Desirable Properties of Universal Formulas for Percolation Thresholds

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Abstract

Percolation models are infinite random graph models which exhibit phase transition behavior at critical values called percolation thresholds. Several so-called "universal formulas" that predict approximate values for percolation thresholds of all periodic graphs have been proposed in the physics and engineering literature. The existing universal formulas have been found to have substantial errors in their predictions for some lattices. This paper proposes a set of desirable criteria for universal formulas to satisfy, and begins to investigate which criteria are satisfied by the formulas in the literature.

1 Introduction

1.1 Percolation Models

The bond percolation model is described briefly as follows: Consider an infinite locally-finite connected graph G. Each edge of G is randomly declared to be open (respectively, closed) with probability p (respectively, $1-p$) independently of all other edges, where $0 \leq p \leq 1$. The corresponding parameterized family of product measures on configurations of edges is denoted by P_p. For each vertex $v \in G$, let $C(v)$ be the open cluster containing v, i.e. the (random) connected component of the subgraph of open edges in G containing v. Let $|C(v)|$ denote the number of vertices in $C(v)$. The *critical probability* or *percolation threshold* of the bond percolation model on G, denoted $p_c(G \text{ bond})$, is the unique real number such that

$$p > p_c(G \text{ bond}) \Longrightarrow P_p(\exists\ v \text{ such that } |C(v)| = \infty) > 0$$

and

$$p < p_c(G \text{ bond}) \Longrightarrow P_p(\exists\ v \text{ such that } |C(v)| = \infty) = 0.$$

Similarly, the site percolation model declares each vertex to be open with probability p, and interest focuses on the connected components of the subgraph induced by the open vertices.

Due to the interpretation of the percolation threshold as a phase transition point, percolation models have applications to thermal phase transitions, spontaneous magnetization, gelation processes, and epidemics. See Grimmett [12] for a comprehensive discussion of mathematical percolation theory, Stauffer [25] for a physical science perspective, and Sahimi [22] for engineering science applications.

Since the origins of percolation theory [3], the determination of critical probabilities has been a challenging problem. Exact solutions have been found only for arbitrary trees [17] and a small number of periodic two-dimensional graphs [14, 15, 29, 30]. For other graphs of interest, the problem has been approached by simulation and estimation, e.g. [25, 26], and through rigorous bounds, e.g. [2, 16, 32, 33].

1.2 Universal Formulas

Research on percolation thresholds attempts to understand the dependence of the critical probability upon the detailed structure of the underlying graph. For over 40 years there have been efforts to find a *universal formula*, based on a small number of features of the underlying lattice, for predicting the values of the percolation threshold for all lattice graphs. We provide a few important examples.

Vyssotsky, Gordon, Frisch, and Hammersley [28] studied bond percolation on eight regular two- and three-dimensional lattices. They commented that "p_c appears to be little affected by differences of lattice type if the number of dimensions and coordination number are the same," and mentioned the approximation

$$p_c = \frac{d}{(d-1)q}, \tag{1}$$

where d is the dimension of the lattice and q is the coordination number (or vertex degree) of the lattice.

For site percolation, the formula

$$p_c = \frac{d}{(d-1)(q-1)} \tag{2}$$

was proposed by Sahimi *et al* [23] for $d \geq 3$. An alternative formula for site models,

$$p_c = \frac{1}{\sqrt{q-1}}, \tag{3}$$

proposed by Galam and Mauger [7, 8] obtained good results in two dimensions, but not for higher dimensions.

Galam and Mauger [9, 10] provided estimates for several lattices using the power law formula

$$p_c = p_0[(d-1)(q-1)]^{-a} d^b, \tag{4}$$

where the parameters p_0, a, and b were determined by a fit to several exactly known or precisely estimated percolation threshold values. The lattices studied were classified into three "universality classes". For the two classes of lattices with dimensions $d \leq 7$, $b = 0$ for site percolation and $b = a$ for bond percolation. One of these classes includes the two-dimensional square, triangular, hexagonal, and dice lattices, for which $p_0 = 0.8889$ and $a = 0.3601$ for site models and $p_0 = 0.6558$ and $a = 0.6897$ for bond models. The other class includes the Kagomé lattice and other lattices with $3 \leq d \leq 7$, for which $p_0 = 1.2868$ and $a = 0.6160$ for site percolation and $p_0 = 0.7541$ and $a = 0.9346$ for bond percolation. (The third class, which we will not consider here, consists of lattices in eight or more dimensions.) For the lattices considered in [9], the maximum deviation of the formula from numerical estimates is ± 0.008. However, the universality classes are not precisely defined in terms of properties of the lattice graphs, so it is unclear to which class a new lattice belongs.

Although the formulas of Galam and Mauger are in extremely good agreement with simulation estimates for the lattices studied, some numerical discrepancies have been noted in the past. Van der Marck [18] noted that if there is to be a universal formula for percolation thresholds, it needs to be based on more information than d and q only. As examples, he provides two three-dimensional lattices with $d = 3$ and $q = 8$, the body centered cubic lattice and the stacked triangular lattice. Their site percolation threshold estimates are .246 and .2623 respectively, with bond percolation estimates of .1803 and .1859 respectively. Babaliewski [1] investigated and confirmed a discrepancy of .020 in the estimate for the value of the bond percolation threshold of the ferrovariant of the dodecagonal lattice. However, much larger errors exist: Wierman [36] pointed out that there are errors in the values of these formulas as large as .3434.

1.3 Desirable Properties of Universal Formulas

This study proposes the following collection of desirable properties for universal formulas, as part of the development of a framework for evaluating various universal formulas.

The ideal universal formula for percolation thresholds will:

- Be well-defined.
- Be easily computable.

- Provide values only between 0 and 1.
- Depend only on the adjacency structure of the lattice.
- Be accurate.
- Be consistent with the duality relationship (for bond models) and the corresponding matching[1] relationship (for site models).
- Be consistent with the containment principle.
- Be consistent with the contraction principle (for bond models).
- Be consistent with subdivision of edges (for bond models).

The first three properties are necessary for any reasonable formula. Accuracy of predictions is perhaps the single most important property. The final four ask that certain theoretical properties that have been proved for percolation thresholds hold for the predictive formulas.

In section 2, we explain the meaning and justification of each property. However, we also devote a separate section, Section 3, to a discussion of accuracy, due to its importance. The results of our evaluation are summarized in section 4.

Sections 2 and 3 also contain evaluations of the four universal formulas that were mentioned in section 1.2: the formulas of Vyssotsky-Gordon-Frisch-Hammersley (VGFH) and Galam-Mauger power law (GM-pl) for bond thresholds, and the Galam-Mauger square root formula (GM-sr) and Galam-Mauger power law (GM-pl) for site thresholds. In this initial investigation, we are limiting ourselves to formulas for two-dimensional infinite graphs, since there is little mathematically rigorous knowledge of percolation threshold values in other dimensions. These four formulas were chosen as the most cited in the literature. All four universal formulas studied here are based on the average degree of the infinite graph. For the average degree to be well-defined, and for appropriateness for applications, we restrict consideration to the prediction of percolation thresholds for infinite two-dimensional periodic graphs, where a periodic graph is as defined by Kesten [15, pp. 10-11]: An infinite graph is *periodic* in d dimensions if it is a connected, locally finite, contains no loops, and may be imbedded in $\mathbf{R}^d$ in such a way that each coordinate vector is a period for the image and every compact set of $\mathbf{R}^d$ intersects only finitely many edges. Furthermore, we consider only graphs which have no pendant vertices, as such vertices cannot contribute to percolative behavior.

[1]The matching relationship is different from the concept of matching pairs of vertices in a graph. See Sykes and Essam [27] for a discussion.

2 Discussion of Properties

We now elaborate on each of the desirable properties for percolation thresholds listed in our evaluation framework.

2.1 Well-defined

A universal formula should give a well-defined unique value for every periodic graph. Due to periodicity, the average degree can be can be defined naturally as the limit of the average degrees for a sequence of finite rectangles expanding in both dimensions, or equivalently as the average degree of the graph in a rectangle of one period width in each dimension. Thus, the VGFH and GM-sr formulas are well-defined. However, the GM-pl formulas do not provide a clear definition of the "universality classes." Since it is not clear which formula applies to any particular graph, the GM-pl formulas do not satisfy this property.

2.2 Easily Computable

A universal formula should be a function of graph parameters that are easily computed. The formulas considered in this initial investigation are all functions of the average degree, which is relatively easily determined, as long as the fundamental periodic region contains few vertices.

However, estimates based on a limit of the number of spanning trees in a region (as the region expands) have been proposed [4, 5, 13], and in early percolation studies connections with the connective constant of self-avoiding walks on the lattice were considered. Both of these values are exactly known for few lattices, and reasonably accurate estimates of the unknown values may take substantial computational effort. Dependence upon such parameters seriously limits the usefulness of the universal formula.

2.3 Values in [0,1]

Since the percolation threshold is a probability value, its value for any lattice graph must lie in the interval [0,1]. One would expect this to be trivially satisfied for any proposed universal formula, and it is for the VGFH and GM-sr universal formulas. However, it fails dramatically for GM-pl formulas as the average degree approaches two: For bond thresholds, the limit as $q \to 2$ is 1.0578 for the Class 1 formula and is 1.4414 for the Class 2 formula. For site thresholds, the limit for the Class 2 formula is 1.2868.

2.4 Adjacency Structure

From the definition of the percolation model, the percolation threshold depends only on the adjacency structure of the lattice. All four universal formulas investigated here are functions of the average degree, and thus satisfy this property. However, there is a proposal in the literature to estimate percolation thresholds on the basis of a filling factor, i.e., the portion of the plane that is covered by certain disks centered at the vertices. A universal formula based on a filling factor would depend on the embedding of the lattice into the plane, which is irrelevant for the percolation model and thus the value of the percolation threshold of the lattice.

2.5 Accuracy

There are several difficulties in assessing the accuracy of universal formulas. One cannot determine the precise error that a formula makes except in the few cases that the critical probability is exactly known. Even when the precise error is known for a set of lattices, there are options of comparing different universal formulas on the basis of maximum error, median error, or mean error. Of course, these values will be dependent upon the set of lattice graphs used for the comparison. In the case of the Galam and Mauger power law estimates, there is the additional complication that the universality class of a lattice is not well-defined. We discuss some approaches for dealing with these issues, and carry out an analysis of accuracy, in Section 3.

2.6 Duality and Matching

An important theorem in percolation theory, due to Kesten [15], establishes the relationship of bond percolation thresholds for a pair of dual planar graphs, G and G^*. If the graphs have an axis of symmetry, then

$$p_c(G) + p_c(G^*) = 1.$$

Thus, it is desirable for a universal formula to provide predictions of the bond percolation thresholds $\hat{p}_c(G)$ and $\hat{p}_c(G^*)$ which satisfy

$$\hat{p}_c(G) + \hat{p}_c(G^*) = 1.$$

For site percolation, Kesten's theorem proves that the same relationship holds for percolation thresholds of pairs of matching graphs, which were introduced by Sykes and Essam [27]. While we will not give the rather complicated definition here, note that the line graphs of a pair of dual planar graphs are a pair of matching graphs. It is also desirable that a universal formula for site percolation thresholds be consistent with the matching relationship.

A graph that is isomorphic to its dual graph is called *self-dual*, and similarly we may define *self-matching.* As examples, the square lattice is self-dual and the triangular lattice is self-matching. Kesten's results imply in both cases that the percolation threshold is equal to one-half. Note that if a universal formula fails to be consistent with the duality (or matching) relationship, it is possible that it is still consistent with self-duality (or self-matching).

Three of the formulas under investigation here fail both the duality or matching relationship and the self-duality or self-matching property, since the estimate for the bond percolation threshold for the square lattice and the site percolation threshold for the triangular lattice are not one-half. The GM-sr formula is particularly poor with an estimate of .4472 for the triangular lattice site threshold. The GM-pl formulas have relatively small errors regarding self-duality and self-matching, but have larger errors when considering dual or matching pairs.

The VFGH formula is perfectly consistent with duality, and thus also self-duality. This was shown by Sykes and Essam [27] using Euler's formula together with the one-to-one correspondences between edges in the graph and its dual and between faces and vertices of the two graphs.

2.7 Containment

Fisher's containment principle [6] states that if G is a subgraph of H, then

$$p_c(G) \geq p_c(H)$$

for both bond and site models. Thus, a universal formula should provide estimates satisfying

$$\hat{p}_c(G) \geq \hat{p}_c(H)$$

as well.

We now show that the VGFH and GM-sr universal formulas are not consistent with the containment principle, since they are monotone non-increasing functions of the average degree: If we take a periodic graph G with average degree greater than 2, we can add sufficiently long paths periodically to obtain a graph H with smaller average degree than G. Then, G is a subgraph of H, so $p_c(G) \geq p_c(H)$ by the containment principle, but $q(G) > q(H)$, so the formulas provide estimates $\hat{p}_c(G) < \hat{p}_c(H)$.

The lack of well-definition of the GM-pl formulas complicates the analysis. The argument above is valid for the formula within each class. However, it is possible that there is a graph G which is a subgraph of H where the two graphs are in different classes, so that different formulas apply. Without a specification of the GM classes, we have not constructed an example to show that the GM-pl formulas are inconsistent with the containment principle.

2.8 Contraction

The contraction principle, introduced by Wierman [31], states that if G is obtained by contracting edges in H, then

$$p_c(G) \leq p_c(H)$$

for bond models. Thus, a universal formula should provide estimates satisfying

$$\hat{p}_c(G) \leq \hat{p}_c(H)$$

as well.

We now show that the VGFH and GM-sr universal formulas are not consistent with the contraction principle, since they are monotone nonincreasing functions of the average degree: In Figure 1, we show a graph with average degree 3.75 in which edges can be contracted to obtain a graph (the hexagonal lattice) with average degree 3. Thus, the estimated value for the threshold of the contraction graph is less than the estimate for the threshold of the original graph.

The lack of well-definition of the GM-pl formulas affects this analysis in the same way as for containment.

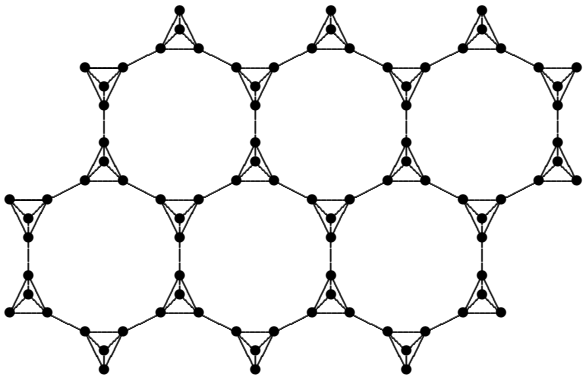

Figure 1: Contraction Example: Contract all six edges in each of the triangles. The original graph has average degree 3.75, while the contracted graph is the hexagonal lattice, which has average degree 3.

2.9 Subdivision

Given a graph G, let G_k denote the graph obtained by subdividing each edge of G into k edges, i.e. by replacing each edge of G by a path of k

edges. For bond percolation

$$p_c(G_k) = [p_c(G)]^{\frac{1}{k}} ,$$

so it is desirable that a universal formula provide estimates satisfying

$$\hat{p}_c(G_k) = [\hat{p}_c(G)]^{\frac{1}{k}} .$$

Consider the subgraph of G contained in a rectangular region, containing n vertices and e edges and thus average degree $q = \frac{2e}{n}$. Then G_k contains $n^* = n + (k-1)e$ vertices and $e^* = ke$ edges, so

$$q(G_k) = \frac{2e^*}{n^*} = \frac{2kq}{2 + (k-1)q}.$$

Taking limits as the rectangular region expands provides the relationship for the average degrees of the infinite graphs.

To investigate consistency of the VGFH formula with subdivision, we compare $[\hat{p}_c(G)]^{\frac{1}{k}}$ with

$$\hat{p}_c(G_k) = \frac{2/q + k - 1}{k}.$$

There are large discrepancies between the two formulas, since as $q \to \infty$, $(\hat{p}_c(G))^{\frac{1}{k}} = \left(\frac{2}{q}\right)^{\frac{1}{k}}$ converges to zero, while $\hat{p}_c(G_k)$ converges to $\frac{k-1}{k}$.

For the GM-pl formula, we compare

$$(\hat{p}_c(G))^{\frac{1}{k}} = p_0 \left(\frac{2}{(q-1)} \right)^{\frac{a}{k}}$$

with

$$\hat{p}_c(G_k) = p_0 \left(\frac{4(k-1)/q + 2}{(2k - 1 - 2(k-1)/q)} \right)^{a},$$

Again, as $q \to \infty$, the first quantity converges to zero, while the second tends a positive constant.

Therefore, neither the VGFH or GM-pl universal formulas provide adequate approximations under subdivision.

3 Accuracy

Our evaluation of the accuracy of the universal formulas has two components: (1) For a selection of graphs, we determine the errors made by each formula, and consider the maximum, median, and average error. (2) We

consider theoretical evidence regarding the maximum errors made by the formulas.

Throughout this discussion, if a lattice graph has been identified by Galam and Mauger as being in a specific class, we provide only the formula value for that class. For other lattices, we provide the formula values for both classes of low-dimensional lattices, and compute errors using the formula closest to the exact value, bounds, or estimates used as the "correct" value. We recognize that this procedure gives an advantage (perhaps significant) to the Galam and Mauger formula.

3.1 Accuracy of Bond Model Formulas

For the evaluation of the bond percolation threshold formulas, we selected nine lattice graphs. The exact critical probability value is known for five of the graphs – the square [14], triangular and hexagonal [29], and bowtie and bowtie dual lattices [30]. The other four graphs are the only lattices in the physical science literature for which the bond percolation thresholds are nearly exactly known [19] [20] [35], i.e., bounded in an interval of length less than 0.01.

Table 1 provides the numerical comparisons of the VGFH and GM-pl bond threshold formulas. The maximum error is smaller for the GM-pl formula than for the VGFH formula: .0598 versus .0728. The median error for the VGFH formula (.0140) is much larger than that of the GM-pl formula (.0054). The average error of the VGFH formula (.0251) is more than twice as large as the GM-pl formula (.0124), mainly since two errors are quite large rather than just one. Note that even allowing the Galam and Mauger estimate to use the formula for the class that gives the closest result, the VGFH formula still has a smaller error for 2 of the 9 lattices in Table 1.

Wierman [34] has shown that there exist graphs with average degree 6 that have bond percolation thresholds arbitrarily close to zero, and by duality, graphs with average degree 3 that have bond percolation thresholds arbitrarily close to one. Thus, the VGFH formula makes errors of at least .3333 and the GM-pl formula makes errors of at least .3206.

Based on this evidence, we consider the accuracy of these formulas to be comparable, and characterize their accuracy as only "Fair."

3.2 Accuracy of Site Model Formulas

There are only three lattices in the physical science literature for which the exact site percolation threshold is known, and there are no nearly exactly solved cases. For this reason, we must use simulation estimates as a standard for evaluating site threshold formulas. We will rely on high

Lattice Name	Value or Bounds	q	VGFH Value	VGFH Error	GM-pl Value	GM-pl Error
$(3, 12^2)$	.7395,.7415	3	.6667	.0728	.6558,.7541	.0126
Hexagonal	.6527	3	.6667	.0140	.6558	.0031
D(Bowtie)	.5955	10/3	.6000	.0045	.5897,.6529	.0058
Kagomé	.5216,.5277	4	.5000	.0216	.5162	.0054
Square	.5000	4	.5000	.0000	.4958	.0042
Dice	.4723,.4784	4	.5000	.0216	.4958	.0174
Bowtie	.4045	5	.4000	.0045	.4066,.3945	.0021
Triangular	.3473	6	.3333	.0140	.3486	.0013
$D(3, 12^2)$	.2585,.2605	6	.3333	.0728	.3486,.3203	.0598

Table 1: Numerical Comparison of Bond Percolation Formulas

precision simulation estimates of the site percolation thresholds of the 11 Archimedean lattices by Suding and Ziff [26].

Table 2 provides the numerical comparisons of the GM-sr and GM-pl site threshold formulas. The GM-sr formula underestimates the percolation threshold for 10 of the 11 graphs. It also has nearly twice as large a maximum error (.1008) than the GM-pl formula (.0552). The median errors are strikingly different: .0502 for GM-sr versus .0048 for GM-pl. The average error of the GM-sr formula (.0485) is almost three times larger than that of the GM-pl formula (.0179).

Based on this evidence, we consider the accuracy of the GM-pl formula to be comparable to that of the bond percolation threshold formulas, and characterize its accuracy as "Fair." We consider the accuracy of the GM-sr formula to be "Poor."

4 Evaluation of Universal Formulas

We summarize our evaluation of the four universal formulas in Table 3. For each of the nine desirable properties, we give our evaluation.

Question marks included on items for the GM-pl formulas indicate that the answer depends on the specification of classes of graphs used to make the formula well-defined. The formulas are computable if it is a simple computation to determine which class a graph is in. The adjacency property will be satisfied if the definition of classes depends only on the adjacency structure. Consistency with the containment and contraction priniciples depends on whether or not the related graphs can be of different classes. As shown in Section 3, the accuracy depends heavily on the class definition,

Lattice Name	Suding-Ziff Estimate	q	GM-sr Value	GM-sr Error	GM-pl Value	GM-pl Error
$(3, 12^2)$	.8079...	3	.7071	.1008	.6926,.8396	.0317
$(4, 6, 12)$	.7478	3	.7071	.0307	.6926,.8396	.0552
$(4, 8^2)$	.7297	3	.7071	.0226	.6926,.8396	.0371
Hexagonal	.6970	3	.7071	.0101	.6926	.0044
Kagomé	.6527...	4	.5774	.0753	.6540	.0013
$(3, 4, 6, 4)$	.6218	4	.5774	.0444	.5985,.6540	.0233
Square	.5937	4	.5774	.0163	.5985	.0048
$(3^4, 6)$	.5795	5	.5000	.0795	.5396,.5478	.0317
$(3^2, 4, 3, 4)$	.5508	5	.5000	.0508	.5396,.5478	.0030
$(3^3, 4^2)$	.5502	5	.5000	.0502	.5396,.5478	.0024
Triangular	.5000...	6	.4472	.0528	.4979	.0021

Table 2: Numerical Comparison of Site Percolation Formulas

since the formula values for the two classes may differ by as much as .1470 for site models and as much as .0983 for bond models.

The rating for duality and matching are to be interpreted as follows: "Yes" indicates that the VGFH has been proved to be consistent with duality. "Fair" indicates that the formula is not consistent with either duality or self-duality (for bond models) or with either matching or self-matching (for site models), but the difference from one of the critical probability sums of appropriate pairs of graphs is relatively small. "Poor" is not consistent, as for "Fair," but with relatively large numerical errors. Although not used here, "Good" would indicate that the formula is not consistent with duality, but is consistent with self-duality, for example.

The evaluation shows that there is considerable room for improvement in universal formulas for percolation thresholds. All the formulas studied fail at least three of the nine criteria, and none is rated better than Fair on accuracy. Although much older, the VGFH satisfies more of the properties than the GM-pl formula for bond thresholds. There is no clear choice between the two site threshold formulas, since the GM-pl formula fails more properties but is more accurate than the GM-sr formula.

5 Future Research Directions

The work described in this article is just a preliminary investigation of a few universal formulas. There are three related directions, described

	Bond Model Formulas		Site Model Formulas	
Property	VGFH	GM-pl	GM-sr	GM-pl
Well-defined	Yes	No	Yes	No
Computable	Yes	Yes?	Yes	Yes?
Values in [0,1]	Yes	No	Yes	No
Adjacency	Yes	Yes?	Yes	Yes?
Accuracy	Fair	Fair?	Poor	Fair?
Duality & Matching	Yes	Fair	Poor	Fair
Containment	No	No?	No	No?
Contraction	No	No?	N.A.	N.A.
Subdivision	No	No	N.A.	N.A.

Table 3: Summary of Evaluations of Universal Formulas

briefly in the following subsections, for future research that are particularly interesting to the authors.

5.1 Other Formulas or Methods

Researchers have considered other means of developing universal formulas for the percolation threshold, based on a minimal spanning tree approach [4, 5, 13], lattice Green functions [23], filling factor [24], and preferred directions for cluster formation [21]. We would like to evaluate these approaches using the framework of this paper.

5.2 Relationship Between Formulas for Bond and Site Models

In [10], Galam and Mauger extended their formula via the use of an effective parameter q_{eff} to replace the average coordination number q. They suggest that their formula has predicting ability for percolation thresholds which have not yet been computed. For example, if the site threshold of a lattice has been estimated, q_{eff} can be computed from the formula for site thresholds, and can be used to predict the bond threshold from the formula for bond thresholds.

The Galam & Mauger extension raises the issue of evaluating the relationship between formulas for the bond threshold and the site threshold. We have already identified some desirable properties:

(1) The universal formulas should be consistent with the bond-to-site transformation, satisfying

$$\hat{p}_c(G \text{ bond }) = \hat{p}_c(L(G) \text{ site }),$$

where $L(G)$ denotes the line graph (also called the covering graph in the physical science literature) of G. This equality holds for percolation thresholds by an early result of Fisher [6].

(2) For every graph, the universal formulas should satisfy

$$\hat{p}_c(G \text{ bond }) \leq \hat{p}_c(G \text{ site }).$$

Hammersley [11] proved that this holds for percolation thresholds of every infinite graph.

(3) The universal formulas should not imply that two graphs must have their bond percolation and site percolation thresholds in the same order. While this was commonly believed for many years, Wierman [38] provided examples of pairs of graphs with the site percolation thresholds in the opposite order than their bond percolation thresholds.

5.3 Developing Improved Universal Formulas

The ultimate goal of this research program is to develop improved universal formulas. This is a challenging problem, since it is clear from our analysis that the existing universal formulas leave much to be desired.

A way must be found to incorporate other information besides the dimension and the average degree of the lattice. This can be seen in Table 2, since graphs with the same degree have different percolation thresholds. For example, the four Archimedean lattices which are regular with degree 3 have estimated site percolation thresholds ranging from .6970 to .8079.

The major challenges are to identify other features of the graphs that play a role in determining the percolation threshold, and to formulate improved universal formulas which satisfy more of the desirable properties than the existing universal formulas.

As one possibility, Wierman and Vahidi [37] suggested that the variability of the vertex degree has an effect on the percolation threshold, with higher variability leading to lower critical probability values. For planar lattice bond percolation models, the variability of degree of the dual lattice corresponds to the variability of the number of sides of faces in the original lattice, so characteristics of the faces may play a role. Formulation of these ideas will necessarily involve investigation of alternative measures of variability of degree and numbers of edges of faces.

An interesting idea is to employ the existing formulas in developing improved formulas. For example, since the VGFH formula satisfies more properties than the others, we plan to study the conversion of the VGFH formula via the bond-to-site transformation to derive a site percolation threshold formula which might outperform the GM-sr and GM-pl formulas.

Acknowledgement: The authors gratefully acknowledge financial support from the Johns Hopkins University's Acheson J. Duncan Fund for the Advancement of Research in Statistics.

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