Homework No.2, 553.793, Due February 11, 2022.

1. This problem studies the local conservation laws of classical molecular fluids.

(a) Derive the local conservation of momentum

$$\partial_t \hat{\mathbf{g}}(\mathbf{x},t) + \boldsymbol{\nabla}\boldsymbol{\cdot}\hat{\mathbf{T}}(\mathbf{x},t) = \mathbf{0},$$

where $\mathbf{T}$ is the stress-tensor given in class and $(\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{T})_i = \partial_j T_{ij}$. *Hint:* Use Newton's third law, and also use the fundamental theorem of calculus to write

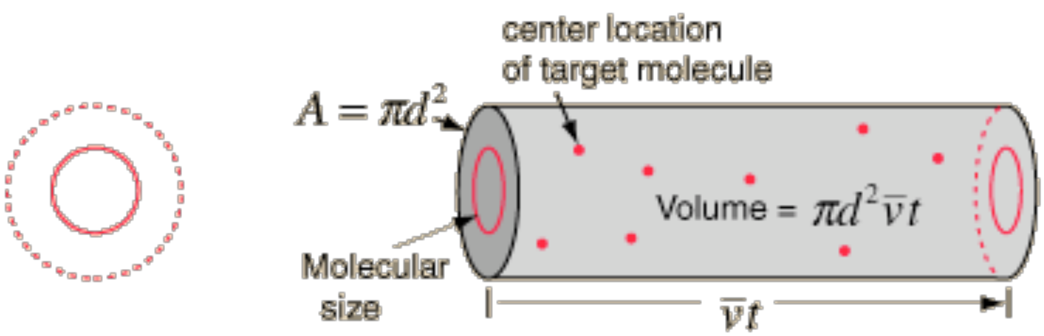
$$(\mathbf{r}_{nm}\boldsymbol{\cdot}\boldsymbol{\nabla})\int_0^1 ds\ \delta^3(\mathbf{x}-\mathbf{r}_n+s\mathbf{r}_{nm}) = \delta^3(\mathbf{x}-\mathbf{r}_m) - \delta^3(\mathbf{x}-\mathbf{r}_n).$$

(b) Use a similar argument to derive the local conservation of energy

$$\partial_t \hat{e}(\mathbf{x},t) + \boldsymbol{\nabla}\boldsymbol{\cdot}\hat{\mathbf{s}}(\mathbf{x},t) = 0$$

where $\hat{\mathbf{s}}$ is the energy current given in class.

2. The concept of the *mean free path length* is a very important one in the molecular theory of fluids, which we explore in this problem.



(a) The above figure illustrates the notion of the "collision cylinder" swept out in time t by a spherical particle of diameter d moving at speed $\bar{v}$ in a gas of similar particles at rest. The points in the figure represent the centers of the other particles in the gas. The mean-free-time τ_{mfp} is defined to be the time for one collision to occur, i.e. for one particle to reside in the collision cylinder. Show that

$$\tau_{mfp} = \frac{1}{n\pi d^2 \bar{v}}$$

where n is the density of particles per unit volume. The corresponding distance traveled is $\lambda_{mfp} = \bar{v}\tau_{mfp} = 1/n\pi d^2$.

(b) In reality, all of the particles in the fluid are moving with a random velocity chosen from an ensemble with a probability density $p(\mathbf{v})$. This will ordinarily be Maxwellian, but we assume here only that the mean velocity is zero and that distinct particles have statistically independent and identically distributed velocities. Show that the *relative* velocity $\mathbf{v}-\mathbf{v}'$ of any pair of particles has the root-mean-square value $v_{rel}=\sqrt{\langle|\mathbf{v}-\mathbf{v}'|^2\rangle}=\sqrt{2}v_{rms}$ where $v_{rms}^2=\int d^3v\ |\mathbf{v}|^2p(\mathbf{v})$. The velocity $\bar{v}$ in the previous argument should be replaced by v_{rel}, whereas the distance traveled by the particle in a mean-free-time is $\lambda_{mfp}=v_{rms}\tau_{rms}$. Use these ideas to derive

$$\lambda_{mfp}=\frac{1}{n\sqrt{2}\pi d^2}.$$

(d) Estimate the mean-free-path length of air at atmospheric pressure $P=1.1013\times 10^5$ Pa and $T=300°$K. Use the ideal gas law $P=nk_BT$ to obtain the density n and estimate $d=2R$ where $R=1.55\times 10^{-10}$ m is the van der Waals radius of molecular nitrogen. Calculate the ratio $\epsilon=R/\lambda_{mfp}$, which is the relevant small parameter for the validity of the Boltzmann kinetic equation.

3. (a) Using the mass and momentum equations of the compressible Navier-Stokes system, derive the balance equation for kinetic energy:

$$\partial_t\left(\frac{1}{2}\rho v^2\right)+\boldsymbol{\nabla}\boldsymbol{\cdot}\left[\left(\frac{1}{2}\rho v^2+P\right)\mathbf{v}+\mathbf{T}^{(1)}\boldsymbol{\cdot}\mathbf{v}\right]=-Q+P(\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{v})$$

where $Q=-\mathbf{T}^{(1)}\boldsymbol{:}\boldsymbol{\nabla}\mathbf{v}=2\eta S^2+\zeta D^2\geq 0$ is the rate of kinetic energy dissipation per unit volume, with $D=\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{v}$ the velocity-divergence/dilatation and $\mathbf{S}=\frac{1}{2}\left[\boldsymbol{\nabla}\mathbf{v}+(\boldsymbol{\nabla}\mathbf{v})^\top-(2/3)D\mathbf{I}\right]$ the velocity strain-rate tensor.

Hint: It is useful to show first that

$$\rho D_t\mathbf{v}+\boldsymbol{\nabla}P+\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{T}^{(1)}=0$$

with $D_t=\partial_t+\mathbf{v}\boldsymbol{\cdot}\boldsymbol{\nabla}$ the Lagrangian or material time-derivative.

(b) Use part (a) and conservation of total energy to derive the balance equation for internal energy density:

$$\partial_t u+\boldsymbol{\nabla}\boldsymbol{\cdot}\left[u\mathbf{v}+\mathbf{q}\right]=Q-P(\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{v}),$$

where $\mathbf{q}=-\kappa\boldsymbol{\nabla}T$ is the heat transport vector due to thermal conduction, or, alternatively,

$$D_tu+u(\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{v})+\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{q}=Q-P(\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{v}),$$

(c) Use the balance equation for conservation of mass to show that the velocity-divergence is given by

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{v}=-D_t\rho/\rho=D_t\mathscr{V}/\mathscr{V}$$

where $\mathscr{V}=1/n$ is the *specfic volume*, or volume per particle. Use this result to show that the pressure-dilatation term is given by

$$P(\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{v})=-\frac{1}{\mathscr{V}}D_tW$$

where $D_tW\doteq -PD_t\mathscr{V}$ is the rate at which work is performed by the velocity field acting against pressure to compress the fluid.

(d) The entropy per volume $s(u,\rho)$ is a thermodynamic function which satisfies the first law of thermodynamics in the form

$$ds=(du-\mu\ d\rho)/T$$

where $T(u,\rho)$ is the absolute temperature and $\mu(u,\rho)$ is the chemical potential per mass. These thermodynamic state functions also satisfy the *homogeneous Gibbs relation*

$$u + P = sT + \mu\rho.$$

Using these standard facts from thermodynamics and the system of compressible Navier-Stokes equation, derive the balance equation for entropy density

$$\partial_t s + \boldsymbol{\nabla\cdot} [s\mathbf{v} + \mathbf{q}/T] = \frac{Q}{T} + \frac{\kappa|\boldsymbol{\nabla} T|^2}{T^2} \geq 0.$$

This equation is a precise statement of the *second law of thermodynamics* for a single-component compressible fluid.

Hint: It is helpful to use the relation

$$D_t s = (1/T)D_t u - (\mu/T)D_t\rho$$

that follows from the first law of thermodynamics.