# Homework No.1, 550.794, Due February 10, 2023.

**Problem 1**. In turbulent flow with bulk velocity $U$ parallel to a straight wall, the *friction velocity* $u_\tau$ is defined as the square-root of the mean viscous stress at the wall, i.e. by

$$\nu \frac{\partial \overline{u}}{\partial n}\bigg|_{\text{wall}} := u_\tau^2.$$

and the *viscous wall unit* is then defined as

$$\delta_\nu := \nu / u_\tau.$$

(a) Derive the relation between the wall unit and the mean-free-path length $\ell_{mf}$:

$$\ell_{mf}/\delta_\nu \sim Ma\sqrt{f}$$

where $Ma = U/c_s$ is the bulk Mach number defined in terms of the sound speed $c_s$ and where $f$ is the friction factor.

(b) Using Navier's law for the slip velocity $\delta u_s$ at a solid wall, show that the mean velocity slip near the wall similarly satisfies

$$\delta\overline{u}_s/u_\tau \sim Ma\sqrt{f}.$$

**Problem 2**. In this problem we shall consider several "theorems" and examples in the paper of D. Rempfer, "On boundary conditions for incompressible Navier-Stokes problems," Appl. Mech. Rev. **59** 107–125 (2006).

(a) In his "Theorem 1", Rempfer claims that the boundary conditions on the pressure are specified by the requirement that $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$ on the boundary $\Gamma$ of the domain $\Omega$. Is this correct? If not, what condition does set the boundary condition on the pressure?

(b) In his "Theorem 2", Rempfer claims that the initial-boundary-value problem for the Navier-Stokes system, his equations (23)-(28), can be solved in a velocity-pressure formulation $(\mathbf{u}, p)$ that uses *arbitrary* Dirichlet boundary conditions

$$p(\mathbf{x}, t) = p_\Gamma(\mathbf{x}, t), \quad \mathbf{x} \in \Gamma$$

in the Poisson equation for the pressure $p$. However, must the resulting fields $(\mathbf{u}, p)$ satisfy all of the equations (23)-(28), as Remper claims? You may consider the special case of stick boundary conditions, i.e. $\mathbf{u}_\Gamma = \mathbf{0}$.

(c) Discuss the example in Rempfer's section 3.2.5.1 in light of the results of Temam (1982) on "compatibility conditions" for smooth solutions at time $t = 0$. If Rempfer's example were used as an initial condition at time $t = 0$ for solving the Navier-Stokes equation with stick b.c., would there be any inconsistency at times $t > 0$ between the pressure calculated with the Neumann b.c. (22) or with the Dirichlet b.c. (53)?

(d) (**BONUS**) The example in Rempfer's section 3.2.5.2 considers the Stokes equation in a bounded domain with non-zero velocities at the boundary, the so-called "non-homogeneous Stokes problem". This may be transformed into a *homogenous* Stokes problem with a suitable body-force $\mathbf{f}$ by using the method of Temam, "Navier-Stokes Equations" (North-Holland, 1984), Theorem I.2.4 for the steady case. The resulting homogeneous Stokes problem has a unique solution by the Theorem III.1.1 in Temam (1975). What is the Neumann boundary condition on the pressure in this solution? Is it the condition assumed by Rempfer in his equation (62)?

**Problem 3**. Batchelor's equation for the material evolution of an infinitesimal area element $\delta\mathbf{A}$ states that

$$\frac{d}{dt}\delta\mathbf{A} = (\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{u})\delta\mathbf{A} - (\boldsymbol{\nabla}\mathbf{u})\delta\mathbf{A}.$$

One way to derive this result is to note that $\delta\mathbf{A} = \delta\boldsymbol{\ell}\times\delta\boldsymbol{\ell}'$ where $\delta\boldsymbol{\ell}$, $\delta\boldsymbol{\ell}'$ are any two non-parallel, material line elements. Using the fact that both of these infinitesimal line elements satisfy the equation

$$\frac{d}{dt}\delta\boldsymbol{\ell} = (\delta\boldsymbol{\ell}\boldsymbol{\cdot}\boldsymbol{\nabla})\mathbf{u},$$

derive Batchelor's equation.

**Problem 4**. (a) If $\Omega$ is a bounded domain in $\mathbb{R}^3$ which rigidly rotates with angular velocity vector $\boldsymbol{\Omega}$ and if $\boldsymbol{\omega}$ is the vorticity field for the Navier-Stokes solution with stick b.c. in $\Omega$, then show that

$$\frac{1}{|\Omega|}\int_\Omega \boldsymbol{\omega}\, dV = 2\boldsymbol{\Omega}$$

where $|\Omega|$ is the volume of $\Omega$. Note that $\Omega$ represents physically a closed container of fluid spinning on an axis. You may use the fact that $\boldsymbol{\omega} = 2\boldsymbol{\Omega}$ on the surface $\partial\Omega$ that represents the walls of the container.

(b) Show that Lyman and Panton definitions of the vorticity source are related by

$$\boldsymbol{\sigma}^L = \boldsymbol{\sigma}^P - \nu\hat{\mathbf{n}}\frac{\partial\omega_n}{\partial n} + \nu(\boldsymbol{\nabla}\hat{\mathbf{n}})\boldsymbol{\omega},$$

where $\hat{\mathbf{n}}$ is the outward-pointing normal to the wall. Show also that $\hat{\mathbf{n}}\boldsymbol{\cdot}\boldsymbol{\sigma}^P = \nu(\partial\omega_n/\partial n)$.

**Problem 5**. In this problem we consider a simple exact solution which illustrates the generation of vorticity at an accelerating boundary. We solve the Navier-Stokes equation for the velocity $\mathbf{u} = (u, v, w)$ in the upper half-space $\mathbb{R}^3_+ = \{(x, y, z) : \; y > 0\}$, with initial condition $\mathbf{u}_0 = \mathbf{0}$ at time $t = 0$ and boundary condition

$$\mathbf{u} = (U, 0, 0) \;\text{ at } y = 0$$

for times $t > 0$. This problem corresponds to a fluid above an infinite, flat plate which is impulsively accelerated from rest at times $t \leq 0$ to a constant velocity $U$ along the $x$-direction at times $t > 0$.

(a) Verify that the exact solution of the Stokes equation $\partial_t \mathbf{u} = -\boldsymbol{\nabla} p + \nu \triangle \mathbf{u}$ for this problem is given by a constant pressure $p$ and $\mathbf{u} = (u(y, t), 0, 0)$ with

$$u(y, t) = U \mathrm{erfc}\left(\frac{y}{\sqrt{4\nu t}}\right)$$

in terms of the complementary error function $\mathrm{erfc}(s) = \frac{2}{\sqrt{\pi}} \int_s^\infty e^{-\sigma^2} \, d\sigma$. Show that $(\mathbf{u} \boldsymbol{\cdot} \boldsymbol{\nabla})\mathbf{u} = \mathbf{0}$ for this velocity, so that $(\mathbf{u}, p)$ also solves the Navier-Stokes equation.

(b) Calculate the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}$. Show that it takes on its maximum magnitude at the wall for all times $t > 0$ and that this maximum at $y = 0$ is infinite at time $t = 0$ and decaying monotonically for $t > 0$. Calculate also the vorticity source density $\boldsymbol{\sigma}$ at the wall

$$\boldsymbol{\sigma} = \mathbf{n}' \times \nu (\boldsymbol{\nabla} \times \boldsymbol{\omega}), \quad y = 0$$

and show that $\boldsymbol{\sigma} = \mathbf{0}$ for all times $t > 0$. Conclude that all the vorticity is created at the wall at the instant $t = 0$ of impulsive acceleration and thereafter is diffused into the interior by viscosity.

(c) Calculate the circulation $\oint_C \mathbf{u}(t) \boldsymbol{\cdot} d\mathbf{x}$ around the rectangular circuit $C$ that runs in straight lines from $(0, 0, 0) \to (L_x, 0, 0) \to (L_x, L_y, 0) \to (0, L_y, 0) \to (0, 0, 0)$ and show that

$$\lim_{L_y \to \infty} \oint_C \mathbf{u}(t) \boldsymbol{\cdot} d\mathbf{x} = U L_x.$$

Does the fact that this circulation is independent of time $t > 0$ agree with your conclusions in part (b)?

**Problem 6**. In this problem we consider a simple exact solution which illustrates the generation of vorticity by a tangential pressure-gradient at a boundary. We consider *plane Poiseuille flow* in the channel $\Omega = \{(x, y, z) : \; |y| < h\}$ with constant pressure gradient $\boldsymbol{\nabla} p = (-\gamma, 0, 0)$.

(a) Verify that the steady solution of the Navier-Stokes equation for stick boundary conditions is given by $\mathbf{u} = (u(y), 0, 0)$ with

$$u(y) = \frac{\gamma}{2\nu}(h^2 - y^2)$$

and $p = p_0 - \gamma x$ for an arbitrary constant $p_0$.

(b) Calculate the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{u}$ and discuss its general features. What is the direction of vorticity? Where does it take on its maximum magnitude? Calculate also the vorticity source density $\boldsymbol{\sigma} = \mathbf{n}'\times\nu(\boldsymbol{\nabla}\times\boldsymbol{\omega})$ at the walls and show that there is vorticity generation constant in time, in opposite directions at the two walls. What happens to this vorticity after it is created?

(c) Calculate the circulation $\oint_C \mathbf{u}\cdot d\mathbf{x}$ around the rectangular circuit $C$ that runs in straight lines from $(0,-h,0) \to (L_x,-h,0) \to (L_x,h,0) \to (0,h,0) \to (0,-h,0)$ and show that

$$\oint_C \mathbf{u}\cdot d\mathbf{x} = 0.$$

Does this result agree with your conclusions in part (b)?

**Problem 7**. Suppose that $\Omega \subset \mathbb{R}^3$ is the exterior domain outside a finite body $B$ with smooth surface $\partial B$. We assume that the body $B$ has no handles, so that the flow domain $\Omega$ is simply-connected. Given a vorticity distribution $\boldsymbol{\omega}$ in $\Omega$, we consider the "div-curl problem" to find a velocity field $\mathbf{u}$ so that $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$ and $\boldsymbol{\nabla}\times\mathbf{u} = \boldsymbol{\omega}$ in $\Omega$, and $\mathbf{u}\cdot\hat{\mathbf{n}} = 0$ at $\partial\Omega$.

(a) Show that a solution exists by the recipe of Lighthill (1963):

$$\mathbf{u} = \tilde{\mathbf{u}} - \boldsymbol{\nabla}\phi$$

where $\tilde{\mathbf{u}}$ is given by the Biot-Savart formula

$$\tilde{\mathbf{u}}(\mathbf{x}) = \int_\Omega \frac{\boldsymbol{\omega}(\mathbf{x}')\times\boldsymbol{\rho}}{4\pi\rho^3}\, dV', \qquad \boldsymbol{\rho} = \mathbf{x} - \mathbf{x}'$$

and $\phi$ solves the Neumann problem

$$\triangle\phi = 0 \text{ in } \Omega; \qquad \frac{\partial\phi}{\partial n} = \tilde{\mathbf{u}}\cdot\hat{\mathbf{n}} \text{ on } \partial\Omega.$$

You may assume without proof that the solution $\phi$ of the latter problem exists and is unique up to a constant when $\Omega$ is simply-connected.

(b) Assuming that $\boldsymbol{\omega}$ is well-localized, e.g. compactly supported, give arguments why $\mathbf{u} = O(1/r^3)$ as $r \to \infty$. (It is OK if you cannot give a rigorous proof. We'll discuss later some tools such as Helmholtz decomposition which help in this.)

(c) Show that the solution $\mathbf{u}$ is unique. *Hint:* Consider $\mathbf{w} := \mathbf{u} - \mathbf{u}'$ where $\mathbf{u}'$ is another possible solution.