Ordering Bond Percolation Critical Probabilities

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1 Introduction

Since the origins of percolation theory in the 1950s, the determination of critical probabilities has been an important and challenging problem. To date, exact solutions have been found only for arbitrary trees [10] and a small number of periodic two-dimensional graphs [6, 7, 15, 16]. For other graphs of interest, the problem has been approached by simulation and estimation, and through rigorous bounds. An additional goal of these lines of research is to understand the dependence of the critical probability upon the detailed structure of the underlying graph and possibly to find accurate approximation formulae based on graph properties.

The bond percolation model is described as follows. Consider an infinite locally-finite connected graph G. Each edge of G is randomly declared to be open (respectively, closed) with probability p (respectively, $1-p$) independently of all other edges, where $0 \leq p \leq 1$. The corresponding parameterized family of product measures on configurations of edges is denoted by P_p. For each vertex $v \in G$, let $C(v)$ be the open cluster containing v, i.e. the connected component of the subgraph of open edges in G containing v. Let $|C(v)|$ denote the number of vertices in $C(v)$. The critical probability of the bond percolation model on G, denoted $p_c(G \text{ bond})$, is the unique real number such that

$$p > p_c(G \text{ bond}) \Longrightarrow P_p(\exists\, v \text{ such that } |C(v)| = \infty) > 0$$

and

$$p < p_c(G \text{ bond}) \Longrightarrow P_p(\exists\, v \text{ such that } |C(v)| = \infty) = 0.$$

See Grimmett [3] and Hughes [5] for comprehensive discussions of mathematical percolation theory, Stauffer [14] for a physical science perspective, and Sahimi [13] for engineering science applications.

Motivated by applications to bond percolation, site percolation, self-avoiding walks, and first-passage percolation, Parviainen and Wierman [12] determined the subgraph partial order for the Archimedean and Laves lattices. A *regular tiling* is a tiling of the plane which consists entirely of regular polygons. An *Archimedean lattice* is the graph of vertices and edges of a regular tiling which is vertex-transitive, i.e., for every pair of vertices, u and v, there is a graph isomorphism that maps u to v. A proof that there are exactly 11 Archimedean lattices is given in Grünbaum and Shephard [4, Ch. 2].

A notation for Archimedean lattices, which can also serve as a prescription for constructing them, is given in Grünbaum and Shephard: Around any vertex (since all are equivalent, by vertex-transitivity), starting with the smallest polygon touching the vertex, list the number of edges of the successive polygons around the vertex. For convenience, an exponent is used to indicate that a number of successive polygons have the same size.

Since the Archimedean lattices are planar graphs, each has a planar dual graph. The duals of the Archimedean lattices have applications in crystallography, where they are called Laves lattices [8, 9]. The square lattice, (4^4), is self-dual, and the triangular, (3^6), and hexagonal, (6^3), lattices are a dual pair of graphs. The other 8 Archimedean lattices have dual graphs that are not Archimedean. Thus, there are a total of 19 different Archimedean and Laves lattices. We will denote the dual of an Archimedean lattice G by $D(G)$.

Fisher's containment principle [2] states that if G is a subgraph of H, then $p_c(H) \leq p_c(G)$, so the subgraph partial ordering of the Archimedean and Laves lattices establishes inequalities between the bond percolation critical probabilities of these lattice graphs. However, there are two additional graphs with exact critical probabilities known, the bowtie lattice and its dual graph. In Section 2, we incorporate the bowtie lattice and its dual into the subgraph partial order, in order to gain more information about the values of bond percolation critical probabilities. Note that by results of Aizenman and Grimmett [1], each of the resulting inequalities are actually strict inequalities.

In Section 3, we combine the subgraph partial order results with information about bond percolation critical probabilities obtained by other means, such as exact solutions, the substitution method, and Kesten's duality theorem.

The inequalities between bond percolation thresholds are illustrated by a diagram in Figure 1. In the diagram, if ovals are connected by a line segment, the lattice in the higher oval has a larger bond percolation threshold than the lattice in the lower oval. Inequalities that are implied by combinations of other inequalities are not represented by line segments in the diagram, since they are easily seen by sequences of line segments. Such a diagram, called a *Hasse diagram*, is a convenient way to visualize a partially ordered set. However, we do not claim that our class of lattices is partially ordered by bond percolation critical probabilities, since they may not satisfy the anti-symmetry condition: It is possible that two different lattices may have equal bond percolation critical probability values. In Figure 1, on the left side we have drawn a vertical chain of lattices for which the exact critical probability is known or for which very accurate bounds have been established, then related the other lattices to these

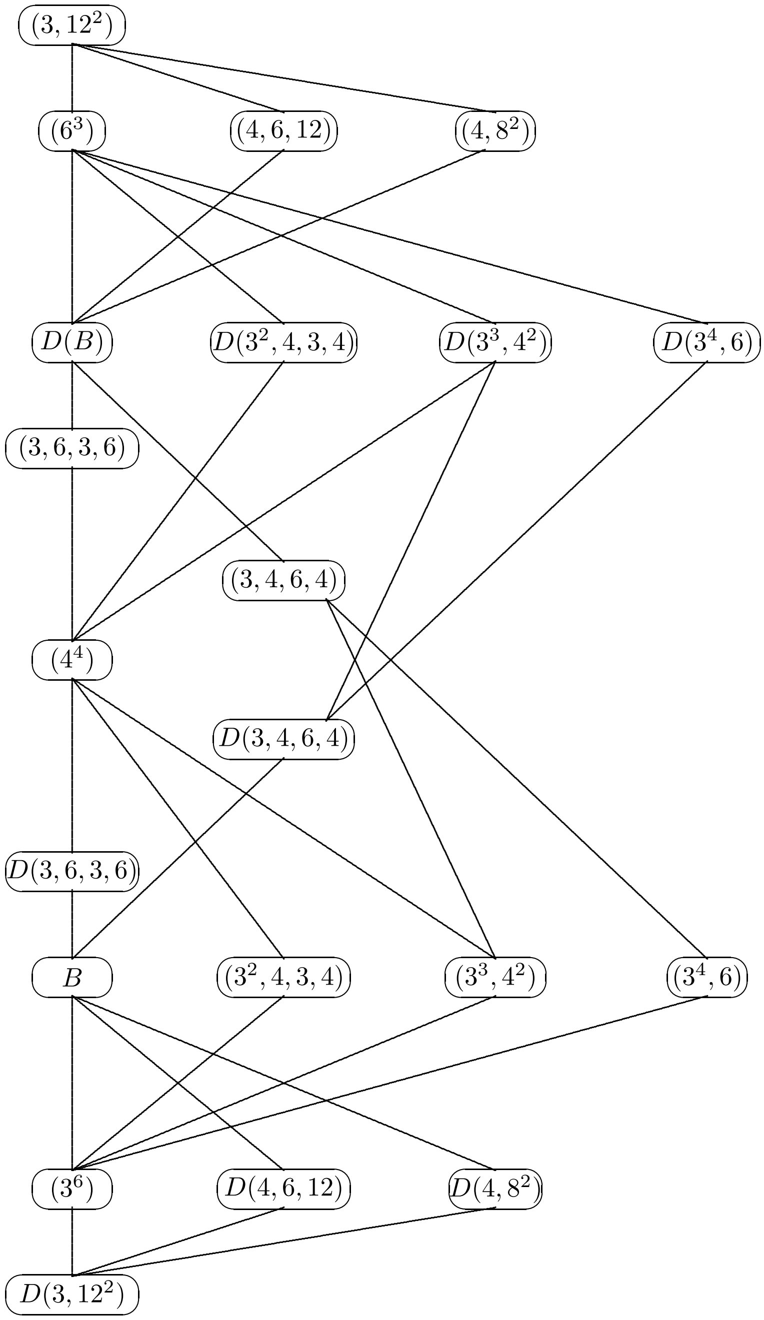

Figure 1: A diagram describing current knowledge of the bond percolation threshold order of the Archimedean and Laves lattices. Edges of the diagram indicate covering relationships, in which the lattice higher in the diagram has a larger percolation threshold than the lattice lower in the diagram. Additional subgraph relationships, valid by transitivity, are implied, but not shown.

reference lattices.

A collection of the current knowledge of numerical values and bounds for bond percolation critical probabilities is given in Table 1. Note that exact bond percolation critical values are known for only five of the lattices: the hexagonal (6^3), bowtie dual $D(B)$, square (4^4), bowtie B, and triangular (3^6). Nearly-exact values, within .01, are known for four others: the $(3, 12^2)$, Kagomé $(3, 6, 3, 6)$, and their duals lattices. On the other hand, the least accurate bounds, separated by .2483, are for the $(3, 4, 6, 4)$ lattice and its dual. The authors are not aware of any previous non-trivial bounds for the $(3, 4, 6, 4)$, $(3^4, 6)$, $(3^3, 4^2)$, and $(3^2, 4, 3, 4)$ lattices and their duals, although it has been possible to calculate crude bounds based on the vertex degree or numbers of self-avoiding walks.

Lattice	Bounds
$(3, 12^2)$	(7393, .7418)
$(4, 8^2)$	(.6281, .7201)
$(4, 6, 12)$	(.6430, .7376)
(6^3)	$\approx$ .652704
$D(B)$	$\approx$ .595482
$D(3^2, 4, 3, 4)$	(.5000, .6528)
$D(3^3, 4^2)$	(.5000, .6528)
$D(3^4, 6)$	(.4045, .6528)
$(3, 6, 3, 6)$	(.5209, .5291)
$(3, 4, 6, 4)$	(.3472, .5955)
(4^4)	=.500000
$D(3, 4, 6, 4)$	(.4045, .6528)
$D(3, 6, 3, 6)$	(.4709, .4791)
$(3^4, 6)$	(.3472, .5000)
$(3^3, 4^2)$	(.3472, .5000)
$(3^2, 4, 3, 4)$	(.3472, .5000)
B	$\approx$ .404518
(3^6)	$\approx$ .347296
$D(4, 6, 12)$	(.2624, .3570)
$D(4, 8^2)$	(.2799, .3719)
$D(3, 12^2)$	(.2582, .2607)

Table 1: Numerical bounds for bond percolation thresolds of the Archimedean and Laves lattices, the bowtie lattice and its dual lattice. Exacly solved cases are given rounded to six digits. Bounds are indicated by an interval, with values are rounded up or down to four digits for upper and lower bounds, respectively.

2 The Bowtie Lattice and Its Dual

In this section, we determine all inclusion relationships between the bowtie lattice and the Archimedean and Laves lattices and between the dual of the bowtie lattice and the Archimedean and Laves lattices. For each relevant pair of lattices, we either prove that one contains the other as a subgraph or prove that neither contains the other as a subgraph. A compilation of these results is given in Table 2.

	B		$D(B)$	
Lattice	$\subseteq$	$\supseteq$	$\subseteq$	$\supseteq$
$(3, 12^2)$	T	Φ	ϕ	Φ
$(4, 6, 12)$	T	Φ	S	Φ
$(4, 8^2)$	T	Φ	S	Φ
(6^3)	T	Φ	S	ϕ
$D(3^2, 4, 3, 4)$	S	Φ	Φ	ϕ
$D(3^3, 4^2)$	S	Φ	Φ	ϕ
$D(3^4, 6)$	S	Φ	Δ	ϕ
$(3, 6, 3, 6)$	+	Φ	ϕ	C
$(3, 4, 6, 4)$	S	Φ	ϕ	+
(4^4)	+	ϕ	Φ	+
$D(3, 4, 6, 4)$	S	ϕ	Δ	S
$D(3, 6, 3, 6)$	S	ϕ	Δ	S
$(3^4, 6)$	A	Δ	Δ	T
$(3^3, 4^2)$	A	Δ	Δ	T
$(3^2, 4, 3, 4)$	S	Δ	Δ	T
(3^6)	Φ	+	Δ	T
$D(4, 8^2)$	Δ	+	Δ	T
$D(4, 6, 12)$	Δ	V	Δ	S
$D(3, 12^2)$	Δ	T	Δ	T
B			Δ	+

+	Inclusion
T	Transitivity
ϕ	Minimum polygon size
Φ	Maximum polygon size
C	Combining polygons
Δ	Maximum degree
V	Variation in degree
A	Adjacent polygons
S	Special case

Table 2: All inclusions and non-inclusions between B or $D(B)$ and the Archimedean or Laves lattices. For each lattice listed at the top, there are two columns. In the left column, each entry indicates if the lattice listed at the top is included in the lattice listed at the left, and if not, indicates the reasoning that proves non-inclusion. The right column indicates if the lattice listed at the top includes the lattice at the left, and, if not, provides the reason. The key at the right provides an interpretation for each symbol used in the table. A "+" indicates an inclusion which is demonstrated in the figures in Subsection 2.1. A "T" indicates an inclusion which is valid by transitivity. An "S" indicates that non-inclusion is proved in a special argument given in Subsection 2.3. All other symbols refer to a lemma or method in Subsection 2.2 for proving non-inclusion.

2.1 Inclusion Proofs

The graphs shown in Figure 2 demonstrate the following inclusions between the bowtie lattice or its dual lattice and the set of Archimedean and Laves lattices:

$$(4^4) \subseteq B,$$
$$(3,6,3,6) \subseteq B,$$
$$B \subseteq (3^6),$$
$$B \subseteq D(4,8^2),$$
$$D(B) \subseteq B,$$
$$D(B) \subseteq (4^4),$$
$$D(B) \subseteq (3,4,6,4).$$

The following sequences show the inclusions by transitivity, indicated by "T" in Table 2.

$$(3,12^2) \subseteq (3,6,3,6) \subseteq B$$
$$(4,6,12) \subseteq (4^4) \subseteq B$$
$$(4,8^2) \subseteq (4^4) \subseteq B$$
$$(6^3) \subseteq (4^4) \subseteq B$$
$$B \subseteq (3^6) \subseteq D(3,12^2)$$
$$D(B) \subseteq (3,4,6,4) \subseteq (3^4,6)$$
$$D(B) \subseteq (4^4) \subseteq (3^3,4^2)$$
$$D(B) \subseteq (4^4) \subseteq (3^2,4,3,4)$$
$$D(B) \subseteq B \subseteq (3^6)$$
$$D(B) \subseteq (4^4) \subseteq D(4,8^2)$$
$$D(B) \subseteq (3^6) \subseteq D(3,12^2)$$

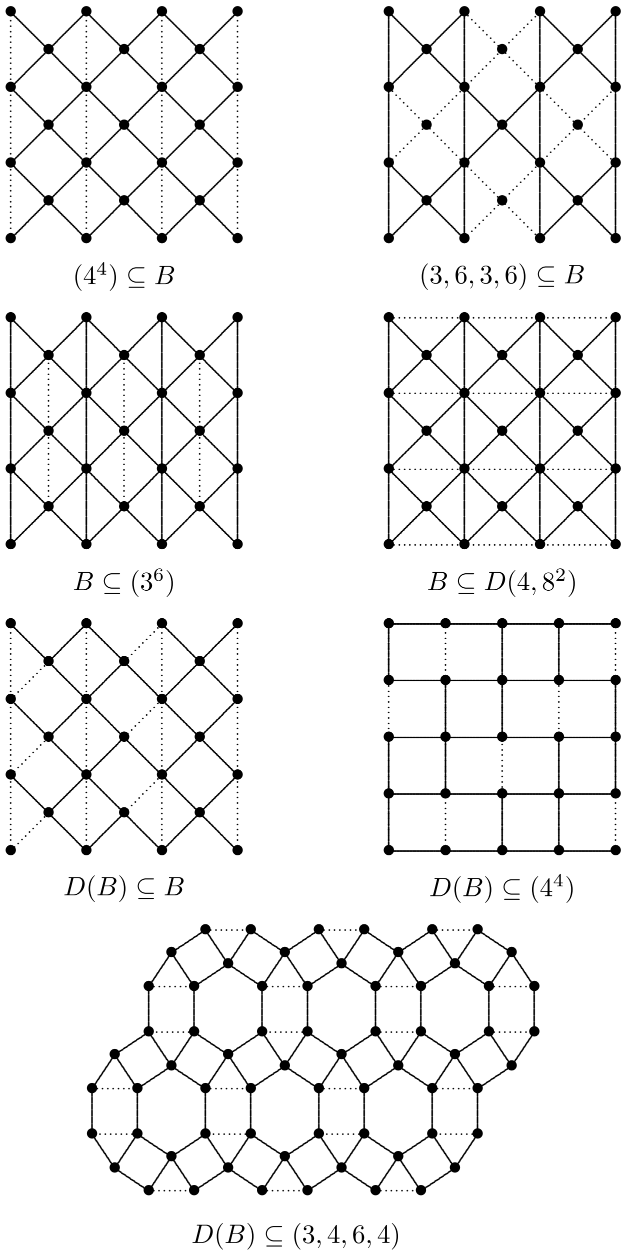

Figure 2: Inclusions between the bowtie lattice B, its dual lattice D(B), and the Archimedean and Laves lattices. Solid lines indicate edges of the subgraph, and dotted lines indicate edges of the supergraph that are not in the subgraph.

2.2 Basic Non-Inclusion Proofs

We now recall several conditions, discussed in [12], under which one lattice cannot be a subgraph of another. These allow us to prove the majority of the non-inclusion results involving the bowtie lattice and its dual, although there are still a number of special cases which require individualized reasoning, which are discussed in subsection 2.3.

Let $\mathcal{S}$ denote the set of graphs consisting of the Archimedean lattices, the Laves lattices, the bowtie lattice and the dual of the bowtie lattice.

We use the symbol $\subseteq$ to denote the inclusion relationship, letting $G \subseteq H$ denote that G is isomorphic to a subgraph of H. In many cases we will write as if H is a subgraph of G, rather than a separate graph on a different set of vertices. To create subgraphs of a given graph, we will delete vertices and edges. When we refer to deleting a vertex, we mean that the vertex and all edges incident to it are deleted from the graph.

For any graph G, let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. Let $d_G(v)$ denote the degree of vertex v in graph G. A vertex with degree n will be called an n-vertex.

For a graph G, denote the maximum degree by $\Delta(G) = \max_{v \in G} d_G(v)$ and the minimum degree by $\delta(G) = \min_{v \in G} d_G(v)$.

Since each $G \in \mathcal{S}$ is a 3-connected planar graph, it has a unique dual graph G^*. Consequently, any plane representation of G determines a set of faces of G, denoted $F(G)$, and each vertex $v \in V(G^*)$ corresponds to a unique face $f \in F(G)$.

We will refer to a face as a polygon, to the number of sides of a polygon as the *size* of the polygon, and to a polygon of size k as a k-gon. For a graph G, we denote the maximum polygon size by $\Phi(G)$ and the miminum polygon size by $\phi(G)$. Similarly, we will refer to cycle of length k as a k-cycle. The term polygon is only used for cycles with empty interior.

We will say that two polygonal faces are *i*ncident if they share a common vertex but have no common edge.

We will say that two polygonal faces are adjacent if they share a common edge. Let $A(G)$ denote the maximum number of $\phi(G)$-gons that are adjacent to a $\phi(G)$-gon in G.

Similarly to the notation for Archimedean lattices, we will say a vertex is of type $(a_1, a_2, \ldots)$ if successive faces around the vertex are size a_1, a_2, etc. A polygon is said to be adjacent to each of its vertices. An edge of a polygon is said to be adjacent to the polygon.

Polygon sizes in lattices we are considering satisfy a monotonicity property: Let H be constructed from $G \in \mathcal{S}$ by deleting a set of vertices and edges. If F_H denotes the face of H containing the face F in G, then the size of F_H is greater than or equal to the size of F. [Since the Archimedean and Laves lattices are all periodic, one only needs to check a sufficiently large bounded region to verify this property.] Note that deleting a set of edges which includes an edge of a face need not strictly increase the size of the face: A triangular face may be obtained by deleting edges in the $(3, 12^2)$ lattice.

The non-inclusions in Table 2 are establihsed using the following criteria:

Lemma Δ: If $\Delta(H) > \Delta(G)$, then $H \not\subseteq G$.

Lemma ϕ: Let $G, H \in \mathcal{S}$. If $\phi(H) < \phi(G)$, then $H \not\subseteq G$.

Lemma Φ: Let $G, H \in \mathcal{S}$. If $\Phi(H) < \Phi(G)$, then $H \not\subseteq G$.

Lemma A: Let $G, H \in \mathcal{S}$, $G \neq D(3, 12^2)$. If $\phi(H) = \phi(G)$ and $A(H) > A(G)$, then $H \not\subseteq G$.

Lemma C : Let $G, H \in \mathcal{S}$. Suppose H contains k-gons and G does not. If deleting edges from any n-gons in G with $n < k$ produces only n-gons with $n > k$, then $H \not\subseteq G$.

Approach V: For several pairs of lattices, there is a large disparity between the maximum degrees, so that lowering the large degree in one lattice creates faces of a larger size than exist in the other lattice.

We finish this subsection with the one individual argument using Approach V that we need, to show that $B \not\subseteq D(4, 6, 12)$: To obtain B by deletions from $D(4, 6, 12)$, each 12-vertex in $D(4, 6, 12)$ must either be deleted or have its degree lowered to at most 6. Deleting the vertex creates an n-gon with $n \geq 12$, which cannot exist in B, so the degree must be lowered by deleting at least 6 incident edges. However, deleting more than 2 edges creates either an n-gon with $n \geq 5$ or more than 2 4-gons incident to the vertex.

2.3 Non-Inclusion Proofs for Special Cases

2.3.1 $D(3^2, 4, 3, 4) \not\subseteq B$, $D(3^3, 4^2) \not\subseteq B$, and $D(3^4, 6) \not\subseteq B$.

All faces in $D(3^2, 4, 3, 4)$, $D(3^3, 4^2)$, and $D(3^4, 6)$ are 5-gons. A 5-gon can be obtained by deletion in B only by deleting an edge that is adjacent to a 3-gon and a 4-gon. There are 3 other 3-gons adjacent to the 4-gon, each of which must combine with a 4-gon that is incident to the original 4-gon. For one of these 3-gons, the resulting 5-gon must share 2 edges with the original 5-gon, which does not occur in $D(3^2, 4, 3, 4)$, $D(3^3, 4^2)$, or $D(3^4, 6)$. See Figure 3.

2.3.2 $(3, 4, 6, 4) \not\subseteq B$.

In $(3, 4, 6, 4)$, every 3-gon is adjacent to 3 4-gons. In B, every 3-gon is adjacent to 2 4-gons and one 3-gon, which can be enlarged only to a 5-gon or larger. Thus, deletions in B cannot produce a 3-gon which is adjacent to 3 4-gons.

2.3.3 $D(3, 4, 6, 4) \not\subseteq B$ and $D(3, 6, 3, 6) \not\subseteq B$.

In $D(3, 4, 6, 4)$, all faces are 4-gons, each of which is incident to 5 others. In $D(3, 6, 3, 6)$, all faces are 4-gons, each of which is incident to 6 others. In B, deletion of an edge of a 4-gon would create an n-gon with $n > 4$, so only the

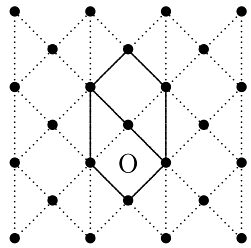

Figure 3: The original 4-gon is indicated by O. It is extended to become a 5-gon. The adjacent 3-gon A must combine with the incident 4-gon B, which produces a 5-gon which shares 2 edges with the 5-gon containing O.

edges between adjacent 3-gons may be deleted. This produces the square lattice, in which each 4-gon is incident to 4 others. Thus, deletions in B cannot produce a graph with all faces being 4-gons, each incident to 5 others or 6 others.

2.3.4 $(3^2, 4, 3, 4) \not\subseteq B$.

Both lattices have only 3-gons and 4-gons as faces. Deletion of an edge from B would create an n-gon with $n \geq 5$, so no edges may be deleted, so it suffices to show that the lattices are not isomorphic. However, B has a vertex of type $(3^2, 4, 3^2, 4)$, while $(3^2, 4, 3, 4)$ does not.

2.3.5 $(4, 6, 12) \not\subseteq D(B)$.

Deletion of any edge of $D(B)$ creates a polygon which shares two edges with another polygon, which does not occur in $(4, 6, 12)$. Thus, these two shared edges must also be deleted. Whether the first edge deleted is between 2 6-gons or between a 6-gon and a 4-gon, this process leads to a 10-gon. (See Figure 4.) The 10-gon can be enlarged to a 12-gon only by deleting an edge from an adjacent 4-gon. However, this creates a 12-gon which shares 2 edges with another polygon.

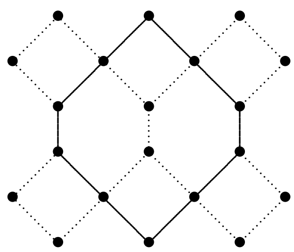

Figure 4: The 10-gon that is obtained by deleting edges from $D(B)$.

2.3.6 $(4, 8^2) \not\subseteq D(B)$.

Since $(4, 8^2)$ contains no 6-gons, at least one edge must be deleted from each 6-gon in $D(B)$. Since such a deletion must not create an n-gon with $n > 8$, the deleted edge must be adjacent to a 6-gon and a 4-gon. This process creates 8-gons which are adjacent to at most 3 4-gons, while each 8-gon in $(4, 8^2)$ is adjacent to 4 4-gons.

2.3.7 $(6^3) \not\subseteq D(B)$.

Since (6^3) contains no 4-gons, at least one edge must be deleted from each 4-gon in $D(B)$. However, deleting an edge from a 4-gon creates an 8-gon or larger, so (6^3) cannot be obtained by deletions from $D(B)$.

2.3.8 $D(B) \not\subseteq D(3, 4, 6, 4)$.

In $D(B)$, the degree sequence of any 4-cycle is 3,4,3,4. In $D(3, 4, 6, 4)$, the degree sequence of any 4-cycle is 3,4,6,4. Thus, to obtain $D(B)$ from $D(3, 4, 6, 4)$ by deletions, 3 edges must be deleted from some vertices of degree 6 in $D(3, 4, 6, 4)$. Deleting 2 or 3 consecutive edges creates an n-gon with $n \geq 8$, which does not exist in $D(B)$. Deleting every other edge creates 3 6-gons which share a common vertex, which does not occur in $D(B)$.

2.3.9 $D(B) \not\subseteq D(3, 6, 3, 6)$.

$D(B)$ contains a pair of adjacent 6-gons which are both adjacent to two 4-gons, where no 2 of the polygons share more than one edge. By deleting edges in $D(3, 6, 3, 6)$, two adjacent 6-gons must be isomorphic to those shown in Figure 5, but there can be only one 4-gon adjacent to both 6-gons.

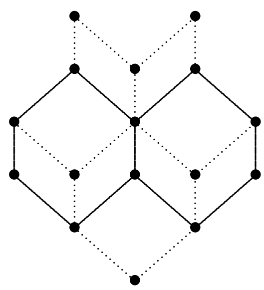

Figure 5: Two adjacent 6-gons in $D(3, 6, 3, 6)$, with only one 4-gon that is adjacent to both.

2.3.10 $D(B) \not\subseteq D(4, 6, 12)$.

In $D(B)$, the smallest polygons are 4-gons, which have degree sequence 3,4,3,4. In $D(4, 6, 12)$, only 3 types of 4-gons can be created, as illustrated in Figure 6.

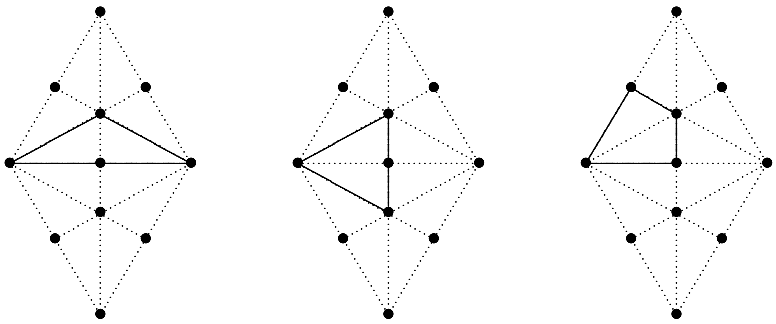

Figure 6: In $D(4,6,12)$, only three types of 4-gons can be created.

At least one vertex of each 4-gon was a 12-vertex in $D(4,6,12)$.

In Case 1, for the 4-gon to have the correct degree sequence, the 12-vertex must have its degree reduced to 4. Since the 4-gon includes 2 incident edges, all but 2 other edges must be deleted. If no n-gon with $n > 6$ is to be created, at least 2 6-gons and a 5-gon (which must be enlarged to a 6-gon) must be created. However, these share a common vertex, which cannot occur in $D(B)$.

In Case 2, the 12-vertex must have its degree reduced to 3, which implies that an n-gon with $n \geq 7$ must be created.

In Case 3, there are two possibilities. If the 12-vertex has its degree reduced to 3, an n-gon with $n \geq 7$ must be created, as in Case 2. If the 12-vertex has its degree reduced to 4, it must become of type 4,6,4,6 in the subgraph created. However, the 6-gons created cannot have degree sequence 3,3,4,3,3,4 as required in $D(B)$, since a neighboring vertex (v in Figure 7) can have degree at most 3.

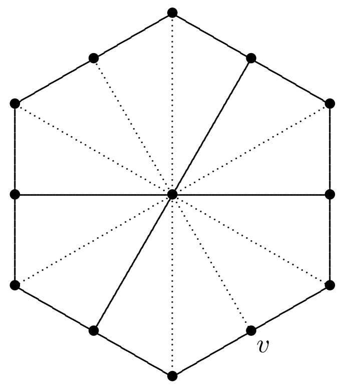

Figure 7: The central vertex is a 12-vertex reduced to degree 4. Vertex v can have degree at most 3.

3 Combination with Other Methods

As mentioned earlier, the inclusion and non-inclusion results above were intended to supplement the information gained from the investigation of the subgraph partial ordering of the Archimedean and Laves lattices by Parviainen and Wierman. The subgraph ordering inequalities provide a starting point, but, by themselves, do not produce any numerical bounds. In many cases, a lattice does not have a subgraph or supergraph in the class, so the ordering cannot provide an upper bound or lower bound respectively. In the following subsections we will recall and utilize other methods to assemble the most accurate bounds known for these lattices. In this section, we combine the Parviainen and Wierman results and the results in Section 2 with previous results, including the few exact percolation threshold solutions, Kesten's duality theorem, and the substitution method. In the course of this section, the numerical values in Table 1 are justified.

3.1 Exact Solutions and Duality

The first periodic lattice to be solved exactly for the bond percolation critical probability was the square lattice (4^4). Due to its self-duality property, its bond percolation threshold is one-half [6].

The triangular and hexagonal lattice bond percolation critical probabilities were determined in 1981 [15], using the star-triangle transformation and the fact that they are dual lattices. The critical probability of the triangular lattice is the solution of the equation $1 - 3p - p^3 = 0$, which is exactly $2\sin(\pi/18)$, while the hexagonal lattice threshold is $1 - 2\sin(\pi/18)$, leading to the approximations in Table 1.

Also as a consequence of duality and the star-triangle transformation, the bowtie lattice and its dual were exactly solved in 1984 [16]. The bond percolation threshold of the bowtie lattice is the solution of $1 - p - 6p^2 + 6p^3 - p^5 = 0$ in the interval [0,1], while $p_c(D(B)) = 1 - p_c(B)$, producing the approximations given in Table 1.

A key to the exact solutions for bond percolation thresholds mentioned above is the use of duality. An important theorem of Kesten [7] states that the bond percolation critical probabilities of a dual pair of lattices sum to one.

3.2 Substitution Method Bounds

The substitution method determines critical probability bounds using stochastic ordering and coupling methods. Recent advances in computational techniques for the substitution method have led to near-exact bounds for two lattices:

$$.520938 < p_c(3,6,3,6) < .529095 \qquad [19]$$

$$.739399 < p_c(3,12^2) < .741757 \qquad [11],$$

which are rounded appropriately to give the four-digit bounds in Table 1.

Although not as accurate, the best known bounds for bond percolation on the $(4,8^2)$ and $(4,6,12)$ were also derived by the substitution method [21]:

$$.643068 < p_c(4,6,12) < .737550.$$

$$.6281893 < p_c(4,8^2) < .720011.$$

By Kesten's duality theorem, these bounds translate into bounds for the duals of the four lattices.

3.3 The Subgraph Partial Order

Together, the results discussed in Subsections 3.1 and 3.2 establish the ordering of critical probabilities of the nine lattices in the vertical sequence at the left side of Figure 1, as well as the relationships of the $(4,6,12)$, $(4,8^2)$, and their dual lattices' critical probabilities to the nine lattices.

To justify the remaining inequalities represented in Figure 1, we rely on the subgraph partial ordering of Archimedean and Laves lattices [12] and the extension to include the bowtie and its dual from Section 2. The Hasse diagram for the subgraph partial ordering of Archimedean and Laves lattices is shown in Figure 8.

The upper bound for the $(3,4,6,4)$ lattice bond percolation threshold follows from the inclusion

$$D(B) \subseteq (3,4,6,4)$$

in Subsection 2.1. The upper bound for the $(3^4,6)$ lattice threshold follows from the previous inclusion and

$$(3,4,6,4) \subseteq (3^4,6),$$

and bounds for the $(3^2,4,3,4)$ and $(3^3,4^2)$ lattices are consequences of the inclusions

$$(4^4) \subseteq (3^2,4,3,4) \subseteq (3^6)$$

and

$$(4^4) \subseteq (3^3,4^2) \subseteq (3^6).$$

Applying the duality theorem produces all the remaining bounds in Table 1.

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References

[1] M. Aizenman and G. Grimmett (1991) Strict monotonicity for critical points in percolation and ferromagnetic models. *Journal of Statistical Physics 63,* 817-835.

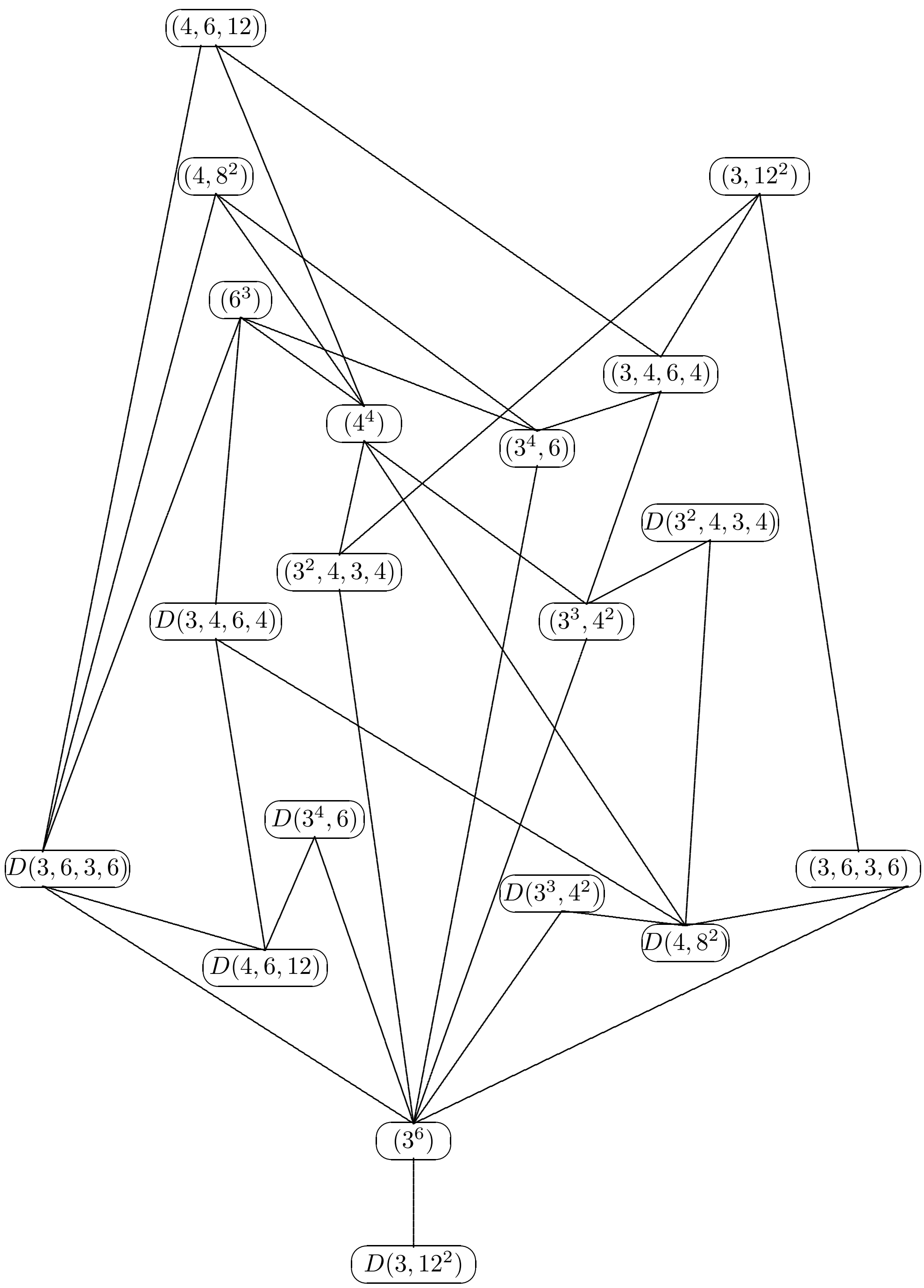

Figure 8: The Hasse diagram of the subgraph order of the Archimedean and Laves lattices. Edges of the diagram indicate covering relationships, in which the lattice higher in the diagram is a subgraph of the lattice lower in the diagram. Additional subgraph relationships, valid by transitivity, are implied, but not shown.

[2] M. E. Fisher (1961) Critical probabilities for cluster size and percolation problems. *Journal of Mathematical Physics 2,* 620-627.

[3] G. Grimmett (1999) *Percolation,* Springer.

[4] Branko Grünbaum and G. C. Shephard (1987) *Patterns and Tilings,* W. H. Freeman.

[5] B. D. Hughes (????) *Random Walks and Random Environments, Volume 2: Random Environments*, Oxford Science Publications.

[6] Harry Kesten (1980) The critical probability of bond percolation on the square lattice equals 1/2. *Commun. Math. Phys. 74*, 41-59.

[7] Harry Kesten (1982) *Percolation Theory for Mathematicians,* Birkhäuser.

[8] Fritz Laves (1930) Die bau-zusammenhänge innerhalb der kristallstrukturen. *Zeitschrift für Kristallographie, Kristallgeometrie, Kristallphysik, Kristallchemie 78,* I. 202-265, II. 275-324.

[9] F. Laves (1931) Ebenenteilung und koordinationszahl. *Zeitschrift für Kristallographie, Kristallgeometrie, Kristallphysik, Kristallchemie 78,* 208-241.

[10] Russell Lyons, Random walks and percolation on trees, *Annals of Probability 18* (1990), 931-958.

[11] William D. May and John C. Wierman (2002) Using symmetry to improve percolation threshold bounds, in preparation.

[12] Robert Parviainen and John C. Wierman (2002) The subgraph partial ordering of Archimedean and Laves lattices, Uppsala University Department of Mathematics Report 2002:13.

[13] Muhammad Sahimi (1994) *Applications of Percolation Theory,* Taylor & Francis.

[14] Dietrich Stauffer and Amnon Aharony (1991) *Introduction to Percolation Theory,* Taylor & Francis.

[15] John C. Wierman (1981) Bond percolation on the honeycomb and triangular lattices. *Advances in Applied Probability 13,* 298-313.

[16] John C. Wierman (1984) A bond percolation critical probability determination based on the star-triangle transformation. *Journal of Physics A: Mathematical and General 17,* 1525-1530.

[17] John C. Wierman (1995) Substitution method critical probability bounds for the square lattice site percolation model. *Combinatorics, Probability, and Computing 4,* 181-188.

[18] John C. Wierman (2001) An improved upper bound for the hexagonal lattice site percolation critical probability. *Combinatorics, Probability and Computing,* to appear.

[19] John C. Wierman (2001) Upper and lower bounds for the Kagomé lattice bond percolation critical probability. *Combinatorics, Probability and Computing,* to appear.

[20] John C. Wierman (2001) Site percolation critical probability bounds for the $(4, 8^2)$ and $(4, 6, 12)$ lattices, *Congressus Numerantium 150*, 117-128.

[21] John C. Wierman (2002) Bond percolation critical probability bounds for three Archimedean lattices, *Random Structures and Algorithms,* to appear.