Example Solutions
Numerical Analysis 550.681
Homework #7
04.11.2008

Kincaid 6.3: 7
6.4: 7,14,17,24

Kincaid 6.3.7 Prove that the Taylor polynomial

$$p(x) = \sum_{j=0}^{k-1} \frac{1}{j!} f^{(j)}(x_0)(x - x_0)^j$$

interpolates f at $x_0, x_0, \ldots, x_0$ (k repetitions).

For any $1 \leq j \leq k-1$,

$$p^{(i)}(x) = \sum_{j=0}^{k-1-i} \frac{1}{j!} f^{(j+i)}(x_0)(x - x_0)^j$$

Thus, $p^{(i)}(x_0) = f^{(i)}(x_0)$. ■

Kincaid 6.4.7 Determine all the values a, b, c, d, e for which the following function is a cubic spline:

$$f(x) = \begin{cases} a(x-2)^2 + b(x-1)^3 & \text{if } x \in (-\infty, 1] \\ c(x-2)^2 & \text{if } x \in [1, 3] \\ d(x-2)^2 + e(x-3)^3 & \text{if } x \in [3, \infty) \end{cases}$$

Next, determine the values of the parameters so that the cubic spline interpolates this table:

x	0	1	4
y	26	7	25

We require

$$\begin{aligned} S_0(1) = S_1(1) &\quad \Rightarrow \quad a = c \\ S_1(3) = S_2(3) &\quad \Rightarrow \quad c = d \end{aligned}$$

$$\begin{aligned} S_0'(1) = S_1'(1) &\quad \Rightarrow \quad 2a(-1) = 2c(-1) \quad \Rightarrow \quad a = c \\ S_1'(3) = S_2'(3) &\quad \Rightarrow \quad 2c = 2d \quad \Rightarrow \quad c = d \end{aligned}$$

$$\begin{aligned} S_0''(1) = S_1''(1) &\quad \Rightarrow \quad 2a = 2c \quad \Rightarrow \quad a = c \\ S_1''(3) = S_2''(3) &\quad \Rightarrow \quad 2c = 2d \quad \Rightarrow \quad c = d \end{aligned}$$

So, our only conditions on a, b, c, d, e that will make $f(x)$ a continuous cubic spline is that $a = c = d$. Thus, any choice of a, b, e in the following will be a cubic spline:

$$S(x) = \begin{cases} a(x-2)^2 + b(x-1)^3 & \text{if } x \in (-\infty, 1] \\ a(x-2)^2 & \text{if } x \in [1, 3] \\ a(x-2)^2 + e(x-3)^3 & \text{if } x \in [3, \infty) \end{cases}$$

■

For $S(x)$ to interpolate the values given, we require

$$\begin{aligned} S_0(0) &= a(-1)^2 + b(-1)^3 = 26 \\ S_0(1) &= a(-1)^2 + b(0)^2 = 7 \\ S_2(4) &= a(2)^2 + e(1)^3 = 25 \end{aligned}$$

Solving this system of equations gives

$$\begin{aligned} a &= 7 \\ b &= 2 \\ e &= -3 \end{aligned}$$

So, the cubic spline which interpolates the given values is

$$S(x) = \begin{cases} 7(x-2)^2 + 2(x-1)^3 & \text{if } x \in (-\infty, 1] \\ 7(x-2)^2 & \text{if } x \in [1, 3] \\ 7(x-2)^2 - 3(x-3)^3 & \text{if } x \in [3, \infty) \end{cases}$$

■

Kincaid 6.4.14 Determine whether this function is natural cubic spline:

$$f(x) = \begin{cases} 2(x+1) + (x+1)^3 & \text{if } x \in [-1, 0] \\ 3 + 5x + 3x^2 & \text{if } x \in [0, 1] \\ 11 + 11(x-1) + 3(x-1)^2 - (x-1)^3 & \text{if } x \in [1, 2] \end{cases}$$

For $f(x)$ to be a cubic spline, we must check continuity up to second derivative.

$$f_1(0) = 2 + 1 = 3$$
$$f_2(0) = 3$$
$$\Rightarrow \quad f_1(0) = f_2(0)$$
$$f_1'(0) = 2 + 3 = 5$$
$$f_2'(0) = 5$$
$$\Rightarrow \quad f_1'(0) = f_2'(0)$$
$$f_1''(0) = 6$$
$$f_2''(0) = 6$$
$$\Rightarrow \quad f_1''(0) = f_2''(0)$$
$$f_2(1) = 3 + 5 + 3 = 11$$
$$f_3(1) = 11$$
$$\Rightarrow \quad f_2(1) = f_3(1)$$
$$f_2'(1) = 5 + 6 = 11$$
$$f_3'(1) = 11$$
$$\Rightarrow \quad f_2'(1) = f_3'(1)$$
$$f_2''(1) = 6$$
$$f_3''(1) = 6$$
$$\Rightarrow \quad f_2''(1) = f_3''(1)$$

For the function to be a natural cubic spline, we also require

$$f_1''(-1) = 0$$
$$f_3''(2) = 0$$

Checking,

$$f_1(x) = 2(x+1) + (x+1)^3$$
$$f_1'(x) = 2 + 3(x+1)^2$$
$$f_1''(x) = 6(x+1)$$
$$f_1''(-1) = 0$$

$$f_3(x) = 11 + 11(x-1) + 3(x-1)^2 - (x-1)^3$$
$$f_3'(x) = 11 + 6(x-1) - 3(x-1)^2$$
$$f_3''(x) = 6 - 6(x-1)$$
$$f_3''(2) = 0$$

Thus, $f(x)$ is a natural cubic spline. ■

Kincaid 6.4.17 Find the natural cubic spline function whose knots are $-1, 0, 1$ and that takes the values $S(-1) = 13, S(0) = 7, S(1) = 9$.

$$S(x) = \begin{cases} ax^3 + bx^2 + cx + d & \text{if } x \in [-1,0] \\ ex^3 + fx^2 + gx + h & \text{if } x \in [0,1] \end{cases}$$

$$S'(x) = \begin{cases} 3ax^2 + 2bx + c & \text{if } x \in [-1,0] \\ 3ex^2 + 2fx + g & \text{if } x \in [0,1] \end{cases}$$

$$S''(x) = \begin{cases} 6ax + 2b & \text{if } x \in [-1,0] \\ 6ex + 2f & \text{if } x \in [0,1] \end{cases}$$

$$\begin{aligned}
x = 0 \quad &\Rightarrow \quad S(0) = 7 = d = h && (1)\\
& \quad\quad S'(0) = c = g && (2)\\
& \quad\quad S''(0) = b = f && (3)\\
x = -1 \quad &\Rightarrow \quad S''(-1) = 0 = -6a + 2b && \\
& \quad\quad\quad \Rightarrow \quad b = 3a && (4)\\
x = 1 \quad &\Rightarrow \quad S''(1) = 0 = 6e + 2f && \\
& \quad\quad\quad \Rightarrow \quad f = -3e && (5)
\end{aligned}$$

$$(4) + (5) \quad \Rightarrow \quad a = -e \qquad (6)$$

$$\begin{aligned}
x = -1 \quad &\Rightarrow \quad S(-1) = 13 = -a + b - c + d && \\
& \quad\quad\quad \Rightarrow \quad 6 = -a + b - c && (7)\\
x = 1 \quad &\Rightarrow \quad S(1) = 9 = e + f + g + h && \\
& \quad\quad\quad \Rightarrow \quad 2 = e + f + g && (8)\\
& \quad\quad\quad \Rightarrow \quad 2 = -a + b + c && (9)
\end{aligned}$$

$$\begin{aligned}
(7) + (9) \quad &\Rightarrow \quad c = -2 && (10)\\
(4) + (7) + (10) \quad &\Rightarrow \quad a = 2 && (11)\\
(4) \quad &\Rightarrow \quad b = 6 && (12)
\end{aligned}$$

Putting this all together gives us

$$S(x) = \begin{cases} 2x^3 + 6x^2 - 2x + 7 & \text{if } x \in [-1,0] \\ -2x^3 + 6x^2 - 2x + 7 & \text{if } x \in [0,1] \end{cases}$$

■

Kincaid 6.4.24 If S is a first-degree spline function that interpolates f at a sequence of knots $0 = t_0 < t_1 < \cdots < t_n = 1$, what is $\int_0^1 S(x)dx$?

First degree spline functions are linear, so we can calculate the area under $S(x)$ by the trapezoidal rule,

$$\int_0^1 S(x)dx = \sum_{i=1}^{n}[(S(t_i) + S(t_{i-1})][\frac{t_i - t_{i-1}}{2}]$$

■