On the Range of Bond Percolation Thresholds for Fully-Triangulated Graphs

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Abstract

This note constructs a sequence of planar periodic fully-triangulated lattices which have bond percolation thresholds tending to zero. As a consequence, we see that bond percolation thresholds of such graphs can range from 0 to $2\sin(\pi/18) \approx .3473$, which is the threshold of the triangular lattice. Rigorous bounds and simulations for other fully-triangulated lattices suggest that the threshold for the triangular lattice is the largest for this class of graphs.

1 Introduction

Periodic fully-triangulated lattices play a special role in percolation theory. Since any fully-triangulated graph is self-matching, the site percolation threshold of any periodic fully-triangulated graph is exactly 1/2. This remarkable fact was derived by Sykes and Essam [19], and proved under symmetry conditions by Kesten [8] and in general by Menshikov [16] and Aizenman and Barsky [1] independently. [Note that there are non-periodic fully-triangulated graphs which have site percolation thresholds different from 1/2. See van den Berg [3] and Wierman [21].] As a consequence, the site percolation threshold of any planar periodic lattice is at least one-half, since additional edges may be inserted to create a periodic fully-triangulated lattice.

Since site percolation thresholds of periodic fully-triangulated lattices are concentrated at a single value, this note investigates the extent of concentration of bond percolation thresholds of such lattices. The surprising result is that bond percolation thresholds of periodic fully-triangulated

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lattices range at least from near zero to $2\sin(\pi/18) \approx .3473$, the threshold of the triangular lattice. Section 2 discusses theoretical bounds for the bond percolation thresholds of certain Archimedean lattices or their duals, which are fully-triangulated. Section 3 provides a construction of a sequence of periodic fully-triangulated lattices which have bond percolation thresholds converging to zero. In section 4, we conjecture that the triangular lattice has the largest bond percolation threshold of any periodic fully-triangulated lattice.

2 Archimedean and Laves Lattices

Archimedean lattices are vertex-transitive graphs with a planar representation that is a tiling of the plane by regular polygons. There are exactly 11 Archimedean lattices [6], which are illustrated in [18]. We denote each Archimedean lattice by a sequence of integers $(n_1^{a_1}, n_2^{a_2}, ...)$, where the n_i denote the number of sides of consecutive faces as one moves around a single vertex, and the a_i give the number of successive faces of the same size. The dual graph of an Archimedean lattice is a regular graph, called a Laves lattice [10, 11]. Several authors [7, 14, 15, 18] have considered various percolation models on Archimedean lattices.

The exact bond percolation threshold for the triangular, or (3^6), lattice, $2\sin(\pi/18) \approx .3473$, was conjectured by Sykes and Essam and proved by Wierman [20]. Recently, Wierman [24] considered bond percolation models on three Archimedean lattices – the $(3, 12^2)$ lattice (also named the "extended Kagomé" lattice [2, 9] and the "star" lattice [18]), the $(4, 6, 12)$ lattice (also called the "cross" lattice [18]), and the $(4, 8^2)$ lattice (also called the "square-octagon," "bathroom tile" or "Briarwood" lattice [18]). The dual Laves lattices, which are fully-triangulated, are illustrated in Figures 1, 2, and 3.

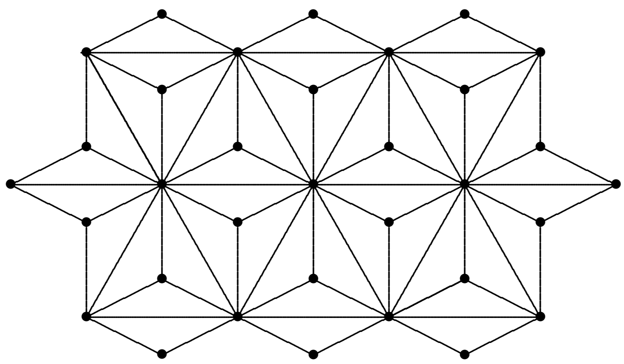

Figure 1: A finite subgraph of the dual of the $(3, 12^2)$ lattice.

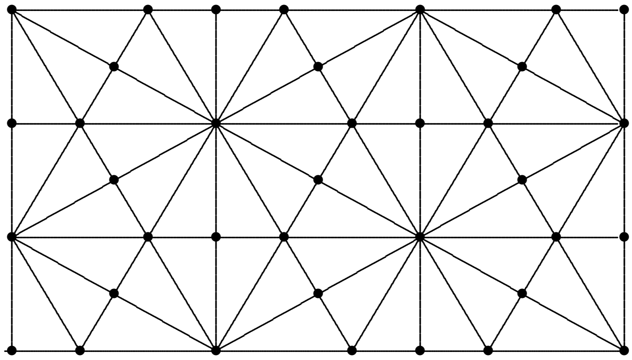

Figure 2: A finite subgraph of the dual of the $(4,6,12)$ lattice.

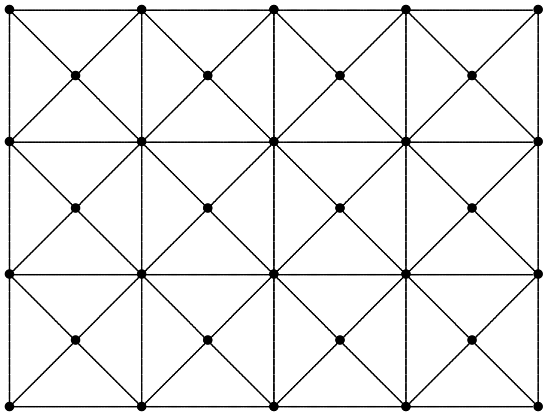

Figure 3: A finite subgraph of the dual of the $(4,8^2)$ lattice.

Rigorous bounds for the bond percolation thresholds of the three Archimedean lattices [24] were obtained by the substitution method [22, 23]:

$$.7385 < p_c((3,12^2)) < .7449,$$
$$.6430 < p_c((4,6,12)) < .7376,$$
$$.6281 < p_c((4,8^2)) < .7201.$$

By Kesten's duality result [8], the bond percolation thresholds of the Archimedean lattice and the dual Laves lattice sum to one. Thus, for the corresponding Laves lattices we have:

$$.2551 < p_c((3,12^2) \text{ dual}) < .2615,$$
$$.2624 < p_c((4,6,12) \text{ dual} < .3570,$$
$$.2799 < p_c((4,8^2) \text{ dual}) < .3719.$$

Combined with the exact solution for the triangular lattice, these results show that bond percolation thresholds of periodic fully-triangulated lattices range at least from .2615 to .3473.

3 Construction

We now construct an infinite sequence of periodic fully-triangulated lattice graphs which have bond percolation thresholds converging to zero. The construction is based on the fact that adding edges and vertices inside a triangular face can lower the threshold of the bond percolation model.

We will construct the sequence by adding edges and vertices to the triangular lattice. Note that the triangular lattice contains triangular faces in two different orientations. In all faces of one orientation, insert k paths of length two between each pair of vertices, to obtain the periodic graph $G(k)$. By inserting additional edges, one obtains a fully-triangulated graph $T(k)$. See Figure 4.

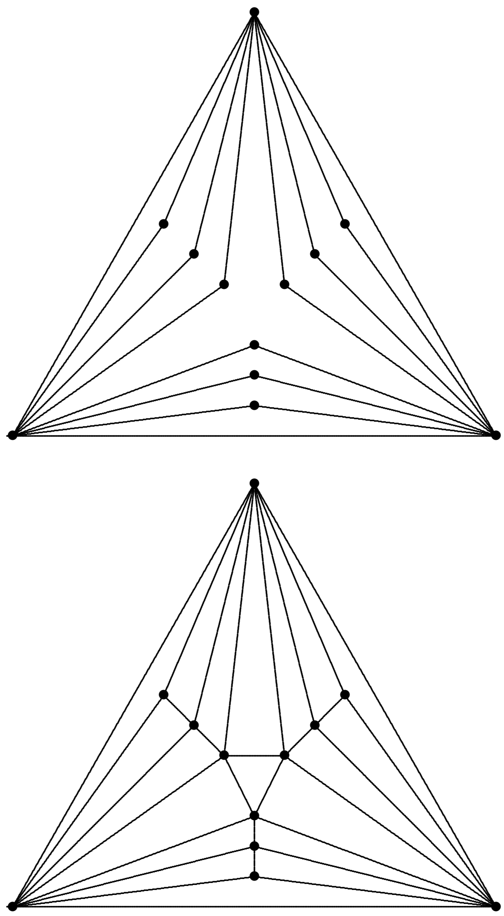

Figure 4: The graph $G(k)$ is constructed by inserting k paths of length two between each pair of vertices in every other face of the triangular lattice. One triangular face of $G(3)$ is shown at the top. Additional edges may be inserted to obtain the subgraph shown at the bottom. Replacing every other face of the triangular lattice with this subgraph produces $T(3)$.

Since $T(k)$ contains $G(k)$, its bond percolation threshold is smaller, by Fisher's containment principle [5].

In the bond percolation model, the probability that there is an open path in $G(k)$ between two vertices of the original triangle that does not pass through the other vertex is

$$1 - (1 - p)(1 - p^2)^k, \tag{1}$$

the probability that at least one of k paths of length two or the original edge of the triangle is open. Thus, the bond percolation model on $G(k)$ is equivalent to a bond percolation model on the triangular lattice with probability $1 - (1 - p)(1 - p^2)^k$ that each edge is open.

If $1 - (1 - p)(1 - p^2)^k > p_c(\text{Triangular})$, then infinite bond percolation open clusters on $G(k)$ will occur when p is the probability of each bond being open, so $p_c(G(k)) < p$. For any fixed $p > 0$, this will happen for k sufficiently large, since $1 - (1 - p)(1 - p^2)^k \longrightarrow 1$ as $k \to \infty$. Therefore, $p_c(G(k))$ converges to zero as $k \to \infty$, which implies that $p_c(T(k))$ converges to zero as $k \to \infty$ also.

Combined with the results of section 2, this shows that bond percolation thresholds of periodic fully-triangulated graphs range (at least) from near zero to $2 \sin(\pi/18) \approx .3473$.

4 Conjecture

We conjecture that the triangular lattice has the largest bond percolation threshold among all periodic fully-triangulated lattices.

From section 2, we have that

$$p_c(\text{Triangular}) > p_c((3, 12^2) \text{ dual}).$$

The bounds do not establish the order of $p_c(\text{Triangular})$ and $p_c((4, 8^2)$ dual), or of $p_c(\text{Triangular})$ and $p_c((4, 6, 12)$ dual). However, early Monte Carlo estimates of $p_c((4, 8^2))$ are .675 by Dean [4] and .684 by Neal [17]. These estimates imply estimates of .325 and .316, respectively, for $p_c((4, 8^2)$ dual). deMagalhães, Tsallis, and Schwachheim [12] used renormalization group methods to obtain an estimate of $.681 \pm .005$ for the $(4, 8^2)$ lattice bond threshold, which is equivalent to an estimate of $.319 \pm .005$ for the $(4, 8^2)$ lattice dual. van der Marck [13] estimated the threshold of the $(4, 8^2)$ lattice dual (named "octagonal" in his paper) as $.3237 \pm .0006$. Together, these bounds and estimates strongly suggest that $p_c(\text{Triangular}) > p_c((4, 8^2)$ dual) also. The bounds for the $(4, 6, 12)$ dual lattice suggest that its threshold is smaller than that of the $(4, 8^2)$ dual lattice, and thus smaller than that of the triangular lattice. We are not aware of any bounds or estimates for bond percolation thresholds which are inconsistent with the conjecture.

A further justification for the conjecture involves a feature of the lattice structure. The triangular lattice is a regular lattice, i.e. all vertices have the same degree. All other fully-triangulated graphs are non-regular. It appears that higher variability of the vertex degrees in these graphs corresponds to lower bond percolation thresholds. If so, the triangular lattice, with no variability of vertex degrees, would be expected to have the largest bond percolation threshold of all fully-triangulated graphs.

To illustrate, we compute the average degree $d(G)$ and variance of the degree $Var(G)$ for each graph G, by weighting each possible value by the proportion of vertices in the graph with degree equal to that value to obtain a probability distribution for the degrees. Using Euler's formula for planar graphs, it is not difficult to show that for any periodic planar graph G,

$$\frac{2}{d(G)} + \frac{2}{d(G \text{ dual})} = 1.$$

Since our graphs are fully-triangulated, $d(G \text{ dual}) = 3$, so $d(G) = 6$, which may also be computed directly in each case. Computing variances produces

$$Var(\text{Triangular}) = 0,$$
$$Var((4, 8^2) \text{ dual}) = 4,$$
$$Var((4, 6, 12) \text{ dual}) = \frac{48}{7},$$
$$Var(T_1) = 12,$$
$$Var((3, 12^2) \text{ dual}) = 18,$$

and

$$Var(T_k) = \frac{36k^2 + 12k + 6}{3k + 1} = 12k + \frac{6}{3k + 1},$$

for $k \geq 2$. Note that the bond percolation threshold decreases as the variance increases, and in particular, that as $k \longrightarrow \infty$, $Var(T_k) \longrightarrow \infty$ while $p_c(T_k) \longrightarrow 0$. This illustration is not meant to claim that the variance is the best measure to use in formulas for estimating the percolation threshold, but to demonstrate that some measure of variability of vertex degrees appears to be correlated with the value of the percolation threshold.

We are not aware of a characterization of all periodic fully-triangulated graphs that would be useful in proving this conjecture. There are certainly infinitely many fully-triangulated graphs, since additional vertices and edges may be inserted in faces of any of the fully-triangulated graphs studied as in the construction in section 3. Graphs constructed in this manner must have smaller bond percolation thresholds than the initial graph, however, by Fisher's containment principle.

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