of vector functions $H = \{\mathbf{v} : \nabla\cdot\mathbf{v} = 0, \ \hat{n}\cdot\mathbf{v}|_{\partial\Omega} = 0\}$.

Since

$$\nabla\cdot[(\mathbf{u}\cdot\nabla)\mathbf{u}] = \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} \neq 0$$

$$(\mathbf{u}\cdot\nabla)\mathbf{u}\,|_{\partial\Omega} = 0 \qquad (\text{by stick b.c.})$$

but

$$\nabla\cdot[\Delta\mathbf{u}] = 0 \qquad (\text{by } \nabla\cdot\mathbf{u} = 0)$$

$$\hat{n}\cdot\Delta\mathbf{u}\,|_{\partial\Omega} \neq 0$$

the Neumann b.c. on $p$ is required so that $(A\mathbf{u})\cdot\hat{n}\,|_{\partial\Omega} = 0$. It has <u>nothing</u> to do with the condition $\nabla\cdot\mathbf{u}|_{\partial\Omega} = 0$!

Note that Leray's construction applies <u>only</u> to initial data such that $u_0 \in H$, which implies that

$$\nabla \cdot u_0 = 0 .$$

Thus, the pressure in Leray's theory is <u>not</u> chosen to ensure $\nabla \cdot u = 0$ but rather to ensure that

$$\nabla \cdot \left( \frac{du}{dt} \right) = 0 , \quad (x,t) \in \Omega \times [0,T]$$

Together with the initial condition, this gives

$$\nabla \cdot u = 0 , \qquad (x,t) \in \Omega \times [0,T]$$

As long as the solution is smooth, $\nabla \cdot u = 0$ also for all $(x,t) \in \partial\Omega \times [0,T]$, by taking limits $x_n \in \Omega \to x \in \partial\Omega$. There is no need to impose $\nabla \cdot u = 0$ separately at the boundary!

(b) The error in Rempfer's "Theorem 2" is closely related to the error in his "Theorem 1". He claims that any Dirichlet condition

$$p(x,t) = p_\Gamma (x,t) \quad , x \in \Gamma$$

on his equation (29) may be imposed, and yet (23)-(28) will still be satisfied. However, in general the velocity boundary conditions (25) will <u>not</u> be satisfied with arbitrary Dirichlet b.c. on the pressure. To see this, consider the simplest case of such b.c. with $u_\Gamma = 0$. In that case, we get

$$\hat{n} \cdot \frac{\partial u_P}{\partial t} = -\frac{\partial p}{\partial n} + \frac{1}{Re} \hat{n} \cdot \Delta u$$

by dotting the NS equation (23) with $\hat{n}$ and taking the limit of a point on the boundary. But we see that

$$\frac{\partial p}{\partial n} \neq \frac{1}{Re} \hat{n} \cdot \Delta u$$

for general Dirichlet data imposed on $p$ and thus

$$\hat{n} \cdot \frac{\partial u_P}{\partial t} \neq 0,$$

which is inconsistent with the requirement that $u_P = 0$ for all time $t \in [0, T]$.

(c) Rempfer's velocity field in his section 3.2.5.1 does not satisfy Temam's 2nd compatibility condition

$$(\hat{n} \times \nabla) p = \hat{n} \times \nu(\Delta u)$$

if the pressure is determined by the (proper) Neumann condition

$$\frac{\partial p}{\partial n} = \hat{n} \cdot \nu(\Delta u).$$

Thus, there will be a discontinuity in $\partial u / \partial t$ at the boundary $\partial\Omega$ at time $t = 0$ if the NS equations are solved with this initial data. However, with such a smooth initial data in 2D, the NS solution will be smooth for times $t > 0$, even with such "improper" data. Thus, for positive times, both the Neumann and Dirichlet conditions will be satisfied!

(d) The problem to be solved is the inhomogeneous Stokes IBVP:

$$
\begin{aligned}
\partial_t \mathbf{u} - \nu \Delta \mathbf{u} &= -\nabla p && \text{on } \Omega \times [0,T] \\
\nabla \cdot \mathbf{u} &= 0 && \text{on } \Omega \times [0,T] \\
\mathbf{u} &= \mathbf{u}_B && \text{on } \partial\Omega \times [0,T] \\
\mathbf{u}(0) &= \mathbf{u}_0 && \text{on } \Omega
\end{aligned}
$$

Temam's theorems guarantee a unique solution $(\mathbf{u}, p)$, up to an additive constant in the pressure $p$. To see this, we use the result proved in Theorem I.2.4 that there exists a $\mathbf{v}$ satisfying

$$\mathbf{v} \in H^1, \quad \nabla \cdot \mathbf{v} = 0, \quad \mathbf{v}|_{\partial\Omega} = \mathbf{u}_B$$

Setting
$$\mathbf{w} \equiv \mathbf{u} - \mathbf{v},$$

we therefore see that $\mathbf{w}$ must satisfy the homogeneous Stokes IBVP:

$$
\begin{aligned}
\partial_t \mathbf{w} - \nu \Delta \mathbf{w} &= -\nabla p + \mathbf{f} && \text{on } \Omega \times [0,T] \\
\nabla \cdot \mathbf{w} &= 0 && \text{on } \Omega \times [0,T] \\
\mathbf{w} &= \mathbf{0} && \text{on } \partial\Omega \times [0,T] \\
\mathbf{w}(0) &= \mathbf{w}_0 && \text{on } \Omega
\end{aligned}
$$

with

$$\mathbf{w}_0 \equiv \mathbf{u}_0 - \mathbf{v}$$

and

$$\mathbf{f} \equiv \nu \Delta \mathbf{v} .$$

Then Temam's Theorem III.1.1 guarantees the existence of a unique solution $(\mathbf{w}, p)$ to the homogeneous problem, where $p$ is the pressure determined by the Neumann problem

$$\begin{cases} -\Delta p = 0 \\ \dfrac{\partial p}{\partial n} = \hat{\mathbf{n}} \cdot (\nu \Delta \mathbf{w} + \mathbf{f}) . \end{cases}$$

Using the definition of $\mathbf{f}$, we see that the proper pressure b.c. for the inhomogeneous problem is

$$\frac{\partial p}{\partial n} = \hat{\mathbf{n}} \cdot \nu \Delta \mathbf{u} .$$

which then has a unique solution $(\mathbf{u}, p)$.

Note, however, that it is <u>not</u> true for this solution that

$$\hat{\mathbf{n}} \cdot \Delta \mathbf{u} \neq 0$$

so that Rempfer's condition

$$\left.\frac{\partial p}{\partial n}\right|_{\partial\Omega} = 0$$

is <u>not</u> correct. The problem is that Rempfer's "solution" does not even satisfy the incompressibility condition $\nabla \cdot \mathbf{u} = 0$! His field gives $\hat{\mathbf{n}} \cdot \Delta \mathbf{u} = 0$, but that is the <u>wrong</u> b.c.

Problem 3. Using

$$\delta A_i = \epsilon_{ijk} \, \delta l_j \, \delta l'_k$$

and $\frac{d}{dt} \delta l_j = \delta l_m \partial_m u_j$ plus the similar equation for $\delta l'_k$ gives

$$\frac{d}{dt} \delta A_i = \epsilon_{ijk} \left( \delta l_m \, \delta l'_k \, \partial_m u_j + \delta l_j \, \delta l'_m \, \partial_m u_k \right)$$

We can next substitute

$$\delta l_m \, \delta l'_k = \mathcal{S}_{mk} + \mathcal{A}_{mk}$$

in terms of symmetric and anti-symmetric parts

$$\mathcal{S}_{mk} = \frac{1}{2} \left( \delta l_m \, \delta l'_k + \delta l_k \, \delta l'_m \right)$$

$$\mathcal{A}_{mk} = \frac{1}{2} \left( \delta l_m \, \delta l'_k - \delta l_k \, \delta l'_m \right).$$

However

$$\epsilon_{ijk} \left( \mathcal{S}_{mk} \, \partial_m u_j + \mathcal{S}_{mj} \, \partial_m u_k \right) = 0$$

because the term in the parentheses is symmetric in $j, k$ and $\epsilon_{ijk}$ is anti-symmetric in $j, k$. Furthermore,

$$\mathcal{A}_{mk} = \frac{1}{2} \epsilon_{mkp} \, \delta A_p$$

Thus,

$$\frac{d}{dt}\delta A_i = \frac{1}{2}\epsilon_{ijk}\Big(\epsilon_{mkp}\,\delta A_p\,\partial_m u_j$$
$$+ \epsilon_{jmp}\,\delta A_p\,\partial_m u_k\Big)$$

$$= -\frac{1}{2}\epsilon_{ijk}\epsilon_{mpk}\,\delta A_p\,\partial_m u_j$$
$$-\frac{1}{2}\epsilon_{ikj}\epsilon_{mpj}\,\delta A_p\,\partial_m u_k$$

and then using

$$\epsilon_{ijk}\epsilon_{mpk} = \delta_{im}\delta_{jp} - \delta_{ip}\delta_{jm}$$

gives easily

$$\frac{d}{dt}\delta A_i = (\partial_j u_j)\,\delta A_i - (\partial_i u_j)\,\delta A_j$$

which is equivalent to Batchelor's equation

$$\frac{d}{dt}\delta \mathbf{A} = (\nabla\cdot\mathbf{u})\,\delta\mathbf{A} - (\nabla\mathbf{u})\,\delta\mathbf{A}\,.$$

QED

Problem 4 (a) Using the identity $\boldsymbol{\omega} = \nabla \cdot (\boldsymbol{\omega} \mathbf{x})$ which follows from $\nabla \cdot \boldsymbol{\omega} = 0$, one gets

$$\int_{\Omega} \boldsymbol{\omega}\, dV = \int_{\Omega} \nabla \cdot (\boldsymbol{\omega} \mathbf{x})\, dV$$

$$= \int_{\partial\Omega} (\hat{\mathbf{n}} \cdot \boldsymbol{\omega})\, \mathbf{x}\, dA$$ by divergence theorem.

If we use the boundary condition

$$\boldsymbol{\omega} = 2\boldsymbol{\Omega} \quad \text{on } \partial\Omega,$$

then

$$\int_{\Omega} \boldsymbol{\omega}\, dV = 2\boldsymbol{\Omega} \cdot \int_{\partial\Omega} \hat{\mathbf{n}} \mathbf{x}\, dA$$ by spatial constancy of $\boldsymbol{\Omega}$

Again by divergence theorem

$$\int_{\partial\Omega} \hat{\mathbf{n}} \mathbf{x}\, dA = \int_{\Omega} \nabla \mathbf{x}\, dV$$

$$= \int_{\Omega} \mathbf{I}\, dV$$ since $\nabla \mathbf{x} = \mathbf{I}$

$$= \mathbf{I} \cdot |\Omega|$$

Thus, $$\frac{1}{|\Omega|} \int_{\Omega} \boldsymbol{\omega}\, dV = 2\boldsymbol{\Omega}.$$ QED

(b) By the definition of Lyman,

$$\sigma_i^L = \nu \left( \frac{\partial \omega_i}{\partial x_j} - \frac{\partial \omega_j}{\partial x_i} \right) \hat{n}_j$$

$$= \nu \frac{\partial \omega_i}{\partial n} - \nu \frac{\partial \omega_j}{\partial x_i} \hat{n}_j$$

$$= \nu \frac{\partial \omega_i}{\partial n} - \nu \frac{\partial}{\partial x_i} ( \omega_j \hat{n}_j ) + \nu \frac{\partial \hat{n}_j}{\partial x_i} \omega_j$$

by product rule

Note however that

$$\omega_n = \omega_j \hat{n}_j = 0 \quad \text{on } \partial\Omega .$$

Since $\partial\Omega$ is a level set of the scalar function $\omega_n$, it follows that the gradient is perpendicular to $\partial\Omega$.

Thus,

$$\frac{\partial}{\partial x_i} ( \omega_j \hat{n}_j ) = \hat{n}_i \frac{\partial \omega_n}{\partial n}$$

so that

$$\sigma_i^L = \underbrace{\nu \frac{\partial \omega_i}{\partial n}}_{\sigma_i^P} - \nu \hat{n}_i \frac{\partial \omega_n}{\partial n} + \nu \partial_i \hat{n}_j \omega_j$$

or, equivalently,

$$\boldsymbol{\sigma}^L = \boldsymbol{\sigma}^P - \nu \hat{\mathbf{n}} \frac{\partial \omega_n}{\partial n} + \nu ( \nabla \hat{\mathbf{n}} ) \boldsymbol{\omega}$$

Finally, note that

$$\hat{\mathbf{n}}\cdot\sigma^P = \nu\hat{\mathbf{n}}\cdot\frac{\partial\boldsymbol{\omega}}{\partial n}$$

$$= \nu\frac{\partial}{\partial n}(\hat{\mathbf{n}}\cdot\boldsymbol{\omega})$$

since $\hat{\mathbf{n}}$ is constant along the normal direction itself, or $\partial\hat{\mathbf{n}}/\partial n = 0$. Thus,

$$\hat{\mathbf{n}}\cdot\sigma^P = \nu\frac{\partial\omega_n}{\partial n},$$

which shows that Panton's definition predicts that wall-normal vorticity is created at the boundary.

BTW, another way to obtain the last result is to use

$$0 = \hat{\mathbf{n}}\cdot\sigma^L = \hat{\mathbf{n}}\cdot\left(\sigma^P - \nu\hat{\mathbf{n}}\frac{\partial\omega_n}{\partial n} + \nu(\nabla\hat{\mathbf{n}})\boldsymbol{\omega}\right)$$

Since $\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}} = \frac{\partial\hat{\mathbf{n}}}{\partial n} = 0$, the last term in the parentheses vanishes and one obtains again

$$\hat{\mathbf{n}}\cdot\sigma^P = \hat{\mathbf{n}}\cdot\nu\hat{\mathbf{n}}\frac{\partial\omega_n}{\partial n} = \nu\frac{\partial\omega_n}{\partial n}.$$

QED

Problem 5 (a) Using $\mathrm{erfc}(0)=1$ and $\mathrm{erfc}(\infty)=0$, we see that

$$u(y,t) = U\,\mathrm{erfc}\left(\frac{y}{\sqrt{4\nu t}}\right)$$

satisfies

$$u(y,0) = U\,\mathrm{erfc}(\infty) = 0$$

and for $t>0$

$$u(0,t) = U\,\mathrm{erfc}(0) = U.$$

BTW, this gives a good example of an initial condition and boundary condition that does not satisfy the first compatibility condition! We therefore see that $u(y,t)$ is not continuous at the boundary $(y=0)$ at time $t=0$. In fact,

$$\lim_{\substack{y\to 0,\, t\to 0 \\ \frac{y}{\sqrt{2\nu t}} = \theta \text{ fixed}}} u(y,t) = U\,\mathrm{erfc}(\theta)$$

which can have any value between $0$ and $U$!

Returning to the verification, we introduce a similarity variable

$$s = y/\sqrt{4\nu t}$$

and note that

$$\frac{d}{ds}\mathrm{erfc}(s) = \frac{-2}{\sqrt{\pi}} e^{-s^2}$$

$$\frac{\partial s}{\partial t} = -\frac{1}{2} t^{-3/2} \frac{y}{\sqrt{4\nu}} = -\frac{s}{2t}$$

$$\frac{\partial s}{\partial y} = \frac{1}{\sqrt{4\nu t}} = \frac{s}{y}$$

Thus,

$$u_t = \frac{-2U}{\sqrt{\pi}} e^{-s^2} \cdot \frac{-s}{2t} = \frac{U}{t} \cdot \frac{s}{\sqrt{\pi}} e^{-s^2}$$

$$u_y = \frac{-2U}{\sqrt{\pi}} e^{-s^2} \cdot \frac{1}{\sqrt{4\nu t}}$$

$$u_{yy} = \frac{-2U}{\sqrt{\pi}} e^{-s^2} (-2s) \cdot \frac{1}{4\nu t} = \frac{U}{\nu t} \cdot \frac{s}{\sqrt{\pi}} e^{-s^2}$$

so that

$$u_t = \frac{U}{t} \cdot \frac{s}{\sqrt{\pi}} e^{-s^2} = \nu u_{yy} .$$

Since

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} = 0 , \quad \Delta \mathbf{u} = u_{yy} \hat{x}$$

we see that the Stokes equation is satisfied, with a constant pressure $p$. Also, $(\mathbf{u} \cdot \nabla)\mathbf{u} = u \frac{\partial}{\partial x} \mathbf{u} = 0$ so Navier-Stokes is satisfied.

(b) The only non-vanishing component of the vorticity is

$$\omega_z = -\frac{\partial u}{\partial y} = \frac{U}{\sqrt{\pi \nu t}} e^{-y^2/4\nu t}$$

Using the results of part (a), This Gaussian function has its maximum at $y=0$ for all times $t>0$, with maximum value

$$\omega_z^{max} = \frac{U}{\sqrt{\pi \nu t}},$$

which $\longrightarrow \infty$ as $t \to 0$ and decays monotonically to 0 as $t \longrightarrow \infty$.

The curl of the vorticity has only an $x$-component:

$$\nabla \times \omega = (\partial_y \omega_z)\hat{x} = -(u_{yy})\hat{x}.$$

Since $n' = \hat{y}$, we see that

$$\sigma = n' \times \nu(\nabla \times \omega)\Big|_{\partial\Omega}$$

$$= (\nu u_{yy})\hat{z}\Big|_{y=0}.$$

From part (a),

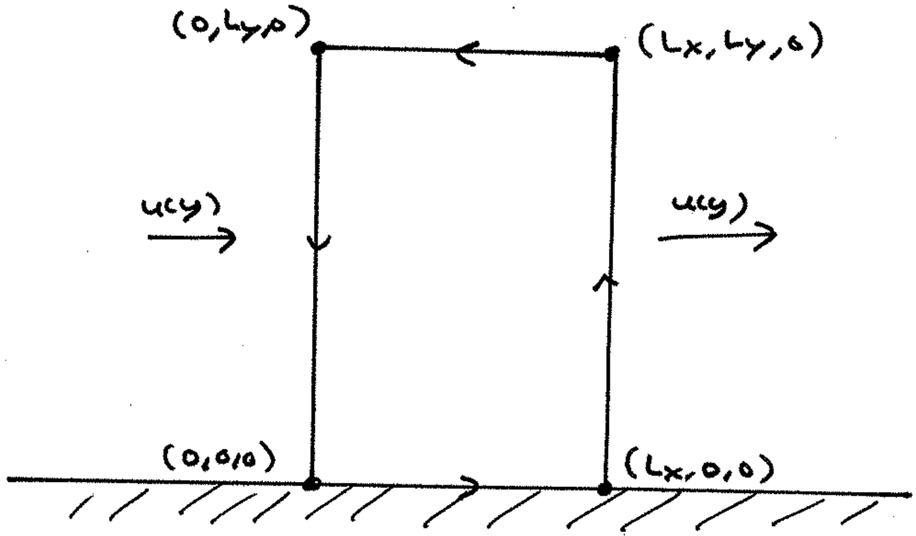
$$\nu u_{yy} = \frac{U}{t} \cdot \frac{y}{\sqrt{4\pi\nu t}} e^{-y^2/4\nu t} \underset{\substack{y \to 0 \\ t > 0}}{\longrightarrow} 0$$

so that

$$\sigma = 0$$

for all times $t > 0$. Thus, all the vorticity creation at the wall is at the initial instant $t=0$, corresponding to an infinite vortex sheet. Subsequently this vorticity is diffused into the fluid by viscosity.

(c) For the stated circuit $C$:




we get

$$\oint_C \mathbf{u}(t)\cdot d\mathbf{x} = U\cdot L_x + 0\cdot L_y + (-u(L_y,t))\cdot L_x + 0\cdot L_y$$

$$= (U - u(L_y,t))\cdot L_x .$$

Thus,

$$\lim_{L_y\to\infty} \oint_C \mathbf{u}(t)\cdot d\mathbf{x} = U\cdot L_x$$

since $u(y,t) \to 0$ as $y \to \infty$. We therefore see that the net circulation around the loop $C$ is invariant in time. Consistent with our earlier conclusion, all of the vorticity is created at the initial instant and is subsequently just diffused to larger distances from the wall.

Problem 6. (a) We see that

$$u(\pm h) = \frac{\gamma}{2\nu}(h^2 - h^2) = 0$$

and

$$u_{yy} = -\frac{\gamma}{\nu}, \quad \partial_x p = -\gamma$$

so that

$$-\partial_x p + \nu \partial_y^2 u = 0 .$$

Thus, the stationary Stokes equation is satisfied. Of course, $\nabla \cdot u = \frac{\partial u}{\partial x} = 0$, so the flow is incompressible. Also, since $(u \cdot \nabla) u = u \frac{\partial}{\partial x} u = 0$, the stationary Navier-Stokes equation is satisfied.

(b) The only non-vanishing component of vorticity is

$$\omega_z = -\frac{\partial u}{\partial y} = \left(\frac{\gamma}{\nu}\right) y .$$

Thus, vortex lines point in the z-direction, positive at the top wall and negative at the bottom wall. The maximum magnitude takes place at the walls. The vorticity source density is

$$\sigma_z = \begin{cases} \nu u_{yy} & y = -h \\ -\nu u_{yy} & y = +h \end{cases} = \begin{cases} -\gamma & y = -h \\ \gamma & y = h \end{cases}$$

since $n' = \hat{y}$ at the bottom wall and $\hat{n} = -\hat{y}$ at the top wall.

We thus see that there is vorticity generation constant in time at the walls, in opposite directions at the top and the bottom. The rate of creation is given by the tangential pressure gradient $\gamma$ at the walls. Once the vorticity is created, it diffuses into the interior of the flow, where it annihilates against vorticity of the opposite sign at the centerline ($y=0$). Note that $\omega_z = 0$ at $y=0$.

(c) Because of the stick b.c. at the walls, one can easily see that

$$\oint_C u \cdot dx = 0 \cdot L_x + 0 \cdot (2h) + 0 \cdot (-L_x) \cdot 0 \cdot (-2h)$$

$$= 0$$

Thus, there is no net circulation around the curve C. This agrees with the conclusion in (b), since the vorticity for $y>0$ is positive and that for $y<0$ is negative and exactly cancel when integrated over the rectangle. The vorticity generation is equal and opposite at the two walls.

Problem 7. (a) By the standard result for the Biot-Savart formula

$$\nabla \times \tilde{\mathbf{u}} = \boldsymbol{\omega} \quad \text{in } \Omega.$$

On the other hand,

$$\nabla \times \nabla \phi = 0$$

so that

$$\begin{aligned} \nabla \times \mathbf{u} &= \nabla \times (\tilde{\mathbf{u}} - \nabla \phi) \\ &= \nabla \times \tilde{\mathbf{u}} \\ &= \boldsymbol{\omega}, \end{aligned}$$

while

$$\begin{aligned} \mathbf{u} \cdot \hat{\mathbf{n}} &= (\tilde{\mathbf{u}} - \nabla \phi) \cdot \hat{\mathbf{n}} \\ &= \tilde{\mathbf{u}} \cdot \hat{\mathbf{n}} - \frac{\partial \phi}{\partial n} = 0 \quad \text{on } \partial \Omega \end{aligned}$$

by the b.c. for $\phi$. Thus, $\mathbf{u}$ is a solution,

(b) We first study

$$\tilde{\mathbf{u}}(\mathbf{x}) = \int_{\Omega} \frac{\boldsymbol{\omega}(\mathbf{x}') \times \boldsymbol{\rho}}{4\pi \rho^3} \, dV', \quad \boldsymbol{\rho} = \mathbf{x} - \mathbf{x}'$$

when $r = |\mathbf{x}| \gg |\mathbf{x}'| = r'$ in the support of the vorticity $\boldsymbol{\omega}$.

Here we can use the standard multipole expansion

$$\frac{1}{\rho} = \sum_{l=0}^{\infty} \frac{r'^{l}}{r^{l+1}} P_l(\cos\theta)$$

where $P_l$ is the degree-$l$ Legendre polynomial and $\theta$ is the angle between $\mathbf{x}$ and $\mathbf{x}'$. Thus,

$$\frac{1}{\rho} = \frac{1}{r} + \frac{\mathbf{x}\cdot\mathbf{x}'}{r^3} + O\left(\frac{1}{r^3}\right)$$

$$\frac{1}{\rho^3} = \frac{1}{r^3} + \frac{3\mathbf{x}\cdot\mathbf{x}'}{r^4} + O\left(\frac{1}{r^5}\right)$$

Together with $\boldsymbol{\rho} = \mathbf{x}\left(1 + O\left(\frac{1}{r}\right)\right)$, we get

$$\tilde{\mathbf{u}} = \left(\int_\Omega \boldsymbol{\omega}(\mathbf{x}')\,dV'\right) \times \frac{\mathbf{x}}{4\pi r^3} + O\left(\frac{1}{r^3}\right)$$

However, Föppl's theorem states that $\int_\Omega \boldsymbol{\omega}(\mathbf{x}')\,dV' = 0$ and thus

$$\tilde{\mathbf{u}} = O\left(\frac{1}{r^3}\right) \text{ for } r \to \infty$$

This component thus decays rapidly.

To make a similar estimate of the potential term $\mathbf{u}_\phi = -\nabla\phi$ is not as easy but can be done using the formula

$$\mathbf{u}_\phi(\mathbf{x}) = \nabla \times \left( \int_{\partial\Omega} \frac{\mathbf{n} \times \mathbf{u}_\phi(\mathbf{x}')}{4\pi|\mathbf{x}-\mathbf{x}'|} \, dA' \right)$$

which follows from the <u>Helmholtz decomposition</u>.

If we again use the multipole expansion of $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$, then the first term can be shown to vanish

$$\int_{\partial\Omega} \mathbf{n} \times \nabla\phi(\mathbf{x}') \, dA' = 0$$

as we shall discuss later. It then follows again that

$$\mathbf{u}_\phi = O\left(\frac{1}{r^3}\right), \quad r \to \infty$$

which corresponds to a "dipole" or "vortex-ring" velocity. Combining the two results,

$$\mathbf{u} = O\left(\frac{1}{r^3}\right), \quad r \to \infty .$$

(c) To prove uniqueness, consider

$$\mathbf{w} := \mathbf{u} - \mathbf{u}'$$

for another possible solution $\mathbf{u}'$. Then,

$$\nabla \times \mathbf{w} = \boldsymbol{\omega} - \boldsymbol{\omega} = 0 \quad \text{in } \Omega.$$

Since $\Omega$ is simply-connected, this implies that

$$\mathbf{w} = \nabla \psi$$

for a scalar function $\psi$. Furthermore,

$$\frac{\partial \psi}{\partial n} = \hat{\mathbf{n}} \cdot \mathbf{w} = \hat{\mathbf{n}} \cdot \mathbf{u} - \hat{\mathbf{n}} \cdot \mathbf{u}' = 0 - 0 = 0 \quad \text{on } \partial\Omega$$

and

$$\Delta \psi = \nabla \cdot \mathbf{w} = \nabla \cdot \mathbf{u} - \nabla \cdot \mathbf{u}' = 0 - 0 = 0 \quad \text{in } \Omega.$$

The solution of this Neumann problem for the Laplace equation in the simply-connected domain $\Omega$ can be only a spatial constant $\psi = c$. Therefore,

$$\mathbf{u} - \mathbf{u}' = \mathbf{w} = \nabla c = 0$$

$$\Longrightarrow \mathbf{u} = \mathbf{u}'.$$

Thus, all solutions are equal to $\mathbf{u}$.

QED