Homework #7

Problem 1, a) Starting from Fourier's equation in the form

$$\partial_t T + \nabla\cdot(\mathbf{v}T - \kappa\nabla T) = \varepsilon/c_p$$

we obtain by coarse-graining

$$\partial_t \overline{T}_\ell + \nabla\cdot\left[\overline{(\mathbf{v}T)}_\ell - \overline{(\kappa\nabla T)}_\ell\right] = \overline{\varepsilon}_\ell/c_p .$$

By the result of Homework #7, Problem 7(b)

$$\lim_{\nu,\kappa\to 0} \nabla\cdot\overline{(\kappa\nabla T)}_\ell = 0$$

so that we obtain, using also $\overline{(\mathbf{v}T)}_\ell = \overline{\mathbf{v}}_\ell \overline{T}_\ell + \tau_\ell(\mathbf{v},T)$,

$$\partial_t \overline{T}_\ell + \nabla\cdot\left[\overline{\mathbf{v}}_\ell \overline{T}_\ell + \tau_\ell(\mathbf{v},T)\right] = \overline{\varepsilon}_\ell/c_p$$

in the limit $\nu,\kappa\to 0$. This can be also written as

$$(\partial_t + \overline{\mathbf{v}}_\ell\cdot\nabla)\overline{T}_\ell + \nabla\cdot\tau_\ell(\mathbf{v},T) = \overline{\varepsilon}_\ell/c_p$$

using the incompressibility condition $\nabla\cdot\overline{\mathbf{v}}_\ell = 0$. Then, dividing this last equation by $\overline{T}_\ell$ and multiplying by ρc_p gives

$$(\partial_t + \overline{\mathbf{v}}_\ell\cdot\nabla)\underbrace{(\rho c_p \ln \overline{T}_\ell)}_{\underline{s}_\ell} + \underbrace{\frac{\rho c_p}{\overline{T}_\ell}\nabla\cdot\tau_\ell(T,\mathbf{v})}_{\frac{1}{\overline{T}_\ell}\nabla\cdot\tau_\ell(u,\mathbf{v})} = \rho\overline{\varepsilon}_\ell/\overline{T}_\ell$$

Then, by writing

$$\frac{1}{\overline{T}_\ell}\nabla\cdot\tau_\ell(u,\mathbf{v}) = \nabla\cdot\left[\frac{1}{\overline{T}_\ell}\tau_\ell(u,\mathbf{v})\right] - \nabla\left(\frac{1}{\overline{T}_\ell}\right)\cdot\tau_\ell(u,\mathbf{v})$$

and again using incompressibility, we get

$$\partial_t \underline{s}_\ell + \nabla\cdot\left[\underline{s}_\ell\overline{\mathbf{v}}_\ell + \underline{\beta}_\ell\tau_\ell(u,\mathbf{v})\right] = \nabla\underline{\beta}_\ell\cdot\tau_\ell(u,\mathbf{v}) + \underline{\beta}_\ell\rho\overline{\varepsilon}_\ell$$

for $\underline{\beta}_\ell := 1/\overline{T}_\ell$, as stated.

(b) Taking $\overline{D}_t = \partial_t + \overline{\mathbf{v}}_\ell\cdot\nabla$, the product rule gives

$$\overline{D}_t(\underline{\beta}_\ell\rho k_\ell) = (\overline{D}_t\underline{\beta}_\ell)\rho k_\ell + \underline{\beta}_\ell(\overline{D}_t\rho k_\ell).$$

The balance equation for ρk_ℓ can be rewritten as

$$\overline{D}_t(\rho k_\ell) + \nabla\cdot\left[\tau_\ell(P,\mathbf{v}) + \tfrac{1}{2}\rho\tau_\ell(v_i,v_i,\mathbf{v})\right] = \rho\Pi_\ell - \rho\overline{\varepsilon}_\ell$$

so that

$$\bar{D}_t(\underline{\beta}_\ell \rho k_\ell) = (\bar{D}_t \underline{\beta}_\ell)\rho k_\ell + \rho\underline{\beta}_\ell \Pi_\ell - \rho\underline{\beta}_\ell \bar{\varepsilon}_\ell$$

$$- \underline{\beta}_\ell \nabla\cdot\left[T_\ell(P,\mathbf{v}) + \tfrac{1}{2}\rho T_\ell(v_i,v_i,\mathbf{v}) \right].$$

Rewriting the last term as

$$\underline{\beta}_\ell \nabla\cdot[\cdots] = \nabla\cdot(\underline{\beta}_\ell[\cdots]) - \nabla\underline{\beta}_\ell\cdot[\cdots],$$

we obtain finally

$$\bar{D}_t(\underline{\beta}_\ell \rho k_\ell) + \nabla\cdot\left[\underline{\beta}_\ell\left(T_\ell(P,\mathbf{v}) + \tfrac{1}{2}\rho T_\ell(v_i,v_i,\mathbf{v})\right)\right]$$

$$= \nabla\underline{\beta}_\ell\cdot\left(T_\ell(P,\mathbf{v}) + \tfrac{1}{2}\rho T_\ell(v_i,v_i,\mathbf{v})\right) - \rho\underline{\beta}_\ell\bar{\varepsilon}_\ell$$

$$+ (\bar{D}_t\underline{\beta}_\ell)\rho k_\ell + \rho\underline{\beta}_\ell\Pi_\ell.$$

Adding this to the balance equation for $\underline{s}_\ell$ in part (a), the term $\rho\underline{\beta}_\ell\bar{\varepsilon}_\ell$ cancels and we obtain

$$\bar{D}_t(\underbrace{\underline{s}_\ell + \underline{\beta}_\ell \rho k_\ell}_{\underline{s}_\ell^*}) + \nabla\cdot(\underline{\beta}_\ell \mathbf{q}_\ell) = \nabla\underline{\beta}_\ell\cdot\mathbf{q}_\ell + (\bar{D}_t\underline{\beta}_\ell)\rho k_\ell + \rho\underline{\beta}_\ell\Pi_\ell$$

as required.

Problem 2. The key observations here are that

$$\nabla\times\overline{\mathbf{u}}_\ell(\mathbf{x}) = \frac{1}{\ell}\int d^d r\,(\nabla G)_\ell(\mathbf{r})\times\delta\mathbf{u}(\mathbf{r};\mathbf{x})$$

and

$$f^s_{\ell i}(\mathbf{x}) = \frac{1}{\ell}\Big\{\int d^d r\,(\partial_j G)_\ell(\mathbf{r})\,\delta u_i(\mathbf{r};\mathbf{x})\,\delta u_j(\mathbf{r};\mathbf{x})$$

$$-\int d^d r\,(\partial_j G)_\ell(\mathbf{r})\,\delta u_i(\mathbf{r};\mathbf{x})\int d^d r'\,G_\ell(\mathbf{r}')\,\delta u_j(\mathbf{r}';\mathbf{x})\Big\}$$

which can be expressed entirely in terms of velocity-increments. Since the latter are scale-local (both UV and IR) so are $\nabla\times\overline{\mathbf{u}}_\ell$ and $\mathbf{f}^s_\ell$ and, thus, Λ_ℓ, when $\mathbf{u}$ is Hölder continuous with exponent $0<h<1$. Therefore,

$$\delta\overline{\mathbf{u}}_\Delta(\mathbf{r}) = O(r\Delta^{h-1})$$

and

$$\delta\mathbf{u}'_\delta(\mathbf{r}) = O(\delta^h)$$

can be used to derive bounds such as

$$\Lambda_\ell(\mathbf{u},\mathbf{u},\overline{\mathbf{u}}_\Delta) = O(\ell^{2h-1}\Delta^{h-1})$$

which vanishes for $\Delta/\ell\to\infty$ and

$$\Lambda_\ell(\mathbf{u},\mathbf{u},\mathbf{u}'_\delta) = O(\ell^{2h-2}\delta^h)$$

which vanishes for $\delta/\ell\to 0$. The same bounds hold no matter which of the three $\mathbf{u}$'s is replaced by $\overline{\mathbf{u}}_\Delta$ or $\mathbf{u}'_\delta$. Thus, helicity cascade is scale-local when $\mathbf{u}$ is Hölder continuous with exponent $0<h<1$.

Problem 3. a) It is enough to observe that

$$\lim_{\nu\to 0} \tanh\left(\frac{Lx}{2\nu t}\right) = \theta(x) - \theta(-x) = \begin{cases} -1 & x<0 \\ 0 & x=0 \\ +1 & x>0 \end{cases}$$

for fixed Lx/t. Thus,

$$\lim_{\nu\to 0} u^{\nu}(x,t) = \frac{x}{t} - \frac{L}{t}\left[-\theta(-x) + \theta(x)\right]$$

$$= \left(\frac{x+L}{t}\right)\theta(-x) + \left(\frac{x-L}{t}\right)\theta(x).$$

b) We note first that

$$\partial_t u(x,t) = \partial_t\left[\left(\frac{x+L}{t}\right)\theta(-x) + \left(\frac{x-L}{t}\right)\theta(x)\right]$$

$$= -\frac{1}{t}u(x,t)$$

since $d/dt\,(1/t) = -1/t^2$. Next,

$$\frac{1}{2}u^2(x,t) = \frac{1}{2}\left(\frac{x+L}{t}\right)^2\theta(-x) + \frac{1}{2}\left(\frac{x-L}{t}\right)^2\theta(x)$$

in the sense of distributions (which allows us to ignore the inequality at $x=0$). Thus, using $\theta'(x) = \delta(x)$,

$$\partial_x\left(\frac{1}{2}u^2(x,t)\right) = \frac{x+L}{t^2}\theta(-x) + \frac{x-L}{t^2}\theta(x)$$
$$+ \frac{1}{2}\left[-\left(\frac{x+L}{t}\right)^2 + \left(\frac{x-L}{t}\right)^2\right]\delta(x)$$

$$= \frac{1}{t}u(x,t) + \frac{1}{2}\underbrace{\cancel{\left[-\left(\frac{L}{t}\right)^2 + \left(\frac{L}{t}\right)^2\right]}}_{\to 0}\delta(x)$$

Thus, finally,

$$\partial_t u(x,t) + \partial_x\left(\tfrac{1}{2}u^2(x,t)\right) = -\frac{1}{t}u(x,t) + \frac{1}{t}u(x,t)$$

$$= 0$$

in the sense of distributions.

c) A more formal calculation which gives the same result is as follows:

$$\int_0^\infty dt \int_{-L}^{L} dx \left[u(x,t)\,\partial_t\varphi(x,t) + \tfrac{1}{2}u^2(x,t)\,\partial_x\varphi(x,t) \right] + \int_{-L}^{L} dx\, u_0(x)\,\varphi(x,0)$$

$$= \int_0^\infty dt \int_{-L}^{L} dx \left[-\partial_t u(x,t)\cdot\varphi(x,t) + \tfrac{1}{2}u^2(x,t)\,\partial_x\varphi(x,t) \right]$$

by integration by parts in t

$$= \int_0^\infty dt \left\{ \int_{-L}^{0} dx \left[\left(\frac{x+L}{t^2}\right)\varphi(x,t) + \frac{1}{2}\left(\frac{x+L}{t}\right)^2 \partial_x\varphi(x,t) \right] + \int_0^{L} dx \left[\left(\frac{x-L}{t^2}\right)\varphi(x,t) + \frac{1}{2}\left(\frac{x-L}{t}\right)^2 \partial_x\varphi(x,t) \right] \right.$$

by substituting $u(x,t)$, $\partial_t u(x,t)$

$$= \int_0^\infty \frac{dt}{t^2} \left\{ \int_{-L}^{0} dx\, \partial_x\left[\tfrac{1}{2}(x+L)^2\varphi(x,t)\right] + \int_0^{L} dx\, \partial_x\left[\tfrac{1}{2}(x-L)^2\varphi(x,t)\right] \right\}$$

$$= \int_0^\infty \frac{dt}{t^2} \left\{ \frac{L^2}{2}\varphi(0^-,t) - \frac{L^2}{2}\varphi(0^+,t) \right\} = 0 \; !$$

Problem 4. a) Using

$$\frac{1}{2}u^2 = \frac{1}{2}\left(\frac{x+L}{t}\right)^2 \theta(-x) + \frac{1}{2}\left(\frac{x-L}{t}\right)^2 \theta(x)$$

$$\frac{1}{3}u^3 = \frac{1}{3}\left(\frac{x+L}{t}\right)^3 \theta(-x) + \frac{1}{3}\left(\frac{x-L}{t}\right)^3 \theta(x)$$

one gets

$$\partial_t\left(\frac{1}{2}u^2\right) = -\left[\frac{(x+L)^2}{t^3}\theta(-x) + \frac{(x-L)^2}{t^3}\theta(x)\right]$$

and

$$\partial_x\left(\frac{1}{3}u^3\right) = \left[\frac{(x+L)^2}{t^3}\theta(-x) + \frac{(x-L)^2}{t^3}\theta(x)\right] + \underbrace{\left[-\frac{1}{3}\left(\frac{x+L}{t}\right)^3 + \frac{1}{3}\left(\frac{x-L}{t}\right)^3\right]\delta(x)}_{= -\frac{2}{3}\left(\frac{L}{t}\right)^3 \delta(x) = -\frac{1}{12}(\Delta u)^3 \delta(x).}$$

Unlike the case for the momentum balance, we see that the delta-functions do not cancel in the energy balance! We finally obtain that

$$\partial_t\left(\frac{1}{2}u^2\right) + \partial_x\left(\frac{1}{3}u^3\right) = -\frac{1}{12}(\Delta u)^3 \delta(x)$$

in the sense of distributions.

(b) Using $\varepsilon^{\nu}(x,t) \cong \frac{L^4}{4\nu t^4} \operatorname{sech}^4\left(\frac{Lx}{2\nu t}\right)$, one gets that

$$\int_{-L}^{L} \varphi(x)\,\varepsilon^{\nu}(x,t)\,dx = \frac{1}{2}\left(\frac{L}{t}\right)^3 \int_{-L^2/2\nu t}^{L^2/2\nu t} \varphi\left(\frac{2\nu t}{L}u\right) \operatorname{sech}^4 u\,du$$

with the change of variables $u = Lx/2\nu t$. Taking the limit $\nu \to 0$

$$\lim_{\nu\to 0} \int_{-L}^{L} \varphi(x)\,\varepsilon^{\nu}(x,t)\,dx = \frac{1}{2}\left(\frac{L}{t}\right)^3 \int_{-\infty}^{+\infty} \varphi(0)\cdot \operatorname{sech}^4 u\,du$$

$$= \frac{2}{3}\left(\frac{L}{t}\right)^3 \varphi(0) \qquad \text{using } \int_{-\infty}^{+\infty} \operatorname{sech}^4 u\,du = \frac{4}{3}$$

$$= \frac{1}{12}(\Delta u)^3 \varphi(0).$$

Problem 5. Using the analytic expression for the sawtooth solution, one easily calculates that

$$\delta u_s^{\nu}(r;t) = u^{\nu}\left(\frac{r}{2},t\right) - u^{\nu}\left(-\frac{r}{2},t\right)$$

$$= \frac{r}{t} - \frac{2L}{t}\tanh\left(\frac{Lr}{4\nu t}\right)$$

$$= \frac{r}{t} - \frac{2L}{t}\tanh\left(\frac{\Delta u\cdot r}{8\nu}\right)$$

with $\Delta u = 2L/t$.

Since

$$\tanh(x) = \mathrm{sign}(x) + O\left(e^{-2|x|}\right), \quad |x| \gg 1,$$

$$\delta u_S^{\nu}(r;t) \cong \frac{r}{t} - \frac{2L}{t}\mathrm{sign}(r) \quad \text{for } |r| \gg \frac{\nu}{\Delta u}$$

If, furthermore, $|r| \ll L$, then

$$\delta u_S^{\nu}(r,t) \cong -\frac{2L}{t}\mathrm{sign}(r) = -(\Delta u)\cdot \mathrm{sign}(r)$$

for $\nu/\Delta u \ll |r| \ll L$.

On the other hand,

$$\tanh(x) = x + O(x^3), \quad |x| \ll 1.$$

Therefore,

$$\delta u_S^{\nu}(r;t) \cong \frac{r}{t} - \frac{2L}{t}\cdot\frac{Lr}{4\nu t}$$

$$= \frac{r}{t}\left(1 - \frac{L^2}{2\nu t}\right)$$

for $|r| \ll \nu/\Delta u$.