

### (C) RANS Equations of Channel Flow

T & L, Section 5.2

We consider turbulent flow between two parallel plates separated at a distance  $2h$ .

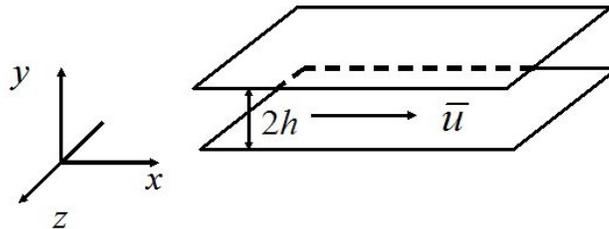


Figure C.1.

The mean flow  $\bar{\mathbf{u}}$  is assumed to be in the direction  $x$ , driven by a pressure-gradient in that direction, parallel to the two plates. The various directions are called

streamwise  $x$ -direction

wall-normal or cross-stream  $y$ -direction

spanwise  $z$ -direction

If the channel is large enough in the streamwise and spanwise directions, then the velocity statistics should become independent of  $x$  and  $z$  and depend only on  $y$ , i.e. become homogeneous in the plane-parallel directions. Furthermore, if the system is evolved long enough, then the velocity statistics should become independent of time  $t$ , i.e. become stationary in time. We shall hereafter assume plates infinitely long and wide, and let the flow evolve infinitely long in time. Thus, all derivatives with respect to  $t$  and  $z$  are zero, and all derivatives with respect to  $x$  are also zero, except for the pressure gradient  $\partial\bar{p}/\partial x$ , which drives the flow against the shear stresses at the walls.

A sketch of the flow is as follows:

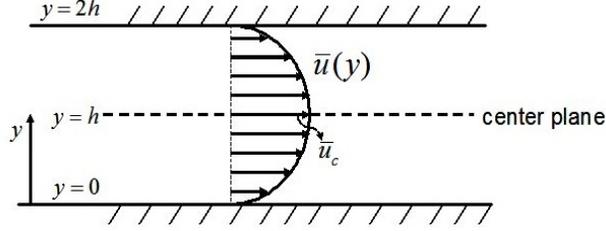


Figure C.2.

With all the above assumptions, the RANS equations become

$$\begin{aligned}
 x\text{-momentum} \quad 0 &= -\frac{\partial \bar{p}}{\partial x} - \frac{\partial}{\partial y} \overline{u'v'} + \nu \frac{\partial^2 \bar{u}}{\partial y^2} \\
 y\text{-momentum} \quad 0 &= -\frac{\partial \bar{p}}{\partial y} - \frac{\partial}{\partial y} \overline{(v')^2} \\
 z\text{-momentum} \quad 0 &= -\frac{\partial}{\partial y} \overline{w'v'} \\
 \text{continuity} \quad 0 &= \frac{\partial \bar{v}}{\partial y}
 \end{aligned} \tag{4}$$

Since  $\bar{v}(0) = 0$  due to the stick b.c. at the wall, the final continuity equation just reproduces  $\bar{v}(y) \equiv 0$ . Since  $\overline{w'v'}(y=0) = 0$  also due to stick b.c., the  $z$ -momentum equation integrates to give  $\overline{w'v'}(y) = 0$ .

The two most important equations are those for the  $x$ - and  $y$ -components of momentum. Since  $(\partial/\partial x)\overline{(v')^2} = 0$  by assumption, the partial  $x$ -derivative of the  $y$ -equation gives

$$0 = -\frac{\partial}{\partial x} \left( \frac{\partial \bar{p}}{\partial y} \right) - \underbrace{\frac{\partial}{\partial x} \frac{\partial}{\partial y} \overline{(v')^2}}_{=0} = -\frac{\partial}{\partial y} \left( \frac{\partial \bar{p}}{\partial x} \right)$$

which shows that streamwise pressure-gradient is  $y$ -independent:

$$\frac{\partial \bar{p}}{\partial x} = \text{constant in } y$$

This fact allows us to integrate the  $x$ -momentum equation in  $y$ . Let us define the friction velocity  $u_*$  in terms of the viscous stress at the bottom wall:

$$-\nu \frac{\partial \bar{u}}{\partial y} \Big|_{y=0} = -u_*^2$$

Since  $\bar{u}(y)$  should be an increasing function near  $y = 0$ ,  $\partial \bar{u}/\partial y(0) > 0$  and this definition makes sense. By symmetry around the centerplane, one must have

$$-\nu \frac{\partial \bar{u}}{\partial y} \Big|_{y=2h} = +u_*^2$$

at the upper wall. Note that  $\overline{u'v'} = 0$  at  $y = 0$  or  $2h$  because of the stick b.c. and thus the viscous stress is the total stress at the two walls.

Using this definition, we may now integrate the  $x$ -equation from  $y = 0$  upward, to yield

$$0 = -y\left(\frac{\partial \bar{p}}{\partial x}\right) - \overline{u'v'}(y) + \nu \frac{\partial \bar{u}}{\partial y}(y) - u_*^2$$

This equation has several interesting special cases. The first is at the centerplane ( $y = h$ ), where the total shear stress must vanish by symmetry:

$$\tau_{xy}^{\text{tot}} = \overline{u'v'} - \nu \frac{\partial \bar{u}}{\partial y} = 0 \quad \text{at } y = h$$

Solving yields the important relation

$$u_*^2 = -h \frac{\partial \bar{p}}{\partial x}$$

The same relation can be obtained by setting  $y = 2h$ , giving

$$2u_*^2 = (2h)\left(-\frac{\partial \bar{p}}{\partial x}\right)$$

which just represents the total balance of  $x$ -momentum. The pressure-gradient integrated across the channel of width  $2h$  balances the stress  $u_*^2$  at each of the 2 walls.

Substituting this result into the equation for general  $y$  gives

$$\tau_{xy}^{\text{tot}} = \overline{u'v'} - \nu \frac{\partial \bar{u}}{\partial y} = -u_*^2\left(1 - \frac{y}{h}\right)$$

This result has the consequence that the stress near the wall is constant (independent of  $y$ ):

$$\tau_{xy}^{\text{tot}} \cong -u_*^2, \quad 0 \leq y \ll h$$

For very small  $y$  the viscous stress dominates, while for larger  $y$  the Reynolds stress dominates (as we shall see). This is often considered to be a “momentum cascade” in space, in analogy to the Kolmogorov energy-cascade in scale where the energy flux plus viscous dissipation is constant for length-scales  $\ell \leq L$ . The stress may also be written in terms of the displacement from the centerplane

$$y' = y - h$$

which gives

$$\tau_{xy}^{\text{tot}} = \overline{u'v'} - \nu \frac{\partial \bar{u}}{\partial y} = u_*^2 \frac{y'}{h}.$$

This formula exhibits the anti-symmetry of stress around the center plane.

energy balance:

The energy balance in the mean field can be obtained by multiplying the  $x$ -momentum equation by  $\bar{u}(y)$ :

$$\bar{u}\left(-\frac{\partial \bar{p}}{\partial x}\right) - \frac{\partial}{\partial y}[\bar{u}\tau_{xy}^{\text{tot}}] = -\overline{u'v'}\frac{\partial \bar{u}}{\partial y} + \nu\left(\frac{\partial \bar{u}}{\partial y}\right)^2 = -\frac{\partial \bar{u}}{\partial y}\tau_{xy}^{\text{tot}}$$

The spatial transport of energy is balanced by the turbulent production and the viscous dissipation. Integrated across the channel, this yields

$$\bar{u}_m\left(-\frac{\partial \bar{p}}{\partial x}\right) = -\frac{1}{2h}\int_0^{2h} dy \frac{\partial \bar{u}}{\partial y}\tau_{xy}^{\text{tot}} = \frac{\bar{u}_m u_*^2}{h}$$

where the second equality used an integration by parts and we have defined

$$\bar{u}_m = \frac{1}{2h}\int_0^{2h} dy \bar{u}(y),$$

the mean value of  $\bar{u}$  over a section of the channel. Thus, the relation  $-\partial \bar{p}/\partial x = u_*^2/h$  also expresses the energy balance in the mean field.

The turbulent kinetic energy balance becomes

$$\frac{\partial}{\partial y}\left[\overline{(p' + \frac{1}{2}q^2)v'} - \frac{1}{2}\nu\frac{\partial q^2}{\partial y}\right] = -\overline{u'v'}\frac{\partial \bar{u}}{\partial y} - \epsilon$$

with, as usual,  $\epsilon = \overline{\nu u'_{i,j}u'_{i,j}}$  the viscous dissipation by turbulence fluctuations. Integrated across the channel, we see that

$$\frac{1}{2h}\int_0^{2h} dy \left[-\overline{u'v'}\frac{\partial \bar{u}}{\partial y}\right] = \frac{1}{2h}\int_0^{2h} dy \epsilon(y).$$

Thus, turbulence production balances turbulent dissipation.

mean vorticity balance

The only non-vanishing component of the mean vorticity is

$$\bar{\omega}_z = -\frac{\partial \bar{u}}{\partial y}$$

From the vorticity conservation  $\partial \bar{\omega} + \nabla \cdot \bar{\Sigma} = 0$  one infers

$$\partial_y \bar{\Sigma}_{yz} = 0$$

with

$$\bar{\Sigma}_{yz} = \overline{v'\omega'_z - w'\omega'_y} - \nu\frac{\partial \bar{\omega}_z}{\partial y}.$$

Thus, the flux of mean  $z$ -vorticity in the  $y$ -direction is constant across the channel, i.e.

$$-\bar{\Sigma}_{yz} = \bar{\Sigma}_{zy} = \sigma_* = \text{const.}$$

Although  $\tau_{xy}^{\text{tot}}$  is only approximately constant,  $\cong -u_*^2$  for  $0 \leq y \leq h$ , the vorticity flux is exactly constant across the whole width of the channel. Thus, there is a “spatial cascade” not only of momentum but also of vorticity.

An important relation for  $\sigma_*$  can be obtained by rewriting the Navier-Stokes equation as

$$\begin{aligned}\partial_t u_k &= \frac{1}{2} \epsilon_{klm} \Sigma_{lm} - \partial_k p_*, \\ p_* &= p + \frac{1}{2} |\bar{\mathbf{u}}|^2.\end{aligned}$$

This result is more or less obvious, because the curl of the NS equation must give  $\partial_t \boldsymbol{\omega} = -\nabla \cdot \boldsymbol{\Sigma}$ .

It is just another form of the equation

$$\partial_t \bar{\mathbf{u}} = \mathbf{u} \times \boldsymbol{\omega} - \nu \nabla \times \boldsymbol{\omega} - \nabla p_*.$$

If we now consider statistically stationary turbulence, the RANS equations yield

$$\partial_k \bar{p}_* = \frac{1}{2} \epsilon_{klm} \bar{\Sigma}_{lm}, \text{ or } \bar{\Sigma}_{ij} = \epsilon_{ijk} \partial_k \bar{p}_*.$$

If this is applied in channel flow, one gets for example

$$\bar{\Sigma}_{yz} = \frac{\partial \bar{p}}{\partial x} = -\frac{u_*^2}{h} < 0.$$

In particular,  $\sigma_* = u_*^2/h$ . Of course, the mean space flux of other vorticity components in other directions will be related also to pressure gradients, but here we focus on  $z$ -vorticity.

The picture is as follows:

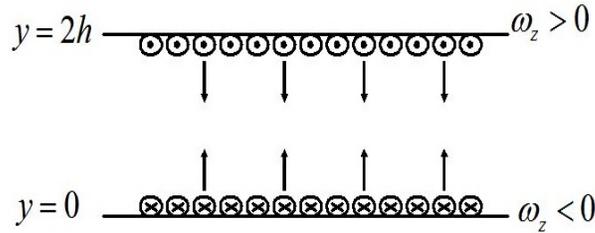


Figure C.3.

Lines of negative  $\omega_z$ -vorticity form at the bottom plate ( $y = 0$ ) and move upward, while lines of positive  $\omega_z$ -vorticity form at the upper plate and move downward. This is only a crude description, of course, accurate in the mean but not for the fluctuations. We shall discuss later the implications for vorticity fluctuations.

The relation  $\bar{\Sigma}_{yz} = \partial\bar{p}/\partial x$  was apparently first derived in RANS theory by

G. I. Taylor, “The transport of vorticity and heat through fluids in turbulent motion,” Proc. Roy. Soc. Lond. A **135** 685-702 (1932)

Of course, we also know from the later work of Lighthill (1963) that  $\partial p/\partial x$  is the source of  $\omega_z$ -vorticity at both the walls. Taylor’s relation implies very important constraints on vorticity dynamics that have not always been observed in phenomenological turbulence models.

One important implication is for energy dissipation. Since energy balance requires that

$$\bar{u}_m(-\frac{\partial\bar{p}}{\partial x}) = \frac{1}{2h} \int_{-h}^h [\epsilon(y) + \nu(\frac{\partial\bar{u}}{\partial y})^2] dy' = \epsilon_m^{\text{tot}}$$

we see that also

$$\bar{u}_m(-\bar{\Sigma}_{yz}) = \epsilon_m^{\text{tot}}.$$

This result implies that organized motion of vorticity is necessary to produce mean energy dissipation in channel flow! We shall discuss in more detail below the precise motions involved.

It is interesting to note that this connection between vortex motion on the one hand and pressure drops & energy dissipation on the other hand has been noted in a rather different context, that of quantum superfluids and superconductors. There it is known as the Josephson-Anderson relation. See especially

P. W. Anderson, “Considerations on the flow of superfluid helium,” Rev. Mod. Phys. **38** 298-310 (1966), Appendix B.

E. R. Huggins, “Energy dissipation theorem and detailed Josephson equation for ideal incompressible fluids,” Phys. Rev. A **1** 332-337 (1970)

The latter paper, although it discusses quantum vortices, uses as a model the classical Navier-Stokes equation and thus applies also to classical fluids. In the quantum context, the relation states that

$$\mu_*(2) - \mu_*(1) = \frac{1}{2} \int_{C:1 \rightarrow 2} \epsilon_{k\ell m} \Sigma_{\ell m} dx_k$$

where  $\Sigma_{\ell m}$  is the flux of the  $m$ th component of vorticity in the  $\ell$ th direction for a quantized vortex-line and

$$\mu_* = \mu + \frac{1}{2}|\mathbf{u}|^2$$

where  $\mu$  is the chemical potential (replacing the pressure  $p$  in the classical analogue). The curve  $C$  is any smooth path starting at point 1 and ending at point 2. If points 1 and 2 are two points in a channel containing a superflow with mass flux  $J$  down the channel, as pictured here:

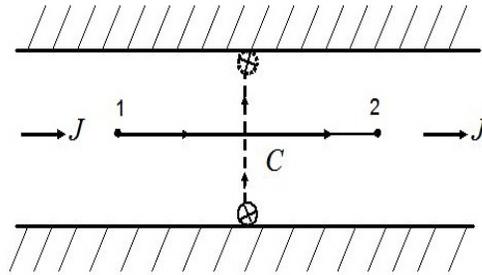


Figure C.4.

and if a quantized vortex line created at the bottom wall migrated across the whole channel width, crossing curve  $C$ , then the loss of energy of the superflow between points 1 and 2 is given by

$$J[\mu_*(2) - \mu_*(1)] = \frac{1}{2}J \int_{C:1 \rightarrow 2} \epsilon_{k\ell m} \Sigma_{\ell m} dx_k = \epsilon^{\text{tot}}.$$

For reviews, see

W. Zimmerman, Jr., “Energy transfer and phase slip by quantum vortex motion in superfluid He4,” *J. Low Temp. Phys.* **93** 1003-1018 (1993)

R. E. Packard, “The role of the Josephson-Anderson equation in superfluid helium,” *Rev. Mod. Phys.* **70** 641-651 (1998)

Also interesting may be the following papers

E. R. Huggins, “Vortex currents in turbulent superfluid and classical fluid channel flow, the Magnums effect, and Goldstone bosom fields, ” *J. Low Temp. Phys.* **96** 317-346 (1994)

G. L. Eyink, “Turbulent flow in pipes and channels as cross-stream ‘inverse cascades’ of vorticity,” *Phys. Fluids* **20** 125101 (2008)

which are concerned with the strong analogies between the superfluid turbulence and classical turbulence flows in a channel. (Note that Huggins was not aware of the much earlier work of G. I. Taylor on the classical case!)

It is interesting to compare the two problems. For example, there is a “drag reduction” problem also in superconductor technology, involving charged superfluids. In that case, supercurrents along a superconducting wire should be resistanceless and without voltage drops. However, in practice, there are voltage drops and energy dissipation associated with nucleation of quantized vortex lines, containing magnetic flux in quantized amounts, that migrate across the current. Applied physicists have found that this problem may be solved by “pinning” the vortices so that they cannot cross the wire. For example, one way to do this is to use not pure crystalline superconductors but instead powdered samples. This introduces pinning sites that trap the vortex lines and allow resistanceless flow! See

D. J. Bishop, P. L. Gammel, and D. A. Huse, “Resistance in high-temperature superconductors,” *Sci. Am.* **268** 48–55 (1993)

Let us return to turbulent channel flow. Up until now we have considered only structures of the ensemble-averaged vorticity. Let us briefly discuss the instantaneous vorticity of the individual realizations. An influential early proposal was made by

T. Theodorsen, “Mechanism of turbulence,” in *Proceedings of the Midwestern Conference on Fluid Mechanics* (Ohio State University, Columbus, OH, 1952)

that vorticity structures near the wall in channel flow have a hairpin structure. Here is the sketch from Theodorsen’s original paper:

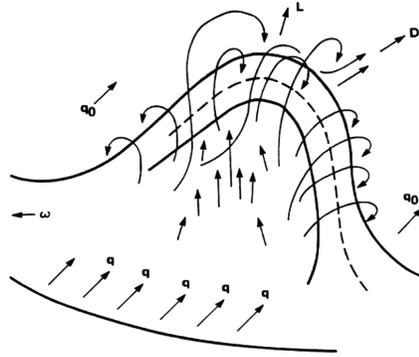


Figure C.4.

Such hairpin vortices are also sometimes called “ $\Lambda$ -vortices” or “ $\Omega$ -vortices”. The legs of the vortex were suggested by Theodorsen to be inclined at  $45^\circ$  with respect to the wall. As you can see from the flows in the sketch, the resulting fluid motions have  $v' < 0$  when  $u' > 0$ , and  $v' > 0$  when  $u' < 0$ . Hence, such structures can contribute to the required negative Reynolds stress  $\overline{u'v'} < 0$  at the bottom wall (and their mirror images about the center-plane will contribute positive stress at the top wall.) The “lifting” of such vortices from the wall will also contribute to the required mean flux of spanwise vorticity in the cross-stream direction.

These ideas of Theodorsen have received some support from numerical simulations and experiments. Detailed visualizations were first possible using simulations. The papers

P. Moin & J. Kim, “The structure of the vorticity field in turbulent channel flow. Part 1. Analysis of instantaneous fields and statistical correlations”, *J. Fluid Mech.* **155** 441–464 (1985)

P. Moin & J. Kim, “The structure of the vorticity field in turbulent channel flow. Part 2. Study of ensemble-averaged fields” , *J. Fluid Mech.* **162** 339–363 (1986)

observed hairpin-like structures both in instantaneous vortex lines:

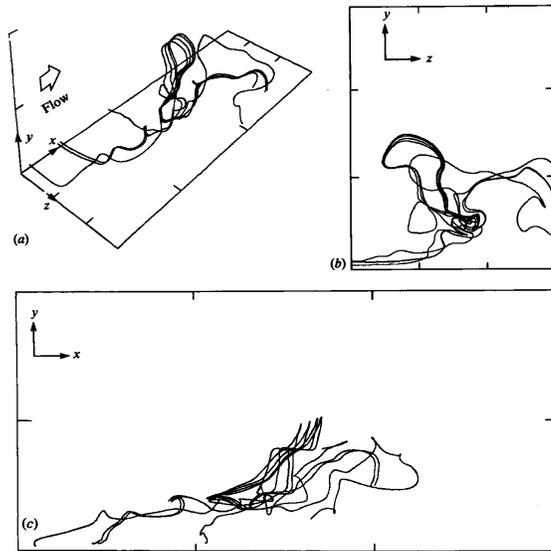


FIGURE 17. A set of vortex lines (vortex filament) displaying a hairpin-like structure. (a) 3-D view, the streamwise extent of the figure is  $1.96\delta$  ( $1257\nu/u_\tau$ ) and its spanwise extent is  $0.74\delta$  ( $471\nu/u_\tau$ ); (b) end view (( $y, z$ )-plane); (c) side view (( $x, y$ )-plane).

Figure C.4.

and in the lines of conditionally averaged vorticity:

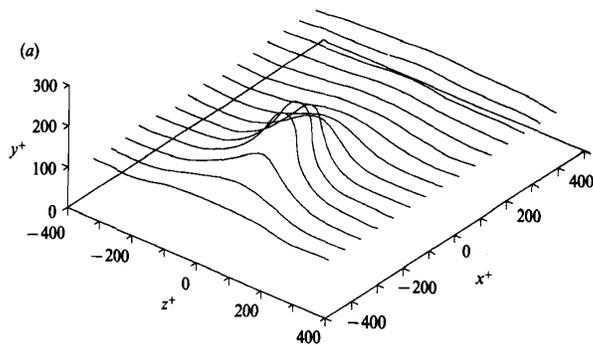


Figure C.5.

The lines above are taken from a “variable-interval space-averaging” (VISA) sampling technique, in which events are chosen so that  $\partial u/\partial x > 0$  and the velocity variance in a local streamwise space-average are greater than  $1.2u_{rms}^2$ . The condition  $\partial u/\partial x > 0$  means that the streamwise velocity is increasing downstream of the test point, which implies the point is involved in a streamwise low-speed fluctuation. This conditions are applied at a distance 100

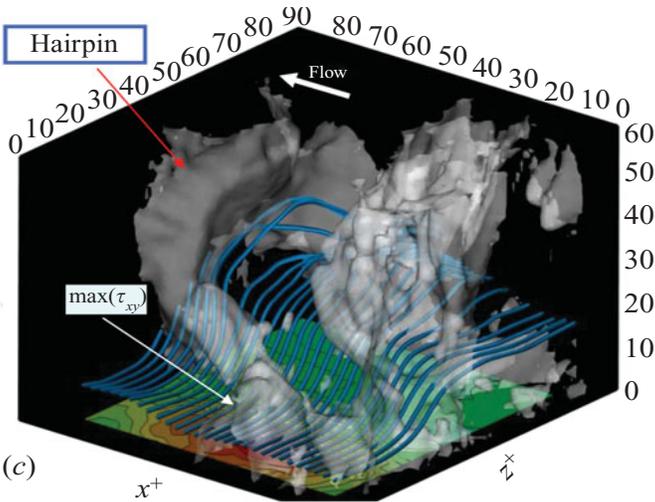
“wall units”  $\nu/u_*$  away from the wall, so that the event should consist of an “ejection” of low-speed fluid away from the wall. Clearly, both instantaneous and conditional vorticity structures are similar to the hairpins predicted by Theodorsen.

Experimental techniques are now also capable of detailed 3D imaging. For example

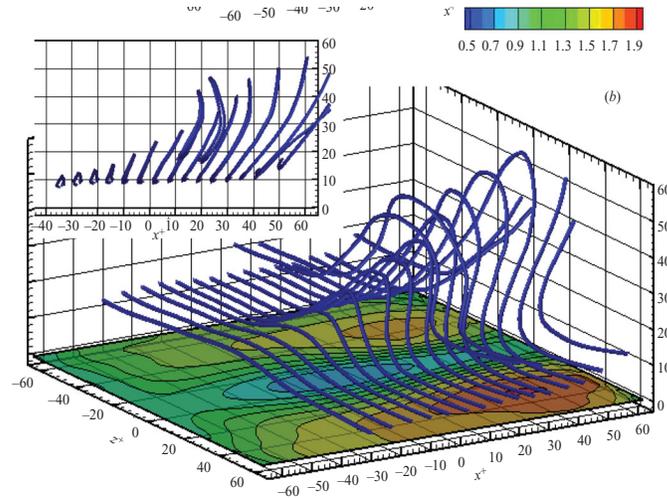
J. Sheng, E. Malkiel & J. Katz, “Buffer layer structures associated with extreme wall stress events in a smooth wall turbulent boundary layer,” *J. Fluid Mech.* **633** 17–60 (2009)

employ digital holographic microscopy to measure 3D velocity fields in the inner part of a turbulent boundary layer. Their measurements are taken at distances less than  $60 \nu/u_*$ , where current numerical simulations have difficulty in achieving good spatial resolution.

Here are some of their plots of both instantaneous vortex lines



and lines of conditionally averaged vorticity:



The vortex field in the second image is averaged over a conditional ensemble given that the instantaneous viscous stress at a point on the wall ( $x = z = 0$ ) has a value less than (more negative than)  $-0.6u_*^2$ . The colors at the walls indicate the values of the stress divided by  $-u_*^2$ .

For a recent more detailed survey of the hairpin picture, see

R. J. Adrian, “Hairpin vortex organization in wall turbulence,” *Phys. Fluids* **19** 041301 (2007)

and for a contrarian view:

P. S. Bernard, “The hairpin vortex illusion,” *13th European Turbulence Conference (ETC13)*, *Journal of Physics: Conference Series* **318** 062004 (2011)