Kincaid 5.3.16 Use the Householder's algorithm to find the QR-factorization of

$$\begin{bmatrix} 0 & -4 \\ 0 & 0 \\ -5 & -2 \end{bmatrix}$$

$$||A_1||_2 = 5$$

$$B_1 = \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1 = \frac{1}{||A_1 - B_1||}(A_1 - B_1) = \frac{1}{5\sqrt{(2)}} \begin{bmatrix} 5 \\ 0 \\ -5 \end{bmatrix}$$

$$Q_1 = I - 2v_1v_1^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$Q_1A = \begin{bmatrix} -5 & -2 \\ 0 & 0 \\ 0 & -4 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

$$v_2 = \frac{1}{||A_2 - B_2||}(A_2 - B_2) = \frac{1}{4\sqrt{(2)}} \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix}$$

$$Q_2 = I - 2v_2v_2^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

So, the QR factorization of A is

$$R = Q_2Q_1A = \begin{bmatrix} -5 & -2 \\ 0 & -4 \\ 0 & 0 \end{bmatrix}$$
$$Q = Q_1Q_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

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Kincaid 5.3.19 Let A be an $m \times n$ matrix, b an m-vector, and $\alpha > 0$. Using the Euclidean norm define

$$F(x) = ||Ax - b||_2^2 + \alpha||x||_2^2$$

Prove that $F(x)$ is a minimum when x is a solution of the equation

$$(A^TA + \alpha I)x = A^Tb$$

If x is a solution of the equation $(A^TA + \alpha I)x = A^Tb$, then for any h,

$$\begin{aligned} F(x+h) - F(x) &= ||(Ax-b) + Ah||_2^2 - ||(Ax-b)||_2^2 + \alpha(||x+h||_2^2 - ||x||_2^2) \\ &= ||Ah||_2^2 + (Ax-b)^TAh + h^TA^T(Ax-b) + \alpha(||h||_2^2 + x^Th + h^Tx) \\ &= ||Ah||_2^2 + \alpha||h||_2^2 + h^T((A^TA+\alpha I)x - A^Tb) + ((A^TA+\alpha I)x - A^Tb)^Th \\ &= ||Ah||_2^2 + \alpha||h||_2^2 \\ &\geq 0 \end{aligned}$$

$$\Rightarrow \quad F(x+h) \geq F(x)$$

Thus, $F(x)$ is a minimum.

Kincaid 5.3.22 Show that in solving the least squares problem for the equation $Ax = b$, we can replace the normal equations by $CAx = Cb$, where C is an $n \times m$ matrix row-equivalent to A^T. Hint: Recall that two matrices G and H are row-equivalent if there is a nonsingular matrix F for which $G = FH$.

Let C be an $n \times m$ matrix which is row equivalent to A^T, so that $C = FA^T$ for some nonsingular matrix F (therefore F is also invertible). Then we can modify the normal equations by multiplying both sides by F on the left,

$$\begin{aligned} & A^TAx = A^Tb \\ \Leftrightarrow \quad & FA^TAx = FA^Tb \\ \Leftrightarrow \quad & CAx = Cb \end{aligned}$$

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Kincaid 5.3.30 Find the least squares solution of the system

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix}$$

If we let

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

then we wish to find the least squares solution to $Ax = b$. Using the normal equations,

$$A^*Ax = A^*b$$

$$A^*A = \begin{bmatrix} 14 & 14 \\ 14 & 17 \end{bmatrix}$$

$$A^*b = \begin{bmatrix} 10 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 14 \\ 14 & 17 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \end{bmatrix}$$

$$L = \begin{bmatrix} 3.7417 & 0 \\ 3.7417 & 1.7321 \end{bmatrix} \qquad L^T = \begin{bmatrix} 3.7417 & 3.7417 \\ 0 & 1.7321 \end{bmatrix} \qquad \text{(Cholesky } LL^T = A^*A\text{)}$$

$$x = 1.3810$$
$$y = -.6667$$

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Kincaid 5.3.37 Find the QR-factorization of the matrix

$$\begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$||A_1||_2 = 5$$

$$B_1 = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$$

$$v = \frac{1}{||A_1 - B_1||}(A_1 - B_1) = \frac{1}{\sqrt{(80)}} \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$Q = I - 2vv^T = \begin{bmatrix} -.6 & -.8 \\ -.8 & .6 \end{bmatrix}$$

$$R = QA = \begin{bmatrix} -5 & -5.2 & -6.6 \\ 0 & 1.4 & 1.2 \end{bmatrix}$$

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