Problem 1 (a) We use the fact that

\[ \ell_{mf} \sim \nu / c_s, \]

since the sound speed \( c_s \sim u_{rM} \), the r.m.s. thermal velocity. In that case,

\[ \ell_{mf} / \delta v \sim \frac{\nu / c_s}{\nu / u_{rM}} = \frac{u_{rM}}{c_s} = \frac{U}{c_s} \cdot \frac{u_{rM}}{U} \sim Ma \sqrt{f}, \]

since \( f = \tau_v / 2U^2 \sim (u_{rM}/U)^2 \).

(b) Ensemble-averaging Navier's law gives

\[ \delta u_s = \ell_{mf} \frac{\partial u}{\partial y} \sim \frac{\nu}{c_s} \frac{\partial u}{\partial y} \]

Using \( \nu \frac{\partial u}{\partial y} = u_{rM}^2 \) thus gives

\[ \frac{\delta u_s}{u_{rM}} = \frac{u_{rM}}{c_s} \sim Ma \sqrt{f}, \]

exactly as before.
Problem 2. (a) The Navier formulation of the NS equation in terms of $u$ alone

$$\frac{du}{dt} + B(u,u) + \nu Au = 0 \quad \text{in } \mathbb{R}^n \times [0,T]$$

is equivalent to a velocity-pressure formulation

$$\frac{d}{dt} u + (u \cdot \nabla) u + \nu \Delta u + \nabla p = 0$$

where the pressure $p$ is the solution of the Poisson equation with Neumann b.c.

$$\Delta p = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \quad \text{in } \Omega$$

$$\frac{\partial p}{\partial n} = \mathbf{n} \cdot \nu \Delta u \quad \text{on } \partial \Omega$$

This follows by the definition of the Helmholtz–Leray projection $P$ which gives

$$B(u,u) = P(u \cdot \nabla) u \quad \text{and} \quad Au = P \Delta u$$

in the space of vector functions $H = \mathbb{R} \quad \text{such that } \nabla \cdot v = 0, \quad \mathbf{n} \cdot v = 0 \quad \text{on } \partial \Omega$.

Since

$$\nabla \cdot [(u \cdot \nabla) u] = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \neq 0$$

$$\left(u \cdot \nabla\right) u \big|_{\partial \Omega} = 0 \quad \text{(by stick b.c.)}$$

but

$$\nabla \cdot [\Delta u] = 0 \quad \text{(by } \nabla \cdot u = 0 \text{)}$$

$$\mathbf{n} \cdot \Delta u \big|_{\partial \Omega} \neq 0$$

the Neumann b.c. on $p$ is required so that $(Au) \cdot \mathbf{n} \big|_{\partial \Omega} = 0$.

It has nothing to do with the condition $\nabla \cdot u \big|_{\partial \Omega} = 0$.
Note that Leray's construction applies only to initial data such that \( u_0 \in H \), which implies that
\[
\nabla \cdot u_0 = 0.
\]

Thus, the pressure in Leray's theory is not chosen to ensure \( \nabla \cdot u = 0 \) but rather to ensure that
\[
\nabla \cdot \left( \frac{du}{dt} \right) = 0, \quad (x,t) \in \Omega \times [0,T]
\]

Together with the initial condition, this gives
\[
\nabla \cdot u = 0, \quad (x,t) \in \Omega \times [0,T]
\]

As long as the solution is smooth, \( \nabla \cdot u = 0 \) also for all \( (x,t) \in \partial \Omega \times [0,T] \), by taking limits \( x_n \in \Omega \rightarrow x \in \partial \Omega \). There is no need to impose \( \nabla \cdot u = 0 \) separately at the boundary!

(c) The error in Renyon's "Theorem 2" is closely related to the error in his "Theorem 1". He claims that any Dirichlet condition
\[
\rho(x,t) = \rho_p(x,t), \quad x \in \Omega
\]

on his counter (29) may be imposed, and yet (23)-(28) will still be satisfied. However, in general the velocity boundary conditions (25) will not be satisfied with arbitrary Dirichlet b.c. on the pressure. To see this, consider the simplest case of stiff b.c. with \( u_p = 0 \). In that case,
we set
\[ \hat{n} \cdot \frac{\partial u_p}{\partial t} = - \frac{\partial p}{\partial n} + \frac{1}{Re} \hat{n} \cdot \Delta u \]

By dotting the NS equation (28) with \( \hat{n} \) and taking the limit of a point on the boundary, but we see that
\[ \frac{\partial p}{\partial n} \neq \frac{1}{Re} \hat{n} \cdot \Delta u \]

For general Dirichlet data imposed on \( p \) and thus
\[ \hat{n} \cdot \frac{\partial u_p}{\partial t} \neq 0, \]

which is inconsistent with the requirement that \( u_p = 0 \)
for all time \( t \in [0, T] \).

(c) Rempele's velocity field in his section 3.2.5.1 does not satisfy Temam's 2nd compatibility condition
\[ (\hat{n} \times \nabla) p = \hat{n} \times \nu(\Delta u) \]

If the pressure is determined by the (proper) Neumann condition
\[ \frac{\partial p}{\partial n} = \hat{n} \cdot \nu(\Delta u) \]

Thus, there will be a discontinuity in \( \partial u_p/\partial t \) at the boundary \( \partial \Omega \) at time \( t = 0 \) if the NS equations are
solved with this initial data. However, with such a smooth initial data in 2D, the NS solution will be smooth for times \( t > 0 \), even with such "improved" data. Thus, for positive times, both the Neumann and Dirichlet conditions will be satisfied!

(d) The problem to be solved is the inhomogeneous Stokes IBVP:

\[
\begin{align*}
\partial_t u - \nu \Delta u &= - \nabla p & \text{ in } & \Omega \times [0,T] \\
\nabla \cdot u &= 0 & \text{ in } & \Omega \times [0,T] \\
u &= u_B & \text{ on } & \partial \Omega \times [0,T] \\
u(0) &= u_0 & \text{ in } & \Omega
\end{align*}
\]

Temam's theorems guarantee a unique solution \((u, p)\), up to an additive constant in the pressure \( p \). To see this, we use the result proved in Theorem 1.2.4 that there exists a \( \mathbf{v} \) satisfying

\[
\mathbf{v} \in H^1, \quad \nabla \cdot \mathbf{v} = 0, \quad \mathbf{v} \big|_{\partial \Omega} = u_B
\]

Setting \( \mathbf{w} = \mathbf{u} - \mathbf{v} \), we therefore see that \( \mathbf{w} \) must satisfy the homogeneous Stokes IBVP:

\[
\begin{align*}
\partial_t \mathbf{w} - \nu \Delta \mathbf{w} &= - \nabla p + \mathbf{f} & \text{ in } & \Omega \times [0,T] \\
\nabla \cdot \mathbf{w} &= 0 & \text{ in } & \Omega \times [0,T] \\
\mathbf{w} &= 0 & \text{ on } & \partial \Omega \times [0,T] \\
\mathbf{w}(0) &= \mathbf{w}_0 & \text{ in } & \Omega
\end{align*}
\]
with \[ w_0 \equiv u_0 - v \]
and \[ f \equiv \nu \Delta v. \]

Then Temam's Theorem III.1.1 guarantees the existence of a unique solution \((W, p)\) to the homogeneous problem, where \(p\) is the pressure determined by the Neumann problem

\[
\begin{align*}
\Delta p &= 0 \\
\frac{\partial p}{\partial n} &= \hat{n} \cdot (\nu \Delta w + f)
\end{align*}
\]

Using the definition of \(f\), we see that the proper pressure \(b.c.\) for the inhomogeneous problem is

\[
\frac{\partial p}{\partial n} = \hat{n} \cdot \nu \Delta u
\]

which then has a unique solution \((u, p)\).

Note, however, that it is not true for this solution that

\[
\hat{n} \cdot \Delta u \neq 0
\]

so that Remper's condition

\[
\frac{\partial p}{\partial n} \bigg|_{\alpha n} = 0
\]

is not correct. The problem is that Remper's "solution" does not even satisfy the incompressibility condition \(\nabla \cdot u = 0\). This field gives \(\hat{n} \cdot \Delta u = 0\), but that is the wrong \(b.c.\).
Problem 3. Using

\[ \delta A_i = \epsilon_{ijk} \delta l_j \delta l_k' \]

and \( \frac{d}{dt} \delta l_j = \delta l_m \partial_{lj} \) plus the similar equation for \( \delta l_k' \) gives

\[ \frac{d}{dt} \delta A_i = \epsilon_{ijk} \left( \delta l_m \delta l_k' \partial_{mj} + \delta l_j \delta l_m' \partial_{mk} \right) \]

We can next substitute

\[ \delta l_m \delta l_k' = \delta_{mk} + \delta_{mk}' \]

in terms of symmetric and anti-symmetric parts

\[ \delta_{mk} = \frac{1}{2} \left( \delta l_m \delta l_k' + \delta l_k \delta l_m' \right) \]

\[ \delta_{mk}' = \frac{1}{2} \left( \delta l_m \delta l_k' - \delta l_k \delta l_m' \right) \]

However

\[ \epsilon_{ijk} \left( \delta_{mk} \partial_{mj} + \delta_{mj} \partial_{mk} \right) = 0 \]

because the term in the parentheses is symmetric in j, k and \( \epsilon_{ijk} \) is anti-symmetric in j, k. Furthermore,

\[ \delta_{mk} = \frac{1}{2} \epsilon_{mkp} \delta A_p \]
Thus,

$$\frac{d}{dt} \delta A_i = \frac{1}{2} \varepsilon_{ijk} \left( \varepsilon_{mpq} \delta A_p \delta m_{uj} \\
+ \varepsilon_{jmp} \delta A_p \delta m_{uk} \right)$$

$$= -\frac{1}{2} \varepsilon_{ijk} \varepsilon_{mpk} \delta A_p \delta m_{uj}$$

$$- \frac{1}{2} \varepsilon_{ikj} \varepsilon_{mpj} \delta A_p \delta m_{uk}$$

and then using

$$\varepsilon_{ijk} \varepsilon_{mpk} = \delta_{im} \delta_{jp} - \delta_{ip} \delta_{jm}$$

gives easily

$$\frac{d}{dt} \delta A_i = (\partial j u_j) \delta A_i - (\partial i u_j) \delta A_j$$

which is equivalent to Batchelor's equation

$$\frac{d}{dt} \delta A = (\nabla \cdot \mathbf{u}) \delta A - (\nabla \mathbf{u}) \cdot \delta A.$$

QED
Problem 4 (a) Using the identity $\omega = \nabla \cdot (\omega x)$ which follows from $\nabla \cdot \omega = 0$, one gets

$$\int_{\Omega} \omega \, dV = \int_{\Omega} \nabla \cdot (\omega x) \, dV$$

$$= \int_{\partial \Omega} (\hat{n} \cdot \omega) \times dA \quad \text{by divergence theorem.}$$

If we use the boundary condition

$$\omega = 2\omega \quad \text{on} \quad \partial \Omega,$$

then

$$\int_{\Omega} \omega \, dV = 2\Omega \cdot \int_{\partial \Omega} \hat{n} \times dA \quad \text{by spatial constancy of} \, \Omega$$

Again by divergence theorem

$$\int_{\partial \Omega} \hat{n} \times dA = \int_{\Omega} \nabla x \, dV$$

$$= \int_{\Omega} I \, dV \quad \text{since} \quad \nabla x = I$$

$$= I |\Omega|$$

Thus,

$$\frac{1}{|\Omega|} \int_{\Omega} \omega \, dV = 2 \Omega, \quad \text{Q.E.D}$$
(b) By the definition of Lyman,

\[ \sigma_i^L = \nu \left( \frac{\partial w_i}{\partial x_j} - \frac{\partial w_j}{\partial x_i} \right) \hat{n}_j \]

\[ = \nu \frac{\partial w_i}{\partial n} - \nu \frac{\partial w_j}{\partial x_i} \hat{n}_j \]

\[ = \nu \frac{\partial w_i}{\partial n} - \nu \frac{\partial}{\partial x_i} (w_j \hat{n}_j) + \nu \frac{\partial \hat{n}_j}{\partial x_i} w_j \]

by product rule.

Note however that

\[ w_n = w_j \hat{n}_j = 0 \quad \text{on} \quad \partial \Omega. \]

Since \( \partial \Omega \) is a level set of the scalar function \( w_n \), it follows that the gradient is perpendicular to \( \partial \Omega \).

Thus,

\[ \frac{\partial}{\partial x_i} (w_j \hat{n}_j) = \hat{n}_i \frac{\partial w_n}{\partial n} \]

so that

\[ \sigma_i^L = \nu \frac{\partial w_i}{\partial n} - \nu \hat{n}_i \frac{\partial w_n}{\partial n} + \nu \hat{n}_i \hat{n}_j w_j \]

\[ \sigma_i^L = \sigma_i^P - \nu \hat{n}_i \frac{\partial w_n}{\partial n} + \nu (\nabla \hat{n}) w \]

or, equivalently, 

\[ \sigma^L = \sigma^P - \nu \hat{n} \frac{\partial w_n}{\partial n} + \nu (\nabla \hat{n}) w \]
Finally, note that

\[ \hat{n} \cdot \sigma^P = \nu \hat{n} \cdot \frac{\partial w}{\partial n} \]

\[ = \nu \frac{\partial}{\partial n} (\hat{n} \cdot w) \]

since \( \hat{n} \) is constant along the normal direction itself, or \( \partial \hat{n} / \partial n = 0 \). Thus,

\[ \hat{n} \cdot \sigma^P = \nu \frac{\partial w_n}{\partial n} \]

which shows that Pantin's definition predicts that wall-normal vorticity is created at the boundary.

BTW, another way to obtain the last result is to use

\[ 0 = \hat{n} \cdot \sigma_L = \hat{n} \cdot (\sigma^P - \nu \hat{n} \frac{\partial w_n}{\partial n} + \nu (\nabla \hat{n}) w) \]

Since \( \hat{n} \cdot \nabla \hat{n} = \frac{\partial \hat{n}}{\partial n} = 0 \), the last term in the parentheses vanishes and one obtains again

\[ \hat{n} \cdot \sigma^P = \hat{n} \cdot \nu \hat{n} \frac{\partial w_n}{\partial n} = \nu \frac{\partial w_n}{\partial n} \]

QED
Problem 5 (a) Using \( \text{erfc}(0) = 1 \) and \( \text{erfc}(\infty) = 0 \), we see that
\[
u(y,t) = \text{U} \text{erfc}\left(\frac{y}{\sqrt{4ut}}\right)
\]
satisfies
\[
u(y,0) = \text{U erfc}(\infty) = 0
\]
and for \( t > 0 \)
\[
u(0,t) = \text{U erfc}(0) = \text{U}
\]
BTW, this gives a good example of an initial condition and boundary condition that does not satisfy the first compatibility condition! We therefore see that \( \nu(y,t) \) is not continuous at the boundary \( y=0 \) at time \( t=0 \). In fact,
\[
\lim_{y\to 0, t\to 0} \nu(y,t) = \text{U erfc}(0)
\]
\[
\frac{y}{\sqrt{4ut}} = \theta \text{ fixed}
\]
which can have any value between 0 and \( \text{U} \)!

Returning to the verification, we introduce a similarity variable
\[
s = y/\sqrt{4ut}
\]
and note that
\[
\frac{d}{ds} \text{erfc}(s) = \frac{-2}{\sqrt{\pi}} e^{-s^2}
\]
\[
\frac{ds}{dt} = -\frac{1}{2} t^{1/2} \frac{y}{\sqrt{4u}} = -\frac{s}{2t}
\]
\[
\frac{\partial s}{\partial y} = \frac{1}{\sqrt{4\pi t^2}} = \frac{s}{y}
\]

Thus,

\[
u_t = \frac{-2u}{\sqrt{\pi}} e^{-s^2} , \quad \frac{-s}{2t} = \frac{U}{t} \cdot \frac{s}{\sqrt{\pi}} e^{-s^2}
\]

\[
u_y = \frac{-2u}{\sqrt{\pi}} e^{-s^2} \cdot \frac{1}{\sqrt{4\pi t}}
\]

\[
u_{yy} = \frac{-2u}{\sqrt{\pi}} e^{-s^2} (-2s) \cdot \frac{1}{4\sqrt{\pi t}} = \frac{U}{\sqrt{\pi t}} \cdot \frac{s}{\sqrt{\pi}} e^{-s^2}
\]

so that

\[
u_t = \frac{U}{t} \cdot \frac{s}{\sqrt{\pi}} e^{-s^2} = \nu_{yy} .
\]

Since

\[
\nabla \cdot u = \frac{\partial u}{\partial x} = 0 , \quad \Delta u = u_{yy} x
\]

we see that the Stokes equation is satisfied, with \( u \) constant pressure \( p \). Also, \((\nabla \cdot \nabla) u = u \frac{\partial^2}{\partial x^2} u = 0 \) so Navier-Stokes is satisfied.

(b) The only non-vanishing component of the vorticity is

\[
\omega_2 = -\frac{\partial u}{\partial y} = \frac{U}{\sqrt{\pi \nu t}} e^{-y^2/4\nu t}
\]

Using the results of part (a), this Gaussian function has its maximum at \( y = 0 \) for all times \( t > 0 \), with
Maximum value

$$w_{z}^{\text{max}} = \frac{U}{\sqrt{\pi v t}}$$

which \( \rightarrow \infty \) as \( t \rightarrow 0 \) and decays monotonically to 0 as \( t \rightarrow \infty \).

The curl of the vorticity has only an \( x \)-component:

$$\nabla \times \mathbf{w} = \left( \partial_y w_z \right) \hat{x} = -\left( u_{yy} \right) \hat{x}.$$  

Since \( n' = \hat{y} \), we see that

$$\sigma = n' \times \nu(\nabla \times w) \big|_{\partial \Omega}$$

$$= (\nu u_{yy}) \hat{z} \bigg|_{y = 0}.$$  

From part (a),

$$u_{yy} = \frac{U}{t} \cdot \frac{y}{\sqrt{4\pi vt}} e^{-y^2/4vt} \xrightarrow{\ y \rightarrow 0\ \ t \rightarrow 0\ } 0$$

so that

$$\sigma = 0$$

for all times \( t > 0 \). Thus, all the vorticity created at the wall is at the initial instant \( t = 0 \), corresponding to an infinite vortex sheet. Subsequently, this vorticity is diffused into the fluid by viscosity.
(c) For the stated circuit C:

\[ \int_C u(x) \cdot dx = U \cdot L_x + 0 \cdot L_y + (-u(L_y^t)) \cdot L_x + 0 \cdot L_y \]

\[ = (U-u(L_y^t)) \cdot L_x. \]

Thus,

\[ \lim_{L_y \to 0} \int_C u(x) \cdot dx = U \cdot L_x \]

since \( u(y, t) \to 0 \) as \( y \to 0^+ \). We therefore see that the net circulation around the loop C is diminishing in time.

Consistent with our earlier conclusion, all of the vorticity is created at the initial instant and is subsequently just diffused to larger distances from the wall.
Problem 6. (a) We see that
\[ y(\pm h) = \frac{x}{2u} (u^2 - h^2) = 0 \]
and
\[ u_{yy} = -\frac{x}{\nu}, \quad \partial_x p = -\gamma \]
so that
\[ -\partial_x p + \nu \partial_y^2 u = 0. \]
Thus, the stationary Navier-Stokes equation is satisfied. Of course,
\[ \nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} = 0, \]
so the flow is incompressible. Also, since
\[ (\mathbf{u} \cdot \nabla) \mathbf{u} = u \frac{\partial u}{\partial x} u = 0, \]
the stationary Navier-Stokes equation is satisfied.

(b) The only non-vanishing component of vorticity is
\[ \omega_2 = -\frac{\partial u}{\partial y} = \left( \frac{x}{\nu} \right) y. \]
Thus, vortex lines point in the \( z \)-direction, positive at the top wall and negative at the bottom wall. The maximum magnitude takes place at the walls. The vorticity source density is
\[ \zeta_2 = \begin{cases} \nu u_{yy} & y = -h \\ -\nu u_{yy} & y = +h \end{cases} = \begin{cases} -\gamma & y = -h \\ \gamma & y = h \end{cases} \]
since \( n' = \hat{y} \) on the bottom wall and \( \hat{n} = -\hat{y} \) on the top wall.
We thus see that there is vorticity generation constant in time at the walls, in opposite directions at the top and the bottom. The rate of creation is given by the tangential pressure gradient \( \gamma \) at the walls.

Once the vorticity is created, it diffuses into the interior of the flow, where it annihilates against vorticity of the opposite sign at the centerline \((y=0)\), where \( u = 0 \) at \( y = 0 \).

(c) Because of the stick b.c. at the walls, one can easily see that

\[
\oint_C u \cdot dx = 0 \cdot Lx + 0 \cdot (2h) + 0 \cdot (-Lx) = 0 \cdot (-2h)
\]

Thus, there is no net circulation around the curve \( C \).

This agrees with the conclusion in (b), since the vorticity for \( y > 0 \) is positive and that for \( y < 0 \) is negative and exactly cancel when integrated over the rectangle. The vorticity generation is equal and opposite at the two walls.
Problem 7. (a) By the standard result for the Biot-Savart formula

\[ \nabla \times \mathbf{u} = \mathbf{w} \text{ in } D \]

On the other hand,

\[ \nabla \times \nabla \phi = 0 \]

so that

\[ \nabla \times u = \nabla \times (\mathbf{u} - \nabla \phi) \]

\[ = \nabla \times \mathbf{u} \]

\[ = \mathbf{w}, \]

while

\[ u \cdot \hat{n} = (\mathbf{u} - \nabla \phi) \cdot \hat{n} \]

\[ = \mathbf{u} \cdot \hat{n} - \frac{\partial \phi}{\partial n} = 0 \text{ on } \partial D \]

by the b.c. for \( \phi \). Thus, \( u \) is a solution,

(b) We first study

\[ \mathbf{u}(x) = \int_{D} \frac{\mathbf{w}(x') \times \mathbf{e}}{4\pi r^3} \, dV', \quad \mathbf{e} = x - x' \]

when \( r = |x| \gg |x'| = r' \) in the support of the vorticity \( \mathbf{w} \).
Here we can use the standard multipole expansion

\[ \frac{1}{\rho} = \sum_{l=0}^{\infty} \frac{r^l}{r^{l+1}} P_l(\cos \theta) \]

where \( P_l \) is the degree-\( l \) Legendre polynomial and \( \theta \) is the angle between \( x \) and \( x' \). Thus,

\[ \frac{1}{\rho} = \frac{1}{r} + \frac{x \cdot x'}{r^3} + O\left(\frac{1}{r^5}\right) \]

\[ \frac{1}{\rho^3} = \frac{1}{r^3} + \frac{3x \cdot x'}{r^4} + O\left(\frac{1}{r^5}\right) \]

Together with \( \rho = x \left(1 + O\left(\frac{1}{r}\right)\right) \), we get

\[ \mathbf{\hat{u}} = \left( \int_{U_2} \mathbf{w}(x') \, dV' \right) \times \frac{x}{4\pi r^3} + O\left(\frac{1}{r^2}\right) \]

However, Föppl's theorem states that \( \int_{U_2} \mathbf{w}(x') \, dV' = 0 \) and thus

\[ \mathbf{\hat{u}} = O\left(\frac{1}{r^3}\right) \text{ for } r \to \infty \]

This component thus decays rapidly.
To make a similar estimate of the potential term \( \mathbf{u}_\phi = -\nabla \phi \) is not as easy but can be done using the formula

\[
\mathbf{u}_\phi(x) = \nabla \times \left( \int_{\Sigma_2} \frac{\mathbf{n} \times \mathbf{u}_\phi(x')}{4\pi |x-x'|} \, dA' \right)
\]

which follows from the Helmholtz decomposition.

If we again use the multipole expansion of \( \frac{1}{|x-x'|} \), then the first term can be shown to vanish

\[
\int_{\Sigma_2} \mathbf{n} \times \nabla \phi(x') \, dA' = 0
\]

d\Sigma_2

as we shall discuss later. It then follows again that

\[
\mathbf{u}_\phi = 0 \left( \frac{1}{r^3} \right), \quad r \to \infty
\]

which corresponds to a "dipole" or "vortex-ring" velocity.

Combining the two results,

\[
\mathbf{u} = 0 \left( \frac{1}{r^3} \right), \quad r \to \infty
\]
(c) To prove uniqueness, consider

\[ w := u - u' \]

for another possible solution \( u' \). Then,

\[ \nabla \times w = w - w = 0 \quad \text{in } \Omega. \]

Since \( \Omega \) is simply-connected, this implies that

\[ w = \nabla \psi \]

for a scalar function \( \psi \). Furthermore,

\[ \frac{\partial \psi}{\partial n} = \hat{n} \cdot w = \hat{n} \cdot u - \hat{n} \cdot u' = 0 - 0 = 0 \quad \text{on } \partial \Omega \]

and

\[ \Delta \psi = \nabla \cdot w = \nabla \cdot u - \nabla \cdot u' = 0 - 0 = 0 \quad \text{in } \Omega. \]

The solution of this Neumann problem for the Laplace equation in the simply-connected domain \( \Omega \) can be only a spatial constant \( \psi = C \). Therefore,

\[ u - u' = w = \nabla C = 0 \]

\[ \implies u = u'. \]

Thus, all solutions are equal to \( u \).

\[ \square \]