

(E) Dynamics of Vortex-Lines in a Turbulent Flow

The conservation of circulations (Kelvin Theorem) is intimately related to the materiality of vortex lines. Many classical arguments in turbulence assume also that vortex lines move as material lines, e.g. Taylor's explanation of growth of $\overline{\omega^2}$. These ideas must therefore also be critically reconsidered.

Let us first review some of the standard theory of vortex lines. By definition, a vortex line is a line everywhere parallel/tangent to the local vorticity vector. Thus, a parameterized curve $\mathbf{x}(\sigma)$ is a vortex line iff for some scalar field $\alpha(\mathbf{x})$

$$\frac{d\mathbf{x}}{d\sigma} = \alpha(\mathbf{x})\boldsymbol{\omega}(\mathbf{x}).$$

Now consider a material line $\mathbf{x}(\sigma, t)$ which satisfies for some velocity \mathbf{u}

$$\frac{d}{dt}\mathbf{x}(\sigma, t) = \mathbf{u}(\mathbf{x}(\sigma, t), t)$$

for all t and σ . If $\mathbf{x}(\sigma, t)$ is a vortex line at the initial instant $t = t_0$, then it remains a vortex line for all other times if and only if

$$(*) \quad \mathbf{0} = \boldsymbol{\omega} \times [D_t\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}] = \boldsymbol{\omega} \times [\partial_t\boldsymbol{\omega} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega})]$$

or, equivalently, if and only if

$$(*) \quad D_t\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} = \partial_t\boldsymbol{\omega} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = \beta\boldsymbol{\omega}$$

for some scalar field $\beta(\mathbf{x}, t)$. This is the so-called Helmholtz-Zorawski condition from

K. Zorawski, "Über die Erhaltung der Wirbelbewegung," Bulletin de l'Académie des Sciences de Cracovie, Comptes Rendus, p. 335 (1900)

This condition is somewhat weaker than the condition for conservation of vorticity flux through a moving circuit $C(t)$. We give a simple proof of this result, following

R. Prim & G. Truesdell, "A derivation of Zorawski's criterion for permanent vector lines," Proc. Amer. Math. Soc. **1** 32 (1950); C. Truesdell, The Kinematics of Vorticity (Indiana University Press, Bloomington, 1954).

The material line is a vortex line if and only if

$$\frac{\partial \mathbf{x}}{\partial \sigma} \times \boldsymbol{\omega} = \mathbf{0}.$$

Note that

$$\frac{d}{dt} \left(\frac{\partial \mathbf{x}}{\partial \sigma} \right) = \left(\frac{\partial \mathbf{x}}{\partial \sigma} \right) \cdot \nabla \mathbf{u}(\mathbf{x}, t)$$

by the chain rule. Now take the time-derivative of $\mathbf{y}(\sigma, t) \equiv \frac{\partial \mathbf{x}}{\partial \sigma}(\sigma, t) \times \boldsymbol{\omega}(\mathbf{x}(\sigma, t), t)$:

$$\begin{aligned} \frac{d}{dt} \mathbf{y}(t) &= \left(\frac{\partial \mathbf{x}}{\partial \sigma} \cdot \nabla \right) \mathbf{u} \times \boldsymbol{\omega} + \frac{\partial \mathbf{x}}{\partial \sigma} \times (D_t \boldsymbol{\omega}) \\ &= \left(\frac{\partial \mathbf{x}}{\partial \sigma} \cdot \nabla \right) \mathbf{u} \times \boldsymbol{\omega} + \frac{\partial \mathbf{x}}{\partial \sigma} \times (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} + \beta \left(\frac{\partial \mathbf{x}}{\partial \sigma} \times \boldsymbol{\omega} \right) \end{aligned} \quad (48)$$

Next substitute into the first two terms the decomposition $\frac{d\mathbf{x}}{d\sigma} = \delta \cdot \boldsymbol{\omega} + \gamma(\boldsymbol{\omega} \times (\frac{d\mathbf{x}}{d\sigma} \times \boldsymbol{\omega}))$ with components parallel and perpendicular to $\boldsymbol{\omega}$, where $\delta = \frac{1}{|\boldsymbol{\omega}|^2} (\boldsymbol{\omega} \cdot \frac{d\mathbf{x}}{d\sigma})$ and $\gamma = \frac{1}{|\boldsymbol{\omega}|^2}$. It is easy to see that the term parallel to $\boldsymbol{\omega}$ gives a zero contribution. The final result is that

$$\frac{d}{dt} \mathbf{y} = \gamma \left((\boldsymbol{\omega} \times \mathbf{y}) \cdot \nabla \right) \mathbf{u} \times \boldsymbol{\omega} + \gamma (\boldsymbol{\omega} \times \mathbf{y}) \times (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} + \beta \mathbf{y}.$$

Although this equation appears complicated, it is just a linear ODE for each value of σ

$$\frac{d}{dt} \mathbf{y} = \mathbf{A}(t) \mathbf{y}$$

with an appropriate matrix $\mathbf{A}(\sigma, t)$. This equation has the exact solution

$$\mathbf{y}(t) = \text{Texp} \left(\int_{t_0}^t ds \mathbf{A}(s) \right) \mathbf{y}(t_0) \quad (49)$$

in terms of a time-ordered exponential. Thus, $\mathbf{y}(t) = \frac{\partial \mathbf{x}}{\partial \sigma} \times \boldsymbol{\omega} = \mathbf{0}$ if the same condition held true at the initial time t_0 . Thus (*) is sufficient for the material line to remain a vortex line. On the other hand, the condition (*) is also clearly necessary, as follows from the first line of (48) by substituting $\frac{\partial \mathbf{x}}{\partial \sigma} = \alpha \boldsymbol{\omega}$. Finally, note by vector calculus identities

$$\begin{aligned} D_t \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} &= \partial_t \boldsymbol{\omega} + (\mathbf{u} \cdot \nabla) \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} \\ &= \partial_t \boldsymbol{\omega} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) - (\nabla \cdot \mathbf{u}) \boldsymbol{\omega} + \underbrace{(\nabla \cdot \boldsymbol{\omega}) \mathbf{u}}_{=0} \end{aligned} \quad (50)$$

so that

$$\boldsymbol{\omega} \times [D_t \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla) \mathbf{u}] = \mathbf{0} \text{ iff } \boldsymbol{\omega} \times [\partial_t \boldsymbol{\omega} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega})] = \mathbf{0} \quad \text{QED}$$

It is worthwhile to remark that this proof did NOT assume that \mathbf{u} is the velocity related to the vorticity itself, i.e.

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}.$$

Material lines for any velocity \mathbf{u} will remain integral lines of $\boldsymbol{\omega}$ /“vortex lines” (if initially so) as long as the condition (*) is satisfied.

We now see that coarse-grained vorticity in a turbulent flow will generally NOT satisfy the required condition (*), which takes the form

$$\bar{\boldsymbol{\omega}}_\ell \times [\nabla \times (\mathbf{f}_\ell^v - \nu \nabla \times \bar{\boldsymbol{\omega}}_\ell + \bar{\mathbf{f}}_\ell^B)] \neq 0.$$

if one chooses the natural advection velocity $\bar{\mathbf{u}}_\ell$. In the inertial range of turbulent flow the dominant term is the vortex force

$$\mathbf{f}_\ell^v = \overline{(\mathbf{u} \times \boldsymbol{\omega})}_\ell - \bar{\mathbf{u}}_\ell \times \bar{\boldsymbol{\omega}}_\ell$$

or, equivalently, the subscale force \mathbf{f}_ℓ^s . It is interesting to decompose this into parts longitudinal and transverse to the vorticity $\bar{\boldsymbol{\omega}}_\ell$:

$$\mathbf{f}_\ell^v = \alpha_\ell \bar{\boldsymbol{\omega}}_\ell + \Delta \mathbf{u}_\ell \times \bar{\boldsymbol{\omega}}_\ell$$

with

$$\alpha_\ell = \bar{\boldsymbol{\omega}}_\ell \cdot \mathbf{f}_\ell^v / |\bar{\boldsymbol{\omega}}_\ell|^2, \quad \Delta \mathbf{u}_\ell = \bar{\boldsymbol{\omega}}_\ell \times \mathbf{f}_\ell^v / |\bar{\boldsymbol{\omega}}_\ell|^2.$$

Clearly, α_ℓ and $\Delta \mathbf{u}_\ell$ are quantities with dimension of velocity.

Let us now consider the physical significance of each of these separate terms. The quantities $\Delta \mathbf{u}_\ell$ may be interpreted as a drift velocity of vortex lines of $\bar{\boldsymbol{\omega}}_\ell$. This can be justified in exact terms if $\alpha_\ell \equiv 0$, in which case

$$\partial_t \bar{\boldsymbol{\omega}}_\ell = \nabla \times (\bar{\mathbf{u}}_\ell \times \bar{\boldsymbol{\omega}}_\ell + \mathbf{f}_\ell^v)$$

becomes

$$\partial_t \bar{\boldsymbol{\omega}}_\ell = \nabla \times (\mathbf{u}_\ell^* \times \bar{\boldsymbol{\omega}}_\ell)$$

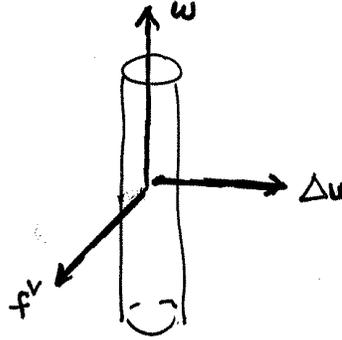
with

$$\mathbf{u}_\ell^* \equiv \bar{\mathbf{u}}_\ell + \Delta \mathbf{u}_\ell.$$

In that case, the condition (*) is satisfied for the net velocity \mathbf{u}_ℓ^* , so that vortex lines may be considered to be material lines for that velocity rather than $\bar{\mathbf{u}}_\ell$. It is interesting that the transverse force can be rewritten then as

$$\mathbf{f}_\ell^{v,\text{trans}} = (\mathbf{u}_\ell^* - \bar{\mathbf{u}}_\ell) \times \bar{\boldsymbol{\omega}}_\ell$$

and interpreted as a subscale Magnus force. This interpretation arises from the well-known Robins-Magnus effect according to which a body in a fluid with a net circulation around it experiences a transverse force when it moves relative to the fluid. First observed experimentally by B. Robins (1761) and G. Magnus (1853), the effect was explained by the Kutta-Joukowski theorem in aerodynamics, derived in the early 20th century to describe the lift force around an airfoil. There are similar forces on vortex lines in a variety of other physical systems, e.g. quantized vortex lines in superfluids. The essential ingredient is that the vortex lines must move not at the fluid velocity $\bar{\mathbf{u}}_\ell$ but instead with a different velocity \mathbf{u}_ℓ^* . The reciprocal effect is that a vortex line subjected to a transverse force $\mathbf{f}^{\text{trans}}$ moves with a velocity not parallel to the force but instead in a perpendicular direction at a velocity $\Delta\mathbf{u} \propto \bar{\boldsymbol{\omega}}_\ell \times \mathbf{f}^{\text{trans}}$. The general picture is:



Compare with T&L, Fig. 3.4.

The longitudinal force

$$\mathbf{f}_\ell^{v,\text{long}} \equiv \alpha_\ell \bar{\boldsymbol{\omega}}_\ell$$

is also physically important. It is a hydrodynamic analogue of the α -effect in MHD:

U. Frisch, Z. S. She & P. L. Sulem, “Large-scale flow driven by the anisotropic kinetic alpha effect,” *Physica D* **28** 382-392 (1987)

It plays an important role in theories of vortex reconnection:

G. Hornig, “Reconnection in magnetic and vorticity fields,” in Tubes, Sheets and Singularities in Fluid Dynamics, ed. K. Bajer (Kluwer, Amsterdam)

The longitudinal force is the only component that contributes to the helicity flux

$$\Lambda_\ell = -2\bar{\omega}_\ell \cdot \mathbf{f}_\ell^v = -2\alpha_\ell |\bar{\omega}_\ell|^2.$$

There is a similar expression for energy cascade when expressed in terms of the work W_ℓ against the vortex-force:

$$W_\ell = \bar{\mathbf{u}}_\ell \cdot \mathbf{f}_\ell^v = \alpha_\ell (\bar{\mathbf{u}}_\ell \cdot \bar{\omega}_\ell) - \Delta \mathbf{u}_\ell \cdot (\bar{\mathbf{u}}_\ell \times \bar{\omega}_\ell)$$

Likewise, the loop-torque can be written as

$$\Gamma_\ell \equiv - \oint_{\bar{C}_\ell(t)} \mathbf{f}_\ell^v \cdot d\mathbf{x} = - \oint_{\bar{C}_\ell(t)} \bar{\omega}_\ell \cdot [\alpha_\ell d\mathbf{x} + d\mathbf{x} \times \Delta \mathbf{u}_\ell] \quad (51)$$

We see that the α -effect creates/destroys circulation when the vorticity $\bar{\omega}_\ell$ has a nonzero component tangent to the loop $\bar{C}_\ell(t)$. The second term represents the rate at which vorticity is being carried across the curve $\bar{C}_\ell(t)$ by the vortex-line motion with relative velocity $\Delta \mathbf{u}_\ell$. With our sign convention, this term is positive when vorticity which is threading the loop in the positive sense (according to the right-hand rule) migrates out of the loop.

This gives us another way to interpret the persistent torques on loops in a turbulent flow. In the figure below we plot the conditional mean torque Γ_ℓ on loops with a given net circulation K , or $\langle \Gamma_\ell | K \rangle$, from the DNS of Chen et. al. (2006). We see that $\langle \Gamma_\ell | K \rangle$ has the same sign as K itself. This shows that the turbulent transport of the vortex-lines is, on average, diffusive. That is, if there is an excess of vorticity of one sign threading the loop, then the turbulence tends to transport this excess out of the loop!

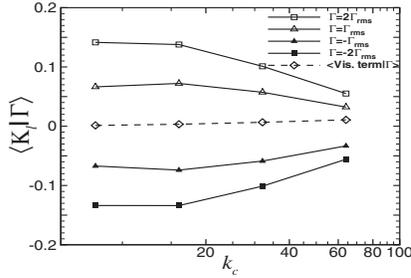


FIG. 3. The conditional average circulation flux from the 3d DNS as a function of k_c , for a square loop with $R = 64$ and for various values of the circulation level (solid lines). The dashed line shows the conditional expectation of the viscosity term, given $\Gamma = \Gamma_{rms}$, as a function of k_c .

Another feature noted in this plot is that, apparently,

$$\lim_{\ell \rightarrow 0} \langle \Gamma_\ell | K \rangle = 0, \quad (52)$$

This shows that the conservation of circulations is restored in a mean sense, as $\ell \rightarrow 0$. This had been conjectured by Eyink (2006) to occur, on various plausible grounds. For example, $\langle \Gamma_\ell \rangle = 0$ in homogeneous turbulence because $\mathbf{f}_\ell^s = -\nabla \cdot \boldsymbol{\tau}_\ell$ has zero average. If Γ_ℓ becomes statistically independent of the circulation on the loop as $\ell \rightarrow 0$, then $\langle \Gamma_\ell | K \rangle \rightarrow 0$. This issue will be discussed more in the next chapter on Lagrangian aspects of turbulence.

A related result is that vortex lines behave more like material lines as $\ell \rightarrow 0$. To see this we estimate

$$\mathbf{f}_\ell^v = O\left(\frac{\delta u^2(\ell)}{\ell}\right), \quad \bar{\boldsymbol{\omega}}_\ell = O\left(\frac{\delta u(\ell)}{\ell}\right)$$

so that

$$\alpha_\ell, \Delta \mathbf{u}_\ell = O^*\left(\frac{\delta u^2(\ell)/\ell}{\delta u(\ell)/\ell}\right) = O^*(\delta u(\ell))$$

This is not a rigorous big-O bound, because we have divided by $|\bar{\boldsymbol{\omega}}_\ell(\mathbf{x}, t)|$ which may become very large at null-points (zeros) of the vorticity vector. However, heuristically, one can expect that $\alpha_\ell, \Delta \mathbf{u}_\ell$ scale essentially as velocity increments $\delta u(\ell) \sim u_0(\ell/L)^h$, and thus vanish as $\ell \rightarrow 0$, as long as $h > 0$. For example, in K41 theory, $\delta u(\ell) \sim (\langle \varepsilon \rangle \ell)^{1/3}$. Thus, we may expect that vortex lines behave more as material lines for small $\ell \rightarrow 0$. The net effect on circulations does not vanish, however, except in some mean sense. We see that the PDF of Γ_ℓ hardly changes

as $\ell \rightarrow 0$. The reason is that $|\bar{\omega}_\ell| \sim \frac{\delta u(\ell)}{\ell}$ tends to diverge as $\ell \rightarrow 0$, as long as $h < 1$. Since $|\mathbf{f}_\ell^v| \propto |\alpha_\ell| \cdot |\bar{\omega}_\ell|$, $|\Delta \mathbf{u}_\ell| \cdot |\bar{\omega}_\ell|$ the combination need not be small, even if $|\alpha_\ell|, |\Delta \mathbf{u}_\ell| \rightarrow 0$ as $\ell \rightarrow 0$.

The effects that we have studied here for the coarse-grained or filtered vorticity could also be studied in the fine-grained vorticity, where the main contribution is from the viscous force

$$\mathbf{f}_\nu^{\text{vis}} = -\nu \nabla \times \boldsymbol{\omega}$$

Thus may likewise be decomposed as

$$\mathbf{f}_\nu^{\text{vis}} = \alpha_\nu \boldsymbol{\omega} + \Delta \mathbf{u}_\nu \times \boldsymbol{\omega}$$

A heuristic estimation gives

$$\alpha_\nu, |\Delta \mathbf{u}_\nu| \sim \delta u(\eta_h) \sim u_0 \left(\frac{\eta_h}{L}\right)^h \sim u_0 Re^{-\frac{h}{1+h}}$$

at a point with local Hölder exponent h . Thus, $\alpha_\nu, |\Delta \mathbf{u}_\nu|$ also tend to vanish as $\nu \rightarrow 0$ or $Re \rightarrow \infty$. Of course, all the effects of non-zero $\alpha_\nu, |\Delta \mathbf{u}_\nu|$ — such as non-conservation of circulations — are not removed in that limit. A number of studies have been made of differences between material lines and vortex-lines in turbulence, by experiment and by simulation. E.g.

M. Guala et al., “On the evolution of material lines and vorticity in homogeneous turbulence,” J. Fluid Mech. **533** 339-359(2005)

and references therein. Guala et al.(2005) find significant differences in the statistical properties of vortex lines and material lines. In particular, at the Reynolds numbers and observation times available to them, they observe a greater tendency of material lines to align with the most extensive strain direction, rather than with the intermediate strain direction as observed from vortex lines.

The work of Constantin & Iyer (2008) sheds fundamental light also on this question of vortex-line motion. Their results show, in a very precise sense, that *vortex-lines for incompressible Navier-Stokes fluids are material lines in a stochastic sense*. To state their result, we must return to the properties of vorticity for smooth solutions of the incompressible Euler equations.

The Helmholtz equation for the vorticity

$$D_t \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u}$$

can be explicitly integrated by the formula⁴

$$\boldsymbol{\omega}(\mathbf{x}, t) = \boldsymbol{\omega}(\mathbf{a}, t_0) \cdot \nabla_{\mathbf{a}} \mathbf{x}(t|\mathbf{a}, t_0) \Big|_{\mathbf{a}(t_0|\mathbf{x}, t)}$$

already derived by Augustin-Louis Cauchy in 1815, published in

A.-L. Cauchy, Théorie de la propagation des ondes à la surface d'une fluide pesant d'une profondeur indéfinie. Oeuvres Completes d'Augustin Cauchy, Série I, Tome I, 5-318, 1882.

Proof: Taking the gradient $\nabla_{\mathbf{a}}$ of the particle equation of motion gives by the chain rule that $\frac{d}{dt} \nabla_{\mathbf{a}} \mathbf{x}(t|\mathbf{a}, t_0) = \nabla_{\mathbf{a}} \mathbf{x}(t|\mathbf{a}, t_0) \cdot \nabla_{\mathbf{x}} \mathbf{u}(\nabla_{\mathbf{a}} \mathbf{x}(t|\mathbf{a}, t_0), t)$. Dotting this equation with $\boldsymbol{\omega}(\mathbf{a}, t_0)$, it follows that the Cauchy formula satisfies the Helmholtz equation. QED.

Constantin & Iyer (2008) showed that the Cauchy formula carries over to the incompressible Navier-Stokes equation in the stochastic form:

$$\boldsymbol{\omega}(\mathbf{x}, t) = \overline{\boldsymbol{\omega}(\mathbf{a}, t_0) \cdot \nabla_{\mathbf{a}} \tilde{\mathbf{x}}(t|\mathbf{a}, t_0)} \Big|_{\mathbf{a}=\tilde{\mathbf{a}}(\mathbf{x}, t|t_0)},$$

where $\tilde{\mathbf{x}}(t|\mathbf{a}, t_0)$ are the stochastic Lagrangian flow maps obtained by solving the SDE (45) forward in time, $\tilde{\mathbf{a}}(\mathbf{x}, t|t_0) := \tilde{\mathbf{x}}(t_0|\mathbf{x}, t)$ are the inverse or “back-to-labels” maps, and overline $\overline{(\dots)}$ is the average over the ensemble of Brownian motions. This formula gives the unique smooth solution (when it exists) to the viscous Helmholtz equation:

$$D_t \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \nu \Delta \boldsymbol{\omega}.$$

⁴From a differential-geometric point of view, this formula means that the vorticity $\boldsymbol{\omega}$ is *Lie-transported* by the fluid velocity \mathbf{u} as a differential 2-form. More concretely, the components $\Omega_{ij} = -\frac{1}{2} \epsilon_{ijk} \omega_k$ of the vorticity 2-form transform in time according to $\Omega_{ij}(\mathbf{x}, t) = \Omega_{nm}(\mathbf{a}, t_0) \frac{\partial a_n}{\partial x_i} \frac{\partial a_m}{\partial x_j}$, the *pullback* by the inverse Lagrangian map. For example, see Nicolas Besse and Uriel Frisch, “Geometric formulation of the Cauchy invariants for incompressible Euler flow in flat and curved spaces.” J. Fluid Mech. **825** 412-478 (2017).

An intuitive way to understand the stochastic Cauchy formula is by means of the “virtual vorticity vectors”

$$\tilde{\omega}(\mathbf{x}, t) = \omega(\mathbf{a}, t_0) \cdot \nabla_{\mathbf{a}} \tilde{\mathbf{x}}(t | \mathbf{a}, t_0) \Big|_{\mathbf{a}=\tilde{\mathbf{a}}(\mathbf{x}, t | t_0)}.$$

The virtual vorticities—by the same argument as above—can be obtained by solving the inviscid Helmholtz equation $d\tilde{\omega}/dt = \tilde{\omega} \cdot \nabla \mathbf{u}$ forward in time along the stochastic Lagrangian trajectories $\tilde{\mathbf{x}}(t)$ which arrive to point \mathbf{x} at time t . In a numerical implementation, this set of stochastic Lagrangian trajectories is instead most easily obtained by integrating the SDE (45) with the final value $\tilde{\mathbf{x}}(t) = \mathbf{x}$ backward to the initial time t_0 . The physical vorticity vector is finally obtained by averaging over the ensemble of virtual vorticities:

$$\omega(\mathbf{x}, t) = \overline{\tilde{\omega}(\mathbf{x}, t)}$$

The different steps in the procedure are illustrated graphically:

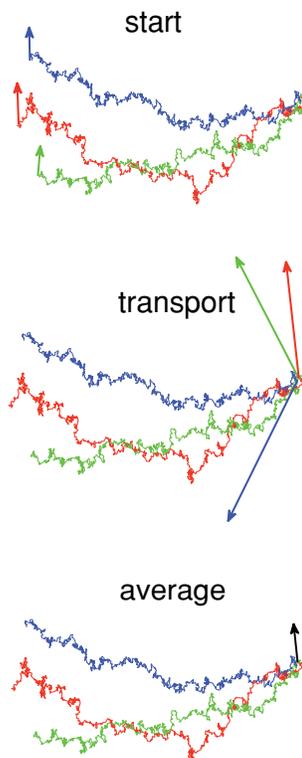


FIGURE. Illustration of the steps in computing vorticity by the stochastic Cauchy formula

What the Constantin-Iyer (2008) results show is that there is a well-defined sense in which vortex lines in a Navier-Stokes fluid do indeed move as material lines. Do these results help to justify Taylor's ideas on vortex-stretching as the origin of turbulent energy dissipation? Perhaps, but the detailed mechanisms are very subtle and many points are unclear, even at a heuristic level. Individual "virtual vortex lines" are stretched by the turbulent flow, which amplifies their strength, but the ensemble average leads to large cancellations (representing viscous dissipation of vorticity). Later we shall present some further ideas and speculations on this subject. However, the whole issue is presently wide open!