Inclusions and non-inclusions of Archimedean and Laves lattices

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Abstract

For each pair of graphs from among the Archimedean lattices and their dual Laves lattices we demonstrate that one is a subgraph of the other or prove that neither can be a subgraph of the other. Therefore, we determine the entire partial ordering by inclusion of these 19 infinite periodic graphs. There are a total of 72 inclusion relationships, of which 35 are covering relations in the partial ordering. Several elementary criteria are developed to prove non-inclusion results. The investigation is motivated by applications to the ordering of critical probabilities in bond and site percolation models, time constants in first-passage percolation theory, and connective constants in self-avoiding walks.

1 Introduction

A *regular tiling* is a tiling of the plane which consists entirely of regular polygons. (A regular polygon is one in which all side lengths are equal and all interior angles are equal.) An *Archimedean lattice* is the graph of vertices and edges of a regular tiling which is vertex-transitive, i.e., for every pair of vertices, u and v, there is a graph isomorphism that maps u to v. There are exactly 11 Archimedean lattices, which are shown in Figure 1. A proof that these are the only vertex-transitive regular tilings is given in Grünbaum and Shephard [10, Ch. 2]. Archimedean lattices were first studied by Kepler [16], and received their name from references by Kepler to Archimedes' description of the related regular solid polyhedra [31].

A notation for Archimedean lattices, which can also serve as a prescription for constructing them, is given in Grünbaum and Shephard. Around any vertex (since all are equivalent, by vertex-transitivity), starting with the smallest polygon touching the vertex, list the number of edges of the successive polygons around the vertex. For convenience, an exponent is used to indicate that a number of successive polygons have the same size.

In addition to the numerical names, there are more descriptive names, sometimes colorful, for all the Archimedean lattices [31]. Naturally, (4^4) is the "square" lattice and (3^6) is the "triangular" lattice, but (6^3) is referred to as both the "hexagonal" and "honeycomb" lattice. The $(3, 4, 6, 4)$ is called the "ruby" [21] or "bounce" lattice, and the $(3, 6, 3, 6)$ is named the "Kagomé" lattice, meaning "woven bamboo pattern" in Japanese. The $(4, 8^2)$ is known as the "bathroom tile" or the "Briarwood" lattice, $(4, 6, 12)$ as the "cross" lattice, and $(3, 12^2)$ as the "extended Kagomé" [4] or "star" lattice. Less common names are "bridge" for $(3^4, 6)$, "puzzle" for $(3^2, 4, 3, 4)$, and "direct" for $(3^3, 4^2)$.

Since the Archimedean lattices are planar graphs, each has a planar dual graph. The square lattice is self-dual, and the triangular and hexagonal lattices are a dual pair of graphs. The other 8 Archimedean lattices have dual graphs that are not Archimedean. These dual graphs are also shown in Figure 1. We will denote the dual of an Archimedean lattice G by $D(G)$. The duals of the Archimedean lattices have applications in crystallography, where they are called Laves lattices [19, 20]. Descriptive names also exist for some of the Laves lattices: $D(3, 6, 3, 6)$ is the "dice" lattice, $D(3^3, 4^2)$ has been called the "pentagonal" lattice, $D(4, 8^2)$ is the "octagonal" [23] or "Union Jack" [29] lattice, and $D(3, 12^2)$ is the "asa no ha" [5] lattice, Japanese for "hemp leaf."

Let $A \subseteq B$ denote that A is isomorphic to a subgraph of B. The relation $\subseteq$ is reflexive and transitive, but is not anti-symmetric in general for infinite graphs. (An example of two non-isomorphic infinite tree graphs which are included in each other is given in [6, p. 231].) In this paper, we determine

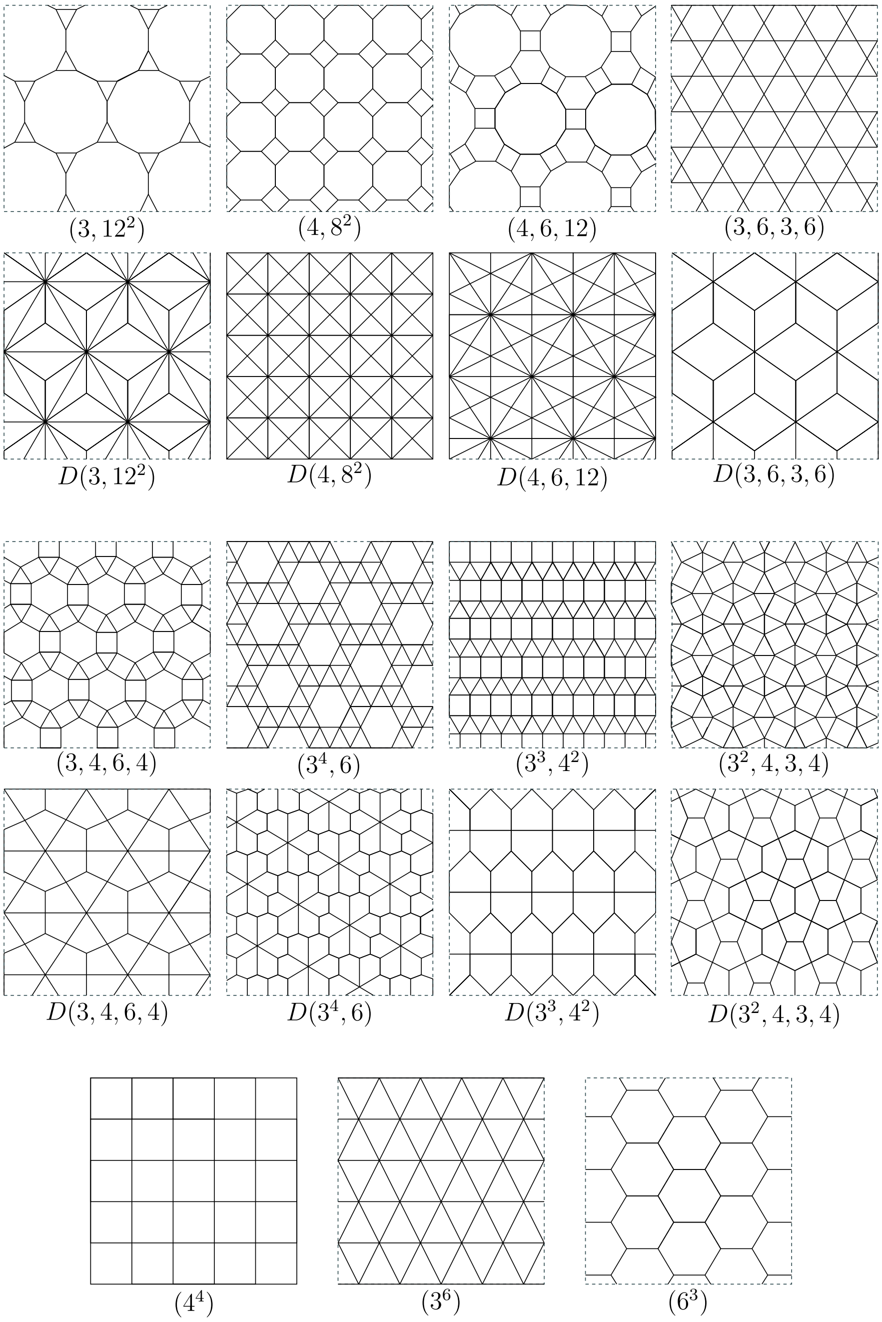

Figure 1: The 11 Archimedean lattices and their dual Laves lattices.

which Archimedean and Laves lattices are subgraphs of others, and show which pairs of them are incomparable. As a consequence, we show that for these lattices, $\subseteq$ is anti-symmetric, and thus is a partial order, which we have determined completely.

This paper is organized as follows: Section 2 provides an introduction to our motivating applications in the theories of classical percolation, first-passage percolation, and self-avoiding walks. A summary of the results are presented in Section 3, in a Hasse diagram showing the subgraph partial ordering and in a Table which also indicates the method of proof. All subgraph inclusion results are demonstrated in Figures in Section 4. In Section 5, several techniques for checking for non-inclusion are described. Non-inclusion proofs that are special cases, not handled by the general techniques, are provided in Section 6.

2 Motivating Applications

This study is motivated by applications to models from probability, combinatorics, and mathematical physics – bond and site percolation, first-passage percolation, and self-avoiding walks. We briefly describe each of these models in the following subsections, then comment on similarities relevant to the subgraph order problem.

2.1 Bond and Site Percolation Models

The two classical percolation models were introduced as models for the spread of fluid through a random medium. The medium is represented by an infinite connected locally finite graph. In the bond percolation model, each edge of the graph is *open* to the flow of fluid with probability p, $0 \leq p \leq 1$. In the site percolation model, each vertex is open with probability p, and fluid is permitted to flow through the subgraph induced by the set of open vertices. The key concept is the *critical probability*, or *percolation threshold*, denoted p_c, such that for $p < p_c$ there are almost surely no infinite connected components of open edges or vertices, and for $p > p_c$ there exists an infinite connected component with probability one. Considerable scientific interest focuses on percolation as a simple mathematical model for a phase transition, which is represented by the critical probability. See [27, 30] for descriptions of applications of percolation in engineering and physics. See Grimmett [9] for the most complete discussion of the mathematical theory.

2.2 First-passage Percolation

In *first-passage percolation*, introduced by Hammersley and Welsh [12], each edge of a graph is open, and associated with a random variable representing the time for the fluid to pass through the edge. Often the graph is taken to be the square lattice $\mathbb{Z}^2$ or some hypercubic lattice $\mathbb{Z}^d$, where it is natural to measure the speed of the spread of the fluid by a *first-passage time*, e.g., the time needed for the fluid to pass from the origin to a point $(n, 0)$ or to a line $x = n$. The *time constant* is defined as the limit of the first-passage time, normalized by the graph distance (number of edges in the shortest path), as the graph distance tends to infinity. The time constant may be interpreted as the reciprocal of the velocity of spread.

2.3 Self-avoiding Walks

A *self-avoiding walk* is a path of adjacent vertices such that no vertex occurs more than once. For a regular graph, the *connective constant* is the limit of the n-th root of the number of self-avoiding walks with n edges (starting at a fixed vertex). Self-avoiding walks have been used as a lattice model for the excluded volume problem in the theory of polymers. The first mathematically rigorous analysis of the subject was by Hammersley and Morton [11] in 1954. Hughes [13, Ch. 7] gives a nice review of the field, while a more substantial treatment is given by Madras and Slade [22].

2.4 Similarities

In the study of percolation models, first-passage percolation, and self-avoiding walks, considerable interest focuses on quantities which depend on the structure of an underlying graph in an extremely complicated fashion, so much so that there are few (if any) exact values known, and only very crude bounds in the many unsolved cases. Rigorous lower and upper bounds can be found for these critical values for several lattice graphs, usually by methods that require extensive computer calculations.

In classical percolation theory, the exact bond model critical probabilities or site model critical probabilities are known for only a few graphs [17, 18, 32, 33], thus making it important to determine rigorous bounds for unsolved graphs [7, 34, 35, 36, 37, 38] . Many simulation studies have estimated critical probabilities of various graphs, in particular the Archimedean lattices [31].

In first-passage percolation, other than its counterpart for infinite trees, the exact value of the time constant is not known for any non-trivial distribution on any non-trivial lattice. Determining rigorous bounds has been very challenging, with some progress by Janson [14] and Alm and Parviainen [3].

The connective constant is not known for any non-trivial lattice, although Nienhuis [24] has, by non-rigorous methods, derived the value $\sqrt{2 + \sqrt{2}}$ for the hexagonal lattice. (Jensen and Guttmann [15] uses this to conjecture that the connective constant for the $(3, 12^2)$ lattice is 1.71104.) See Alm [2], Conway and Guttmann [8], and Pönitz and Tittmann [26] for bounds on the connective constant.

However, for critical probabilities, time constants, and connective constants, the values for two graphs are ordered if one is a subgraph of the other. If H is a subgraph of G, the critical probabilities and the time constant are higher for H than for G, and the connective constant is lower for H than for G. Thus, knowledge of the subgraph order will allow the use of exact values or bounds for some graphs to provide bounds for other graphs. Only a few subgraph relationships, involving the triangular, hexagonal, and square lattices, have been observed and used in these theories in the past. To our knowledge, there has been no systematic study to determine the complete set of of subgraph relationships for any class of lattices.

The Archimedean and Laves lattices include the most common examples of 2-dimensional graphs studied in the three applications above. Thus, they are an appealing starting point when trying to obtain a deeper understanding of dependence of the critical parameters on the properties of the underlying graph.

3 Results

The results of this investigation can easily be summarized in the form of the Hasse diagram of the subgraph ordering, shown in Figure 4.

The Hasse diagram is accompanied by Table 1, which also provides a summary of the proof. Each entry indicates whether the lattice named at the left margin includes the lattice named at the top margin as a subgraph. A "+" or "T" indicates that inclusion holds, while any other symbol indicates that it does not. Entries of "T" indicate that the inclusion holds by transitivity, in which case the inclusions that imply it may be found by consulting the Hasse diagram or the "+" entries in Table 1. Other symbols indicate the lemma or method in Section 5 which is used to prove that inclusion does not hold.

We now note a few observations about the subgraph partial ordering. Of the 19 lattices, more than half are maximal or minimal elements: Three lattices – $D(3,12^2)$, $D(4,6,12)$, and $D(4,8^2)$ – are minimal elements, and the seven lattices – $D(3^4,6)$, $D(3^3,4^2)$, $D(3^2,4,3,4)$, (6^3), $(4,8^2)$, $(4,6,12)$, and $(3,12^2)$ – are maximal elements. The size of the longest chain is 5, and there are 10 different chains of that size. There is an antichain of size 8, consisting of $D(3,6,3,6)$, $D(3,4,6,4)$, (4^4), $(3,4,6,4)$, $D(3^2,4,3,4)$, $(3,6,3,6)$, $D(3^4,6)$, and $D(3^3,4^2)$. Since the partially ordered set may be covered by 8 chains, by Dilworth's Theorem the size of the largest antichain is 8.

Each subgraph inclusion result implies an inequality for the quantity of interest in each of the motivating applications. For critical probabilities in classical percolation theory, the inclusion results established in this paper all

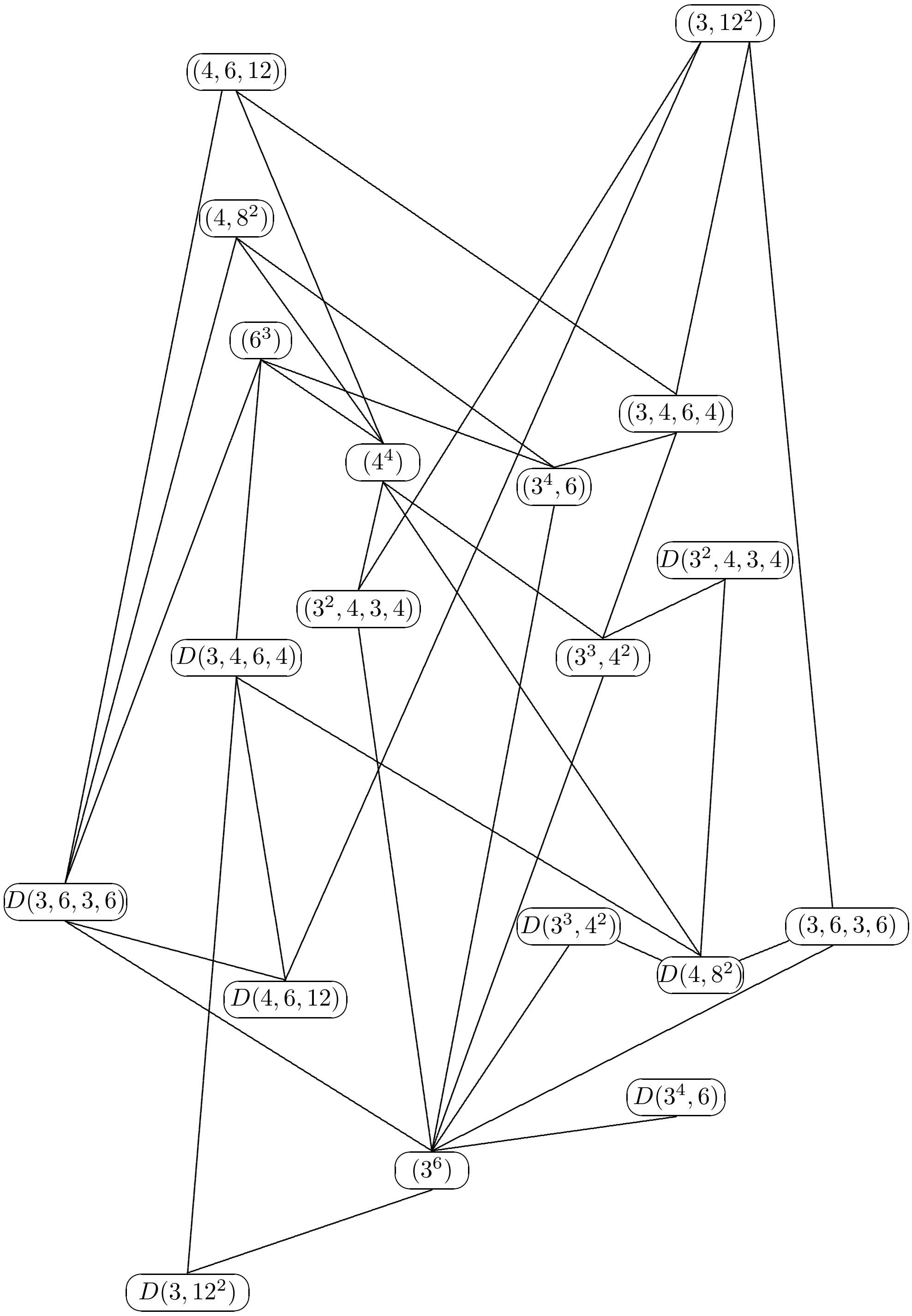

Figure 2: The Hasse diagram of the subgraph order of the Archimedean and Laves lattices. Edges of the diagram indicate covering relationships, in which the lattice higher in the diagram is a subgraph of the lattice lower in the diagram. Additional subgraph relationships, valid by transitivity, are implied, but not shown.

provide strict inequalities for critical probabilities, by a result of Aizenman and Grimmett [1]. For each of these motivating applications, the results of this paper may be combined with results from other techniques to make progress in determining the ordering of the quantities of interest.

	$(3,12^2)$	$(4,6,12)$	$(4,8^2)$	(6^3)	$D(3^2,4,3,4)$	$D(3^3,4^2)$	$D(3^4,6)$	$(3,6,3,6)$	$(3,4,6,4)$	(4^4)	$D(3,4,6,4)$	$D(3,6,3,6)$	$(3^4,6)$	$(3^3,4^2)$	$(3^2,4,3,4)$	(3^6)	$D(4,8^2)$	$D(4,6,12)$	$D(3,12^2)$
$(3,12^2)$		K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K
$(4,6,12)$	K		K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K
$(4,8^2)$	K	K		K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K
(6^3)	K	K	K		K	K	K	K	K	K	K	K	K	K	K	K	K	K	K
$D(3^2,4,3,4)$	ϕ	ϕ	ϕ	C		L	L	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	Δ	Δ	ϕ	Δ
$D(3^3,4^2)$	ϕ	ϕ	ϕ	C	L		L	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	Δ	Δ	ϕ	Δ
$D(3^4,6)$	ϕ	ϕ	ϕ	C	L	L		ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	Δ
$(3,6,3,6)$	+	C	C	C	Φ	Φ	Φ		C	Φ	Φ	Φ	Δ	Φ	Φ	Δ	Δ	Φ	Δ
$(3,4,6,4)$	+	+	S	S	Φ	Φ	Φ	I		Φ	Φ	Φ	Δ	Δ	Φ	Δ	Δ	Φ	Δ
(4^4)	ϕ	+	+	+	χ	χ	χ	χ	χ		L	L	χ	χ	χ	χ	χ	χ	Δ
$D(3,4,6,4)$	ϕ	S	S	+	χ	χ	χ	χ	χ	L		L	χ	χ	χ	χ	χ	χ	Δ
$D(3,6,3,6)$	ϕ	+	+	+	χ	χ	χ	χ	χ	L	L		χ	χ	χ	χ	χ	χ	Δ
$(3^4,6)$	T	T	+	+	Φ	Φ	Φ	S	+	Φ	Φ	Φ		Φ	Φ	Δ	Φ	Φ	Δ
$(3^3,4^2)$	T	T	T	T	+	S	Δ	I	+	+	Δ	Δ	I		I	Δ	Φ	Φ	Δ
$(3^2,4,3,4)$	+	T	T	T	S	S	Δ	S	S	+	Δ	Δ	A	A		Δ	Φ	Φ	Δ
(3^6)	T	T	T	T	T	+	+	+	T	T	S	+	+	+	+		L	L	Δ
$D(4,8^2)$	T	T	T	T	+	+	S	+	+	+	S	S	S	V	V	L		L	Δ
$D(4,6,12)$	+	T	T	T	S	S	S	V	V	V	+	+	V	V	V	L	L		S
$D(3,12^2)$	T	T	T	T	T	T	T	T	T	T	+	T	T	T	T	+	S	S	

+	inclusion
T	transitivity
K	3-connectivity
ϕ	minimum polygon size
Φ	maximum polygon size
C	combining polygons
Δ	maximum degree
V	variation in degree
L	Laves lattice
I	incident polygons
A	adjacent polygons
χ	chromatic number
S	special case

Table 1: All inclusions and non-inclusions among the Archimedean and Laves lattices. Each entry indicates if the lattice listed at the top is included in the lattice listed at the left, and if not, indicates the reasoning that proves non-inclusion. The key at the right provides an interpretation for each symbol used in the table. A "+" indicates an inclusion which is demonstrated in the figures in Section 4. A "T" indicates an inclusion which is valid by transitivity. An "S" indicates that non-inclusion is proved in a special argument similar to those given in Section 6. All other symbols refer to a lemma or method, named by the symbol, in Section 5 for proving non-inclusion.

4 Inclusion Proofs

Figures 2 – 4 demonstrate the 35 subgraph inclusion relationships that are denoted by + entries in Table 1. These are the covering relationships in the Hasse diagram for the subgraph ordering. Since the graphs are periodic, in each case sufficiently large induced subgraphs of the graphs are shown to

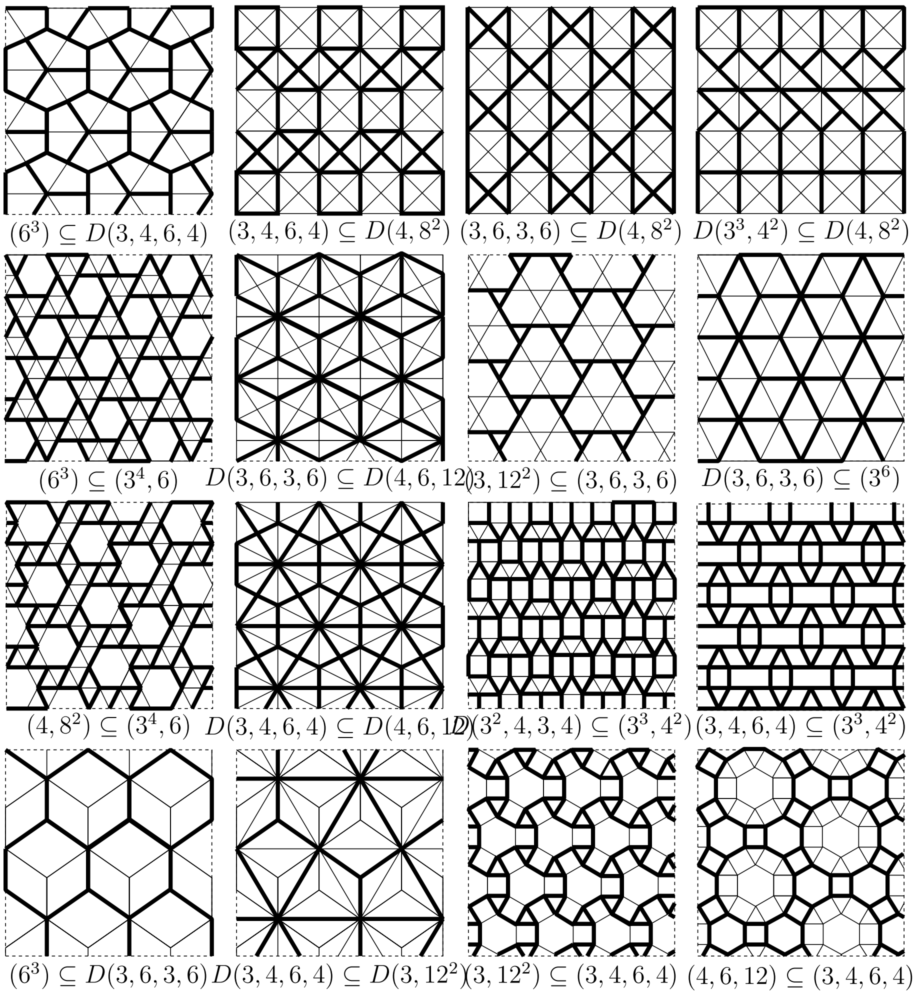

Figure 3: 16 Inclusions. Each drawing shows that one Archimedean or Laves lattice is a subgraph of another. The edges of the subgraph are indicated by thicker lines.

demonstrate that the inclusion relationship can be extended throughout the infinite graphs.

Transitivity implies the remaining 37 inclusions, denoted by T entries in Table 1. In each case, a sequence of covering relationships may be found in the Hasse diagram to demonstrate the inclusion.

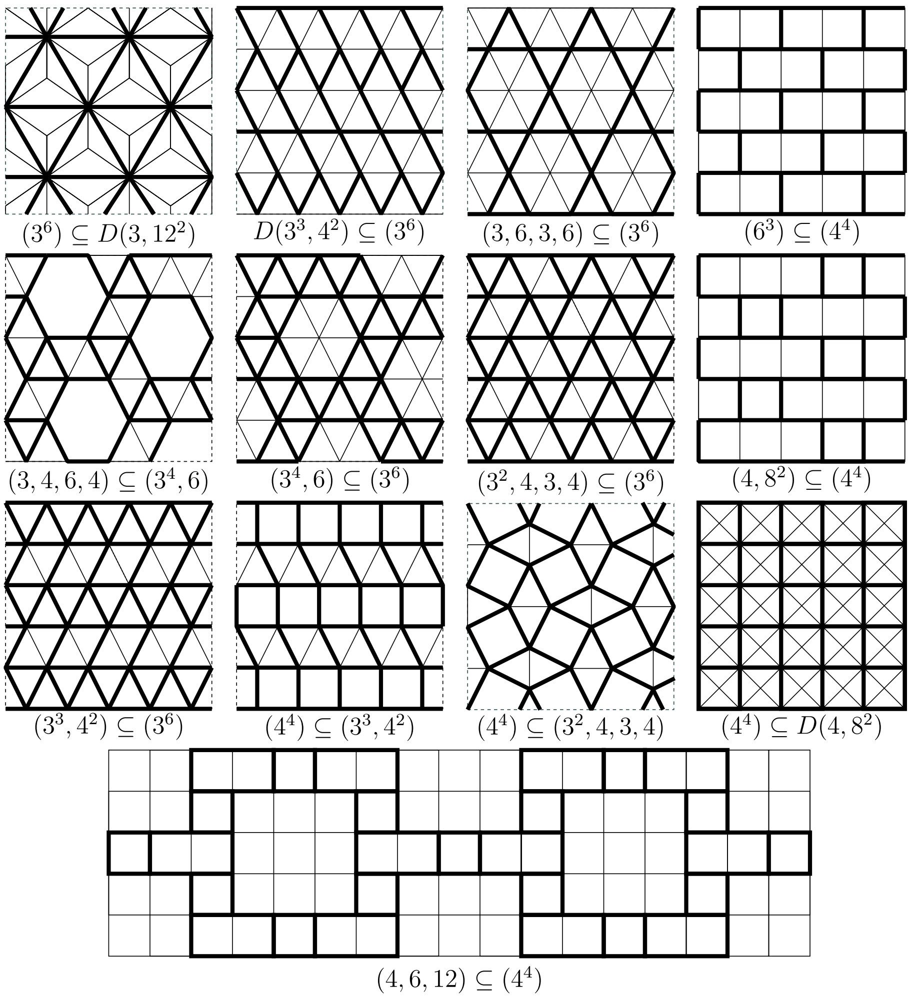

Figure 4: 13 Inclusions. Each drawing shows that one Archimedean or Laves lattice is a subgraph of another. The edges of the subgraph are indicated by thicker lines.

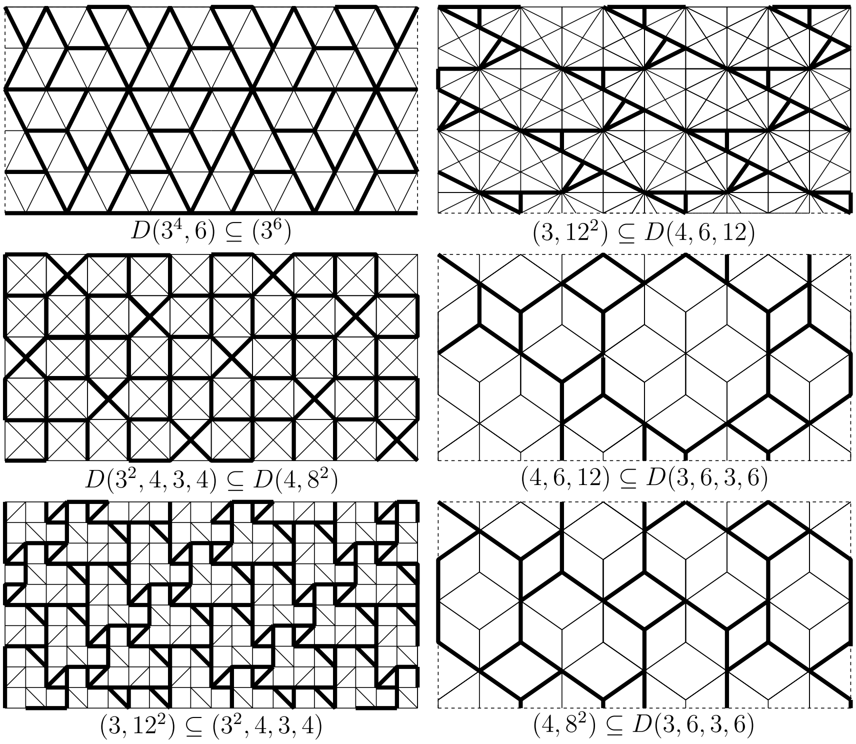

Figure 5: 6 Inclusions. Each drawing shows that one Archimedean or Laves lattice is a subgraph of another. The edges of the subgraph are indicated by thicker lines.

5 Non-inclusion Proofs

This section establishes several conditions under which one lattice cannot be a subgraph of another. While these allow us to prove the majority of the non-inclusion results for the Archimedean and Laves lattices, there are still a number of special cases which require individualized reasoning, which are discussed in Section 6.

Let $\mathcal{A}$ denote the set of Archimedean lattices and $\mathcal{L}$ denote the set of Laves lattices. Note that $\mathcal{A} \cap \mathcal{L} = \{(4^4), (3^6), (6^3)\}$.

We use the symbol $\subseteq$ to denote the inclusion relationship, letting $G \subseteq H$ denote that G is isomorphic to a subgraph of H. In many cases we will write as if H is a subgraph of G, rather than a separate graph on a different set of vertices. To create subgraphs of a given graph, we will delete vertices and edges. When we refer to deleting a vertex, we mean that the vertex and all edges incident to it are deleted from the graph.

For any graph G, let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. Let $d_G(v)$ denote the degree of vertex v in graph G. A vertex with degree n will be called an n-vertex.

For a graph G, denote the maximum degree by $\Delta(G) = \max_{v \in G} d_G(v)$

and the minimum degree by $\delta(G) = \min_{v \in G} d_G(v)$.

Since $G \in \mathcal{A} \cup \mathcal{L}$ is a 3-connected planar graph, it has a unique dual graph G^* by Whitney's 2-Isomorphism Theorem [39]. Consequently, any plane representation of G determines a set of faces of G, denoted $F(G)$, and each vertex $v \in V(G^*)$ corresponds to a unique face $f \in F(G)$.

We will refer to a face as a polygon, to the number of sides of a polygon as the *size* of the polygon, and to a polygon of size k as a k-gon. For a graph G, we denote the maximum polygon size by $\Phi(G)$ and the minimum polygon size by $\phi(G)$. Similarly, we will refer to a cycle of length k as a k-cycle. The term polygon is only used for cycles with empty interior.

Similarly to the notation for Archimedean lattices, we will say a vertex is of type $(a_1, a_2, \ldots)$ if successive faces around the vertex are of size a_1, a_2, etc. A polygon is said to be adjacent to each of its vertices. An edge of a polygon is said to be adjacent to the polygon.

5.1 Degree

Lemma Δ: If $\Delta(H) > \Delta(G)$, then $H \not\subseteq G$.

Proof: If $v \in H \subseteq G$, then $d_H(v) \leq d_G(v)$. Thus, $H \subseteq G$ implies that $\Delta(H) \leq \Delta(G)$.

This criterion is easy to check quickly, and is used in many cases to show that an Archimedean lattice cannot contain a Laves lattice as a subgraph.

Approach V: For several pairs of lattices, there is a large disparity between the maximum degrees, so that lowering the large degree in one lattice creates faces of a larger size than exist in the other lattice. We give an individual argument for each case in the following:

Examples: $(4^4) \not\subseteq D(4,6,12)$. To obtain (4^4) by deletions from $D(4,6,12)$, each 12-vertex in $D(4,6,12)$ must either be deleted or have its degree lowered to 4. Deleting the vertex creates an n-gon with $n \geq 12$, which does not exist in (4^4), so the degree must be lowered by deleting 8 incident edges. However, since each edge-deletion increases an incident polygon by at least one size, only 4 edges can be deleted to create the 4 4-gons needed to create (4^4).

$(3,6,3,6) \not\subseteq D(4,6,12)$. To create 2 3-gons and 2 6-gons around a 12-vertex in $D(4,6,12)$, at most 6 incident edges can be deleted. However, exactly 8 must be deleted to reduce the degree to 4.

$(3^4,6) \not\subseteq D(4,6,12)$. To create 4 3-gons and one 6-gon around a 12-vertex in $D(4,6,12)$, only 3 incident edges can be deleted, but exactly 7 must be to reduce the degree to 5.

$(3,4,6,4) \not\subseteq D(4,6,12)$. At a 12-vertex in $D(4,6,12)$, at most 5, but exactly 8 incident edges must be deleted.

$(3^3, 4^2) \not\subseteq D(4, 6, 12)$ and $(3^2, 4, 3, 4) \not\subseteq D(4, 6, 12)$. At a 12-vertex in $D(4, 6, 12)$, at most 2, but exactly 7 incident edges must be deleted.

$D(4, 8^2) \not\subseteq D(4, 6, 12)$. $D(4, 8^2)$ has 8-vertices surrounded by 3-gons. Deletion of any edge from a 12-vertex in $D(4, 6, 12)$ would create a larger polygon, but 4 edges must be deleted to reduce the degree to 8.

$(3^3, 4^2) \not\subseteq D(4, 8^2)$ and $(3^2, 4, 3, 4) \not\subseteq D(4, 8^2)$. $D(4, 8^2)$ has 8-vertices surrounded by 3-gons. 3 incident edges must be deleted, but only 2 edges can be without creating more or larger polygons than 3 3-gons and 2 4-gons.

5.2 Chromatic Number

A proper vertex coloring of G is a coloring of the vertices of G in which no two adjacent vertices have the same color. The vertex-chromatic number of G, denoted $\chi(G)$, is the minimum number of colors used in any proper coloring.

Lemma χ: If $\chi(H) > \chi(G)$, then $H \not\subseteq G$.

Proof: If C is a proper coloring of G, and $H \subset G$, then the restriction of C to H is a proper coloring of H. Thus, $H \subset G$ implies $\chi(H) \leq \chi(G)$.

Shrock and Tsai [28] give the vertex-chromatic numbers of all Archimedean and Laves lattices. Among these, the only lattice with vertex-chromatic number 4 is $D(3, 12^2)$. The lattices (4^4), (6^3), $(4, 6, 12)$, $(4, 8^2)$, $\mathrm{D}(3, 4, 6, 4)$, and $\mathrm{D}(3, 6, 3, 6)$ have vertex-chromatic number 2, and the remaining lattices have vertex-chromatic number 3. Letting L_k denote any lattice with vertex-chromatic number k, we have the following non-inclusions:

$$D(3, 12^2) \not\subseteq L_3 \not\subseteq L_2.$$

Remark: Similar reasoning is valid for the edge-chromatic number. However, for all $G \in \mathcal{A} \cup \mathcal{L}$, the edge-chromatic number is equal to the maximum degree, so no additional information is gained from edge-chromatic number.

5.3 Edge-Connectivity

A disconnecting set of edges is a set $F \subseteq E(G)$ such that the subgraph of G with edge set $E(G) - F$ has more than one component. A graph is k-edge connected if every disconnecting set has at least k edges. The edge-connectivity of G, denoted $K(G)$, is the minimum size of a disconnecting set, or, equivalently, the maximum k such that G is k-edge-connected.

Each $G \in \mathcal{A} \cup \mathcal{L}$ is 3-edge connected. In particular, each graph $G \in \mathcal{A} \cup \mathcal{L}$ has minimum vertex degree greater than or equal to three. If a vertex or edge is deleted, the remaining graph may have vertices of degree 2 or less. If so, we may continue by repeatedly deleting all vertices of degree two or less (and edges incident to these vertices) until the resulting graph has minimum degree 3 or larger or is the empty graph. We call this process 3-deletion, and denote the graph obtained by 3-deletion of an edge e from G by$(G-e)_{(3)}$.

Lemma K: Let $G, H \in \mathcal{A} \cup \mathcal{L}$. If $(G-e)_{(3)} = \emptyset$ for every $e \in G$, then $H \not\subseteq G$.

Proof: Suppose $H \subset G$ is 3-edge connected. If an edge $e \in G - H$ were deleted, then $H \subset (G-e)_{(3)}$, since all vertices $v \in H$ have $d_H(v) \geq 3$. Thus, if $(G-e)_{(3)} = \emptyset$ for every $e \in G$, then $G \not\supseteq H \in \mathcal{A} \cup \mathcal{L}$.

Since the hexagonal, $(3, 12^2)$, $(4, 8^2)$, $(4, 6, 12)$ lattices are regular with degree three, by Lemma K none can contain any of the other Archimedean or Laves lattices.

5.4 Polygon Size

Polygon sizes in the Archimedean and Laves lattices satisfy a monotonicity property: Let H be constructed from $G \in \mathcal{A} \cup \mathcal{L}$ by deleting a set of vertices and edges. If F_H denotes the face of H containing the face F in G, then the size of F_H is greater than or equal to the size of F. [Since the Archimedean and Laves lattices are all periodic, one only needs to check a sufficiently large bounded region to verify this property.] Note that deleting a set of edges which includes an edge of a face need not strictly increase the size of the face: A triangular face may be obtained by deleting edges in the $(3, 12^2)$ lattice.

Lemma ϕ: Let $G, H \in \mathcal{A} \cup \mathcal{L}$. If $\phi(H) < \phi(G)$, then $H \not\subseteq G$.

Proof: If $H \subseteq G$, then every face of G is entirely contained in some face of H. Since $G \in \mathcal{A} \cup \mathcal{L}$, deletion of vertices or edges does not decrease the polygon size, and the face of H has a larger size. Taking the minimum over all faces of G, $\phi(H) \geq \phi(G)$.

Lemma Φ: Let $G, H \in \mathcal{A} \cup \mathcal{L}$. If $\Phi(H) < \Phi(G)$, then $H \not\subseteq G$.

Proof: If $H \subseteq G$, then every face of G is contained in a face of H. Since $G \in \mathcal{A} \cup \mathcal{L}$, deletion of vertices or edges does not decrease the polygon size, and the face of H has an equal or larger size. Taking the maximum over all faces of G, $\Phi(H) \geq \Phi(G)$.

We will say that two polygonal faces are *incident* if they share a common vertex but have no common edge. Let $I(G)$ denote the maximum number of $\phi(G)$-gons incident to any $\phi(G)$-gon in G.

Lemma I: Let $G, H \in \mathcal{A} \cup \mathcal{L}$, $G \neq D(3, 12^2)$. If $\phi(H) = \phi(G)$ and $I(H) >$

$I(G)$, then $H \not\subseteq G$.

Proof: Let F be a face of size $\phi(G)$ in G which has $I(G)$ incident faces of size $\phi(G)$. Deletion of vertices and edges cannot create any additional faces of size $\phi(G)$. If H is obtained by deletion of vertices or edges of G, the face F may not remain in H, or the face F may remain, in which case the number of faces of size $\phi(H) = \phi(G)$ is less than or equal to $I(G)$. Taking the maximum over all faces of size $\phi(H)$, noting that there may be more faces of size $\phi(G)$ in G than in the subgraph H, we have $I(H) \leq I(G)$.

Remark: The reasoning above is not valid when $G = (3, 12^2)$, since it contains 3-cycles which may become 3-gons in a subgraph. However, since $I(G) = 18$, the hypothesis $I(H) > I((3, 12^2))$ is not satisfied for any $H \in \mathcal{A} \cup \mathcal{L}$, so no conclusion could be drawn anyway.

Examples: Each 3-gon in $(3^3, 4^2)$ is incident to 2 others, but is incident to 3 in $(3, 6, 3, 6)$, so

$$(3, 6, 3, 6) \not\subseteq (3^3, 4^2).$$

Each 3-gon in $(3^3, 4^2)$ is incident to 2 others, but is incident to 3 in $(3^4, 6)$, so

$$(3^4, 6) \not\subseteq (3^3, 4^2).$$

Each 3-gon in $(3, 4, 6, 4)$ is incident to no others, but is incident to 3 in $(3, 6, 3, 6)$, so

$$(3, 6, 3, 6) \not\subseteq (3, 4, 6, 4).$$

Each 3-gon in $(3^3, 4^2)$ is incident to 2 others, but is incident to 4 in $(3^2, 4, 3, 4)$, so

$$(3^2, 4, 3, 4) \not\subseteq (3^3, 4^2).$$

We will say that two polygonal faces are adjacent if they share a common edge. Let $A(G)$ denote the maximum number of $\phi(G)$-gons that are adjacent to a $\phi(G)$-gon in G.

Lemma A: Let $G, H \in \mathcal{A} \cup \mathcal{L}$, $G \neq D(3, 12^2)$. If $\phi(H) = \phi(G)$ and $A(H) > A(G)$, then $H \not\subseteq G$.

Proof: Let F be a face of size $\phi(G)$ in G which has $A(G)$ adjacent faces of size $\phi(G)$. Deletion of vertices and edges cannot create any additional faces of size $\phi(G)$. If H is obtained by deletion of vertices or edges of G, the face F may not remain in H, or the face F may remain, in which case the number of adjacent faces of size $\phi(H) = \phi(G)$ is less than or equal to $A(G)$. Taking the maximum over all faces of size $\phi(H)$, noting that there may be more faces of size $\phi(G)$ in G than in the subgraph H, we have $A(H) \leq A(G)$.

Examples: Each 3-gon in $(3^2, 4, 3, 4)$ is adjacent to one other, but is adjacent to 2 in $(3^3, 4^2)$, so

$$(3^3, 4^2) \not\subseteq (3^2, 4, 3, 4).$$

Each 3-gon in $(3^2, 4, 3, 4)$ is adjacent to one other, but is adjacent to 2 or more in $(3^4, 6)$, so

$$(3^4, 6) \not\subseteq (3^2, 4, 3, 4).$$

Our next condition involves the possible sizes of unions of polygons.

Lemma C : Suppose H contains k-gons and G does not. If deleting edges from any n-gons in G with $n < k$ produces only n-gons with $n > k$, then $H \not\subseteq G$.

Proof: Suppose H is a subgraph of G which contains a face F of size k. F is not a face of G, so it must be a union of faces $F_1, F_2, F_3, \dots, F_l$, where $l \geq 2$. However, by hypothesis, a polygon created by such a union has at least $n > k$ edges. Thus, no subgraph of G can have a face of size k.

Examples: The following examples illustrate the application of Lemma C: Deleting edges of 3-gons in $(3, 6, 3, 6)$ gives 7-gons or larger, not 4- or 6-gons, so

$$(4, 6, 12) \not\subseteq (3, 6, 3, 6).$$

Deleting edges of 3-gons in $(3, 6, 3, 6)$ gives 7-gons or larger, not 4-gons, so

$$(4, 8^2) \not\subseteq (3, 6, 3, 6).$$

Deleting edges of 3-gons in $(3, 6, 3, 6)$ gives 7-gons or larger, and not 6-gons, so

$$(6^3) \not\subseteq (3, 6, 3, 6).$$

Deleting edges of 3-gons in $(3, 6, 3, 6)$ gives 7-gons or larger, and not 4- or 6-gons, so

$$(3, 4, 6, 4) \not\subseteq (3, 6, 3, 6).$$

Deleting edges in (6^3) gives 10-gons or larger, not 5-gons, so

$$(6^3) \not\subseteq D(3^2, 4, 3, 4),$$

$$(6^3) \not\subseteq D(3^3, 4^2),$$

and

$$(6^3) \not\subseteq D(3^4, 6).$$

Lemma L: Let $G, H \in \mathcal{L}$, $G, H \neq D(3, 12^2)$. If $\phi(G) = \phi(H)$, then $G \not\subseteq H$ and $H \not\subseteq G$.

Proof: Since G and H are Laves lattices, all their faces are the same size, say k. Deletion of vertices or edges produces only n-gons, for $n > k$. So neither lattice can be obtained from the other by deletion. Thus, if they are not isomorphic, G and H are incomparable. It is easily checked that all pairs of Laves lattices with a common face size are non-isomorphic.

Remark: Note that $D(3, 12^2)$ could not be included in the group of fully-triangulated lattices in the previous paragraph, because vertices and edges can be deleted to obtain different lattices which are still fully-triangulated. In fact, $D(3, 12^2)$ does contain (3^6) as a subgraph!

6 Non-Inclusion Proofs for Special Cases

There are 21 cases of non-inclusion which are not established by the approaches described in Section 5, and have been proved by special individual arguments. These cases are indicated by an "S" in Table 1. In this section, we give the non-inclusion proofs for these 21 cases.

The form of the typical reasoning for these proofs is as follows: First, we identify a particular structure in the graph H, for instance a cycle with specified degree sequence. (The degree sequence of a cycle or path is the degrees of the vertices as the cycle or path is traversed.) We then show that this structure does not exist in the graph G, nor can be created by deletions from G. Therefore, we conclude that H is not isomorphic to a subgraph of G.

6.1 $D(4, 6, 12) \not\subseteq D(3, 12^2)$

In $D(4, 6, 12)$, all 3-paths with degree sequence (6,4,6) are in the interior of 8-cycles. In $D(3, 12^2)$, there are only 3-vertices and 12-vertices, so we must delete edges from 12-vertices to obtain 4-vertices and 6-vertices. However, the shortest cycle in $D(3, 12^2)$ that surrounds any three 12-vertices has length 9, so $D(4, 6, 12)$ cannot be a subgraph of $D(3, 12^2)$.

6.2 $D(4, 8^2) \not\subseteq D(3, 12^2)$

We begin with any easy case, showing how one can use the cycle structure in a non-inclusion proof.

In $D(4, 8^2)$, each 4-vertex is adjacent to every vertex of a surrounding 4-cycle. In $D(3, 12^2)$, the only vertices in the interior of a 4-cycle are 3-vertices. Therefore, $D(4, 8^2)$ cannot be a subgraph of $D(3, 12^2)$.

6.3 $D(3, 4, 6, 4) \not\subseteq (3^6)$

In $D(3, 4, 6, 4)$, there are 8-cycles which surround 3 vertices. The shortest cycle in 3^6 which has 3 vertices in the interior has length 9, so $D(3, 4, 6, 4)$ cannot be included in (3^6).

6.4 $(6^3) \not\subseteq (3,4,6,4)$

We claim that we cannot create vertices of type 6^3 by deleting edges from $(3,4,6,4)$. Edges of a 6-gon in $(3,4,6,4)$ cannot be deleted, since an n-gon of larger size would be created. New 6-gons can only be created by deleting two opposite edges of a 4-gon. A vertex of a new 6-gon can only be of type $(6^2,4)$, $(6^2,5)$, $(6^2,7)$, or $(6^2,k)$ where $k > 7$, but not of type 6^3. See Figure 6.

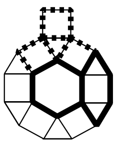

Figure 6: $(6^3) \not\subseteq (3,4,6,4)$

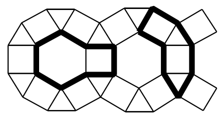

Figure 7: $(4,8^2) \not\subseteq (3,4,6,4)$

6.5 $(4,8^2) \not\subseteq (3,4,6,4)$

In $(4,8^2)$, every other edge of an 8-gon is adjacent to a 4-gon. New 4-gons cannot be created in $(3,4,6,4)$ by deletion. There are only two ways to create an 8-gon from $(3,4,6,4)$, either by deleting an edge in a 6-gon, or by combining 2 3-gons and 2 4-gons. See Figure 7. In neither case does the resulting 8-gon have the property that every other edge is adjacent to a 4-gon.

6.6 $(3,6,3,6) \not\subseteq (3^4,6)$

In $(3,6,3,6)$ each edge in a 6-gon is adjacent to a 3-gon. If any edge in $(3^4,6)$ is deleted, either this property is violated for an existing 6-gon or a polygon with more than 6 edges is created.

6.7 $(3,6,3,6) \not\subseteq (3^2,4,3,4)$

We now show another easy example, in which construction of polygons by deletions leads to contradictions.

In $(3,6,3,6)$, each edge in a 6-gon is adjacent to a 3-gon. To obtain a 6-gon in $(3^2,4,3,4)$, a 4-gon must be enlarged by deletion of 2 edges, or by deletion of one edge and combination with two 3-gons adjacent to each other. However, the resulting 6-gons have at least one edge which is adjacent to a 4-gon or larger.

6.8 $(3,4,6,4) \not\subseteq (3^2,4,3,4)$

In $(3,4,6,4)$, each edge in a 6-gon is adjacent to a 4-gon, and each 6-gon is incident to 6 3-gons. To obtain a 6-gon in $(3^2,4,3,4)$, a 4-gon must be enlarged by deletion of 2 edges, or by deletion of one edge and combination with two 3-gons adjacent to each other. If 2 edges are deleted, the resulting hexagon can be incident to at most 4 3-gons. If the 4-gon is combined with two 3-gons adjacent to each other, and each edge of the resulting 6-gon is adjacent to a 4-gon, then there are no incident 3-gons.

6.9 $D(3^3,4^2) \not\subseteq (3^2,4,3,4)$ and $D(3^2,4,3,4) \not\subseteq (3^2,4,3,4)$

In $D(3^3,4^2)$ there are vertices of type 5^4 and all faces are 5-gons. A vertex of type 5^4 can be obtained from $(3^2,4,3,4)$ by deleting edges in only one way, shown in Figure 8. However, this configuration can not be extended to a graph in which all faces are 5-gons, because the 3-gon which is adjacent to 2 5-gons can be enlarged only to a 4-gon, a 6-gon, or a k-gon with $k > 6$.

The same reasoning is valid for $D(3^2,4,3,4) \not\subseteq (3^2,4,3,4)$, since $D(3^2,4,3,4)$ also has vertices of type 5^4 and all its faces are 5-gons.

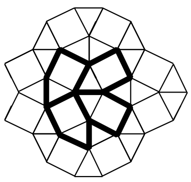

Figure 8: $D(3^3,4^2) \not\subseteq (3^2,4,3,4)$

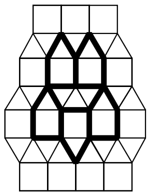

Figure 9: $D(3^3,4^2) \not\subseteq (3^3,4^2)$

6.10 $D(3^3,4^2) \not\subseteq (3^3,4^2)$

Two types of 5-gons may be created in $(3^3,4^2)$, by combining three 3-gons, or by combining a 3-gon and a 4-gon. Both types must occur in any subgraph of only 5-gons. Checking all possible arrangements of adjacent 5-gons verifies that a 5-gon of the first type cannot have degree sequence $(3,3,3,4,4)$, as 5-gons in $D(3^3,4^2)$ have. One arrangement is shown in Figure 10.

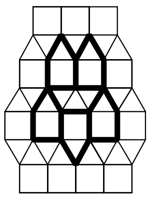

Figure 10: $D(3^3,4^2) \not\subseteq (3^3,4^2)$

6.11 $(4, 6, 12) \not\subseteq D(3, 4, 6, 4)$

In $(4, 6, 12)$, each 12-gon has 12 adjacent 4- and 6-gons and no incident polygons, and is connected by one edge to another 12-gon. Of all possible types of 12-cycles in $D(3, 4, 6, 4)$, only one can have 12 adjacent 4- and 6-gons and no incident polygons, but two disjoint 12-gons of this type cannot be connected by one edge. See Figure 11.

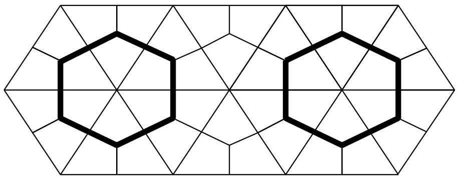

Figure 11: $(4, 6, 12) \not\subseteq D(3, 4, 6, 4)$

6.12 $(4, 8^2) \not\subseteq D(3, 4, 6, 4)$

In $(4, 8^2)$, each 8-gon has all vertices of degree 3 and edges alternatingly adjacent to 4-gons and 8-gons. There are only two types of 8-cycles in $D(3, 4, 6, 4)$ which can satisfy this property. From an 8-gon of either type, constructing the adjacent polygons to satisfy the property necessarily produces the structure shown in Figure 12 (which also illustrates the two types of 8-gons possible). However, the construction cannot be extended in a way that satisfies the property, in areas marked with dotted arrows in the Figure.

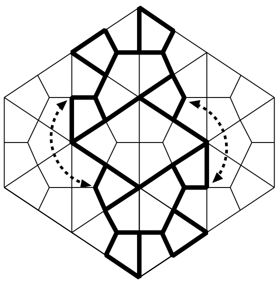

Figure 12: $(4, 8^2) \not\subseteq D(3, 4, 6, 4)$

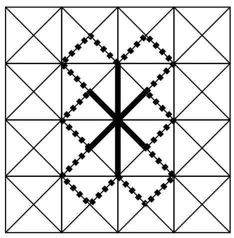

Figure 13: $D(3^4, 6) \not\subseteq D(4, 8^2)$

6.13 $D(3^4, 6) \not\subseteq D(4, 8^2)$

In $D(3^4, 6)$, there exist 6-vertices, and each pair of adjacent 5-gons share only one edge. A 6-vertex can be obtained from $D(4, 8^2)$ only by deleting 2 edges incident to an 8-vertex. There are 6 different types of pairs of edges to delete, but only one type results in each pair of adjacent 5-gons sharing only one edge: delete opposite edges which are incident to 8-vertices. However, since the other 6 edges must be retained, 4-gons will remain in the graph. See Figure 13.

6.14 $D(3,4,6,4) \not\subseteq D(4,8^2)$

This example is similar to the above, but starts by considering vertices instead of polygons.

In $D(3,4,6,4)$, there are 6-vertices which are adjacent to only 4-vertices. A 6-vertex can be obtained from $D(4,8^2)$ only by deleting 2 edges incident to an 8-vertex, while retaining the other 6 incident edges. Thus, at least two edges from the 8-vertex to 4-vertices must be retained. Since 4 additional edges must be retained, the resulting graph must contain 3-gons (See Figure 14), which do not exist in $D(3,4,6,4)$.

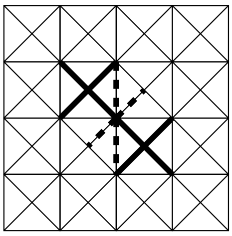

Figure 14: $D(3,4,6,4) \not\subseteq D(4,8^2)$

6.15 $(3^4,6) \not\subseteq D(4,8^2)$

In $(3^4,6)$, all 3-gons contain only 5-vertices, but in $D(4,8^2)$, the 3-gons have degree sequence $(4,8,8)$. Since deletions cannot create new triangles, or raise the degree of the 4-vertices in $D(4,8^2)$, $(3^4,6)$ cannot be a subgraph of $D(4,8^2)$.

6.16 $D(3,6,3,6) \not\subseteq D(4,8^2)$

This example depends on the degree sequences of cycles.

In $D(3,6,3,6)$, each 3-vertex is in the interior of a 6-cycle with degree sequence (3,6,3,6,3,6). In $D(4,8^2)$, no 8-vertex is in the interior of any 6-cycle. Thus, to obtain $D(3,6,3,6)$ from $D(4,8^2)$ by deletion, the 3-vertices in $D(3,6,3,6)$ must be obtained from 4-vertices in $D(4,8^2)$, which is possible only if $D(4,8^2)$ contains 6-cycles around 4-vertices with 4-vertices at every other vertex. There are two different types of 6-cycles around 4-vertices in $D(4,8^2)$, but each type includes four 8-vertices, so $D(3,6,3,6)$ is not a subgraph of $D(4,8^2)$.

6.17 $D(3, 12^2) \not\subseteq D(4, 6, 12)$

$D(3, 12^2)$ contains a 3-gon consisting of 12-vertices. $D(4, 6, 12)$ does not contain such a 3-gon, and since $\Delta((4, 6, 12)) = 12$, none can be created by deletions.

6.18 $D(3^4, 6) \not\subseteq D(4, 6, 12)$

This is perhaps the most complicated special case, with the longest proof.

In $D(3^4, 6)$, around any 6-vertex there is an 18-cycle which separates it from all other 6-vertices.

To obtain $D(3^4, 6)$ from $D(4, 6, 12)$ by deletions, first observe that the 6-vertices in $D(4, 6, 12)$ cannot become 6-vertices in $D(3^4, 6)$: Since all vertices adjacent to the original 6-vertex must become 3-vertices, 3-gons would be retained. Thus, 6-vertices in $D(3^4, 6)$ must be obtained by deleting edges incident to 12-vertices in $D(4, 6, 12)$.

Next, note that a 6-vertex in $D(3^4, 6)$ is connected to 6 other 6-vertices by paths of length 3. These 6-vertices in $D(3^4, 6)$ must be obtained from a 12-vertex and the nearest 6 12-vertices in $D(4, 6, 12)$. However, as shown in Figure 15, the only cycle separating the 6-vertex from the other 6-vertices in $D(3^4, 6)$ is a 12-cycle, not an 18-cycle.

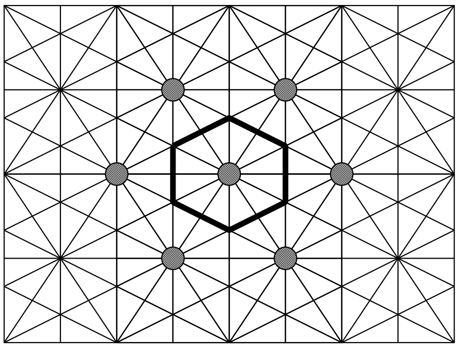

Figure 15: $D(3^4, 6) \not\subseteq D(4, 6, 12)$

6.19 $D(3^3, 4^2) \not\subseteq D(4, 6, 12)$

$D(3^3, 4^2)$ contains doubly-infinite paths of 4-vertices. To obtain $D(3^3, 4^2)$ from $D(4, 6, 12)$ by deletion, first note that 4-vertices in $D(4, 6, 12)$ cannot become 4-vertices in $D(3^3, 4^2)$, since this would create either triangles or 4-gons, as shown in Figure 16. Thus, a doubly-infinite path of 4-vertices must correspond to a path with degree sequence $(\ldots, 12, 6, 12, 6, \ldots)$ in $D(4, 6, 12)$. However, this implies that the resulting graph contains 3-gons. See Figure 16 again, in which the vertices marked with circles are required have degree 3 in $D(3^3, 4^2)$.

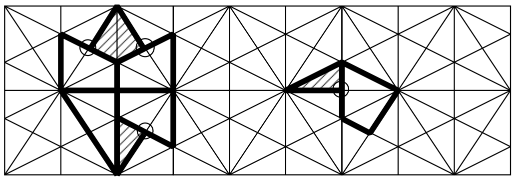

Figure 16: $D(3^3, 4^2) \not\subseteq D(4, 6, 12)$

6.20 $D(3^2, 4, 3, 4) \not\subseteq D(4, 6, 12)$

In $D(3^2, 4, 3, 4)$, each edge that connects two 3-vertices is in the interior of a 10-cycle. In $D(4, 6, 12)$, there are 3 classes of edges; those connecting vertices of degree 12 and 4, 12 and 6, and 6 and 4. Of these, only the last are in the interior of a 10-cycle. Consider the possible configurations of edges adjacent to such an edge, as shown in Figure 17, to see that all require the retention of 3-gons or 4-gons if the construction is extended.

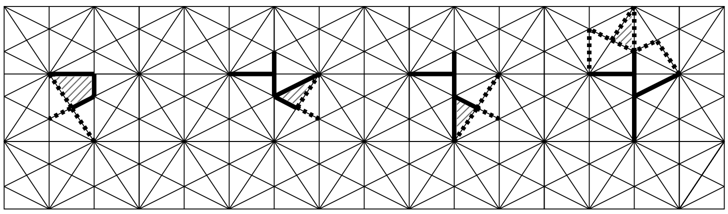

Figure 17: $D(3^2, 4, 3, 4) \not\subseteq D(4, 6, 12)$

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