Homework #5

Problem 1. We let L be the integral length and u_{rms} the rms velocity (which are ℓ and u, respectively, in Tennekes & Lumley). We assume that the radius ℓ of the hot patch satisfies initially

$$\eta \ll \ell \ll L.$$

In that case, we expect that

$$\frac{d\ell}{dt} = \frac{\ell}{\tau_\ell}$$

with

$$\tau_\ell \sim \varepsilon^{-1/3}\ell^{2/3}, \quad \varepsilon \sim u_{rms}^3/L$$

or

$$\frac{d\ell}{dt} = C(\varepsilon\ell)^{1/3}.$$

This equation is easily solved as

$$\ell(t) = \left[\ell_0^{2/3} + \frac{2}{3}C\varepsilon^{1/3}(t-t_0)\right]^{3/2}$$

$$\sim \varepsilon^{1/2}t^{3/2} \quad \text{for } t \gg t_0$$

Assuming that $K = \int \theta(\mathbf{x},t)\,d^dx$ is finite and conserved,

$$K = \int \theta(\mathbf{x},t)\,d^dx \sim \theta_{max}(t)\cdot \ell^d(t)$$

$$\Longrightarrow \quad \theta_{max}(t) \sim K\ell^{-d}(t) \sim K\varepsilon^{-d/2}t^{-3d/2} \quad t \gg t_0$$

An alternative approach is to introduce an effective diffusivity or eddy-diffusivity $D(\ell)$ at length-scale ℓ, so that

$$\frac{1}{2}\frac{d}{dt}\ell^2 \sim D(\ell) \sim C\varepsilon^{1/3}\ell^{4/3},$$

where in the last line we used a K41 dimensional estimate with

$$[\varepsilon^{1/3}\ell^{4/3}] = \left(\frac{(\text{length})^2}{(\text{time})^3}\right)^{1/3}(\text{length})^{4/3} = \frac{(\text{length})^2}{\text{time}}.$$

Since $\frac{1}{2}\frac{d}{dt}\ell^2 = \ell\frac{d\ell}{dt}$, this approach is strictly equivalent to the previous one and gives the same equation

$$\frac{d\ell}{dt} = C(\varepsilon\ell)^{1/3}$$

as before. Note that when $\ell > L$, the integral length scale of the turbulence, then one instead expects that

$$D(\ell) = D_L \quad (\text{constant}) \quad \text{for } \ell > L.$$

Crossover occurs when $\ell(t) = L$ which happens when $t = T_L := \varepsilon^{-1/3}L^{2/3} = L/u_{rms}$, the large-eddy turnover time. Thereafter,

$$\ell^2(t) = L^2 + D_L(t - T_L) \sim D_L t$$

for times $t \gg T_L$.

A more precise approach is to solve a diffusion equation

$$\partial_t \theta(\mathbf{r},t) = \nabla_r \cdot \left[D(r) \nabla_r \theta(\mathbf{r},t) \right]$$

with a scale-dependent eddy-diffusivity

$$D(r) = D_0 \varepsilon^{1/3} r^{4/3}.$$

Assuming, for simplicity, radial symmetry of the temperature distribution of the patch, the equation becomes

$$\partial_t \theta(r,t) = \frac{D_0 \varepsilon^{1/3}}{r^{d-1}} \frac{\partial}{\partial r}\left[r^{d+1/3} \frac{\partial \theta}{\partial r} \right].$$

Now look for a self-similar decay solution

$$\theta(r,t) = \vartheta(t)\, \phi\left(\frac{r}{\ell(t)}\right)$$

where
$$\vartheta(t) = \theta(0,t) = \theta_{max}(t).$$

Substituting into the evolution equation gives (with $\rho = r/\ell(t)$)

$$\left(\frac{\dot{\vartheta}(t)\, \ell^{2/3}(t)}{D_0 \varepsilon^{1/3}\, \theta(t)} \right) \cdot \phi(\rho) + \left(\frac{-\dot{\ell}(t)}{D_0 \varepsilon^{1/3} \ell^{1/3}(t)} \right) \rho \frac{\partial \phi}{\partial \rho}(\rho)$$

$$= \frac{1}{\rho^{d-1}} \frac{\partial}{\partial \rho}\left(\rho^{d+1/3} \frac{\partial \phi}{\partial \rho} \right)$$

Each of the coefficients must be independent of time, giving

$$\frac{\dot{\vartheta}(t)\, l^{2/3}(t)}{D_0 \varepsilon^{1/3} \vartheta(t)} = -\alpha$$

$$\frac{\dot{l}(t)}{D_0 \varepsilon^{1/3} l^{1/3}(t)} = \beta .$$

The second equation is the same as the one derived earlier for $l(t)$ more heuristically. By changing the definition of time from $t \to t/\beta$, the coefficient $\beta \to 1$. Now combining the equations gives

$$\frac{\dot{\vartheta}(t)}{\vartheta(t)} = -\alpha \frac{\dot{l}(t)}{l(t)}$$

$$\Longrightarrow \vartheta(t) = C l^{-\alpha}(t).$$

We now see that $\alpha = d$ in the previous analysis. Thus, we obtain, as before

$$l(t) = \left[l_0^{2/3} + \frac{2}{3} D_0 \varepsilon^{1/3} (t - t_0) \right]^{3/2} \sim \varepsilon^{1/2} t^{3/2} \quad t \gg t_0$$

$$\Theta_{max}(t) \sim C l^{-d}(t) \sim C \varepsilon^{-d/2} t^{-3d/2} \quad t \gg t_0$$

Note that the equation for the form function ϕ becomes

$$\rho^{4/3}\phi''(\rho) + \left[\left(d+\tfrac{1}{3}\right)\rho^{1/3} + \rho\right]\phi'(\rho) + \alpha\phi(\rho) = 0 .$$

For $\alpha = d$, this was solved by G.K. Batchelor, Proc. Camb. Philos. Soc. 48 345-362 (1952), who obtained

$$\phi(\rho) = \exp\left[-\frac{3}{4}\rho^{4/3}\right]$$

in every space dimension d, or

$$\theta(r,t) = C\,\ell^{-d}(t)\exp\left[-\frac{3}{4}\left(\frac{r}{\ell(t)}\right)^{4/3}\right]$$

with $\ell(t)$ given as before. Note that Batchelor's solution is relevant if

$$0 < \left|\int d^d r\,\theta(r,t)\right| < +\infty .$$

The self-similar solutions with ϕ given by the above equation for $\alpha \neq d$ are relevant if

$$\int d^d r\,\theta(r,t) = 0 \quad \text{or} \quad \int d^d r\,\theta(r,t) = \pm\infty .$$

See G.L. Eyink & J. Xin, J. Stat. Phys. 100 679-741 (2000) for an exhaustive analysis of all self-similar decay solutions.

Problem 2 (a) $$\frac{u^3}{L} \sim \varepsilon = \frac{15\nu u^2}{\lambda^2}$$

$$\Longrightarrow \quad \frac{u}{L} \sim \frac{\nu}{\lambda^2}$$

$$\Longrightarrow \quad \frac{uL}{\nu} \sim \left(\frac{L}{\lambda}\right)^2$$

$$\Longrightarrow \quad \underline{\frac{\lambda}{L} \sim (Re)^{-1/2}}$$

$$\therefore \quad Re_\lambda = \frac{u\lambda}{\nu} = \frac{uL}{\nu} \cdot \frac{\lambda}{L} = (Re)\cdot(Re)^{-1/2} = \underline{(Re)^{1/2}}$$

(b) Recall that

$$\eta = \nu^{3/4} \varepsilon^{-1/4}$$

so that, using $\varepsilon \sim u^3/L$,

$$\frac{\eta}{L} = \left(\frac{\nu}{uL}\right)^{-3/4} = (Re)^{-3/4}.$$

Thus,

$$\frac{\eta}{\lambda} = \frac{\eta/L}{\lambda/L} = \frac{(Re)^{-3/4}}{(Re)^{-1/2}} = \underline{(Re)^{-1/4}}.$$

(c) $$u_\lambda = (\varepsilon\lambda)^{1/3} = u\,(\lambda/L)^{1/3} = \underline{u(Re)^{-1/6}}$$

$$\varepsilon_\lambda = \frac{\nu u_\lambda^2}{\lambda^2} = \frac{\nu u^2}{\lambda^2}\cdot(Re)^{-1/3} = \underline{\varepsilon(Re)^{-1/3}}$$

Problem 3.

$$f^{s}_{\ell i} = -\partial_j (T_\ell)_{ij} = -\frac{\partial}{\partial x_j}\left\{ \int d^d r\, G_\ell(\mathbf{r})\, u_i(\mathbf{x}+\mathbf{r})\, u_j(\mathbf{x}+\mathbf{r}) - \int d^d r\, G_\ell(\mathbf{r})\, u_i(\mathbf{x}+\mathbf{r}) \int d^d r'\, G_\ell(\mathbf{r}')\, u_j(\mathbf{x}+\mathbf{r}') \right\}$$

$$= -\left\{ \int d^d r\, G_\ell(\mathbf{r}) \left(\frac{\partial}{\partial x_j} u_i(\mathbf{x}+\mathbf{r})\right) u_j(\mathbf{x}+\mathbf{r}) - \int d^d r\, G_\ell(\mathbf{r}) \left(\frac{\partial}{\partial x_j} u_i(\mathbf{x}+\mathbf{r})\right) \int d^d r'\, G_\ell(\mathbf{r}')\, u_j(\mathbf{x}+\mathbf{r}') \right\}$$ by incompressibility

$$= -\left\{ \int d^d r\, G_\ell(\mathbf{r}) \left(\frac{\partial}{\partial r_j} u_i(\mathbf{x}+\mathbf{r})\right) [\underbrace{u_j(\mathbf{x}+\mathbf{r}) - \underline{u_j(\mathbf{x})}}_{=\,\delta u_j(\mathbf{r};\mathbf{x})}] - \int d^d r\, G_\ell(\mathbf{r}) \left(\frac{\partial}{\partial r_j} u_i(\mathbf{x}+\mathbf{r})\right) \int d^d r'\, G_\ell(\mathbf{r}')\, [\underbrace{u_j(\mathbf{x}+\mathbf{r}') - \underline{u_j(\mathbf{x})}}_{=\,\delta u_j(\mathbf{r}';\mathbf{x})}] \right\}$$ adding & subtracting same term

$$= -\left\{ \int d^d r\, G_\ell(\mathbf{r}) \left(\frac{\partial}{\partial r_j} \delta u_i(\mathbf{r};\mathbf{x})\right) \delta u_j(\mathbf{r};\mathbf{x}) - \int d^d r\, G_\ell(\mathbf{r}) \left(\frac{\partial}{\partial r_j} \delta u_i(\mathbf{r};\mathbf{x})\right) \int d^d r'\, G_\ell(\mathbf{r}')\, \delta u_j(\mathbf{r}';\mathbf{x}) \right\}$$ since $\frac{\partial}{\partial r_j} u_i(\mathbf{x}) = 0$

by incompressibility

$$= -\left\{ \int d^d r\, G_\ell(\mathbf{r}) \frac{\partial}{\partial r_j}\left(\delta u_i(\mathbf{r};\mathbf{x})\, \delta u_j(\mathbf{r};\mathbf{x}) \right) - \int d^d r\, G_\ell(\mathbf{r}) \frac{\partial}{\partial r_j}\left(\delta u_i(\mathbf{r};\mathbf{x})\right) \int d^d r'\, G_\ell(\mathbf{r}')\, \delta u_j(\mathbf{r}';\mathbf{x}) \right\}$$

integrating by parts

$$= \frac{1}{\ell}\left\{ \int d^d r\, (\partial_j G)_\ell(\mathbf{r})\, \delta u_i(\mathbf{r};\mathbf{x})\, \delta u_j(\mathbf{r};\mathbf{x}) - \int d^d r\, (\partial_j G)_\ell(\mathbf{r})\, \delta u_i(\mathbf{r};\mathbf{x}) \int d^d r'\, G_\ell(\mathbf{r}')\, \delta u_j(\mathbf{r}';\mathbf{x}) \right\}$$ QED

Problem 4. Since $\quad -\Delta p = \partial_i \partial_j (u_i u_j)$,

$$-\Delta \bar{p}_\ell(\mathbf{x}) = \partial_i \partial_j \overline{(u_i u_j)}_\ell$$

$$= \frac{\partial^2}{\partial x_i \partial x_j} \int d^d r \, G_\ell(\mathbf{r}) \, u_i(\mathbf{x}+\mathbf{r}) \, u_j(\mathbf{x}+\mathbf{r})$$

$$= \int d^d r \, G_\ell(\mathbf{r}) \Big\{ \frac{\partial^2}{\partial r_i \partial r_j} \left[u_i(\mathbf{x}+\mathbf{r}) \, u_j(\mathbf{x}+\mathbf{r}) \right] \underbrace{- u_i(\mathbf{x}) \cdot \frac{\partial^2}{\partial r_i \partial r_j} u_j(\mathbf{x}+\mathbf{r}) - \frac{\partial^2}{\partial r_i \partial r_j} u_i(\mathbf{x}+\mathbf{r}) \cdot u_j(\mathbf{x})}_{=0 \text{ by incompressibility}} \Big\}$$

$$= \int d^d r \, G_\ell(\mathbf{r}) \frac{\partial^2}{\partial r_i \partial r_j} \Big[u_i(\mathbf{x}+\mathbf{r}) u_j(\mathbf{x}+\mathbf{r}) - u_i(\mathbf{x}) u_j(\mathbf{x}+\mathbf{r}) - u_i(\mathbf{x}+\mathbf{r}) u_j(\mathbf{x}) + u_i(\mathbf{x}) u_j(\mathbf{x}) \Big]$$

$$= \int d^d r \, G_\ell(\mathbf{r}) \frac{\partial^2}{\partial r_i \partial r_j} \left[\delta u_i(\mathbf{r};\mathbf{x}) \, \delta u_j(\mathbf{r};\mathbf{x}) \right]$$

$$= \frac{1}{\ell^2} \int d^d r \, (\partial_i \partial_j G)_\ell(\mathbf{r}) \, \delta u_i(\mathbf{r};\mathbf{x}) \, \delta u_j(\mathbf{r};\mathbf{x})$$

QED

Problem 5. (a) With $e(\mathbf{x}) = \frac{1}{2}|\mathbf{u}(\mathbf{x})|^2$,

$$\begin{aligned}
\delta e(\mathbf{r};\mathbf{x}) &= e(\mathbf{x}+\mathbf{r}) - e(\mathbf{x}) \\
&= \frac{1}{2}|\mathbf{u}(\mathbf{x}+\mathbf{r})|^2 - \frac{1}{2}|\mathbf{u}(\mathbf{x})|^2 \\
&= \frac{1}{2}\left[\mathbf{u}(\mathbf{x}+\mathbf{r}) + \mathbf{u}(\mathbf{x})\right]\cdot\left[\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})\right] \\
&= \mathbf{u}_{av}(\mathbf{x}+\mathbf{r},\mathbf{x})\cdot\delta\mathbf{u}(\mathbf{r};\mathbf{x})
\end{aligned}$$

(b) It follows directly from the identity above that

$$\delta e(\mathbf{r}) = O\left(u_{max}\,\delta u(r)\right).$$

Thus, the experimental observation corresponds to the above upper bound. If one furthermore assumes that

$$\mathbf{u}_{av}(\mathbf{x}+\mathbf{r},\mathbf{x}) \sim \text{dominated by scales} \approx L$$

$$\delta\mathbf{u}(\mathbf{r};\mathbf{x}) \sim \text{dominated by scales} \approx r$$

are statistically independent, then

$$\begin{aligned}
\langle|\delta e(\mathbf{r})|^2\rangle &\sim \langle|\mathbf{u}_{av}|^2\rangle\langle|\delta\mathbf{u}(\mathbf{r})|^2\rangle \\
&\sim u_{rms}^2\cdot C(\varepsilon r)^{2/3} \quad \text{in K41-theory}
\end{aligned}$$

For more discussion of these issues, see R. J. Hill & J. M. Wilczak, "Fourth-order velocity statistics," Fluid. Dyn. Res. 28 1-22 (2001)