Bond Percolation Critical Probability Bounds for Three Archimedean Lattices

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Abstract

Rigorous bounds for the bond percolation critical probability are determined for three Archimedean lattices:

$$.7385 < p_c((3, 12^2) \text{ bond}) < .7449,$$

$$.6430 < p_c((4, 6, 12) \text{ bond}) < .7376,$$

$$.6281 < p_c((4, 8^2) \text{ bond}) < .7201.$$

Consequently, the bond percolation critical probability of the $(3, 12^2)$ lattice is strictly larger than those of the other ten Archimedean lattices. Thus, the $(3, 12^2)$ bond percolation critical probability is possibly the largest of any vertex-transitive graph with bond percolation critical probability that is strictly less than one.

1 Introduction

Since the origins of percolation theory [3], the determination of critical probabilities has been a challenging problem. Exact solutions have been found only for arbitrary trees [16] and a small number of periodic two-dimensional graphs [10, 11, 24, 25]. For other graphs of interest, the problem has been approached by simulation and estimation, e.g. [21, 22], and through rigorous bounds, e.g. [2, 15, 27, 28]. A goal of these lines of research is to understand the dependence of the critical probability upon the detailed structure of the underlying graph and possibly to find accurate approximation formulae based on graph properties. See, e.g. [22] and references therein.

The bond percolation model is described as follows. Consider an infinite locally-finite connected graph G. Each edge of G is randomly declared to be open (respectively, closed) with probability p (respectively, $1 - p$)

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independently of all other edges, where $0 \leq p \leq 1$. The corresponding parameterized family of product measures on configurations of edges is denoted by P_p. For each vertex $v \in G$, let $C(v)$ be the open cluster containing v, i.e. the connected component of the subgraph of open edges in G containing v. Let $|C(v)|$ denote the number of vertices in $C(v)$. The critical probability of the bond percolation model on G, denoted $p_c(G \text{ bond})$, is the unique real number such that

$$p > p_c(G \text{ bond}) \Longrightarrow P_p(\exists\ v \text{ such that } |C(v)| = \infty) > 0$$

and

$$p < p_c(G \text{ bond}) \Longrightarrow P_p(\exists\ v \text{ such that } |C(v)| = \infty) = 0.$$

See Grimmett [7] for a comprehensive discussion of mathematical percolation theory, Stauffer [21] for a physical science perspective, and Sahimi [20] for engineering science applications.

Archimedean lattices are vertex-transitive graphs with a planar representation that is a tiling of the plane by regular polygons. There are exactly 11 Archimedean lattices [8], which are illustrated in [22]. We denote each Archimedean lattice by a sequence of integers $(n_1^{a_1}, n_2^{a_2}, ...)$, where the n_i denote the number of sides of successive faces as one moves around a single vertex, and the a_i give the number of successive faces of the same size. The dual graph of an Archimedean lattice is called a Laves lattice [13, 14]. Several authors [6, 17, 18, 22] have considered various percolation models on Archimedean lattices.

This paper considers bond percolation models on three Archimedean lattices. The $(3, 12^2)$ lattice, also named the "extended Kagomé" lattice [1, 12] and the "star" lattice [22], is illustrated in Figure 1. The $(4, 6, 12)$ lattice, also called the "cross" lattice [22], is illustrated in Figure 2. The $(4, 8^2)$ lattice, also called the "square-octagon," "bathroom tile" or "Briarwood" lattice [22], is shown in Figure 3.

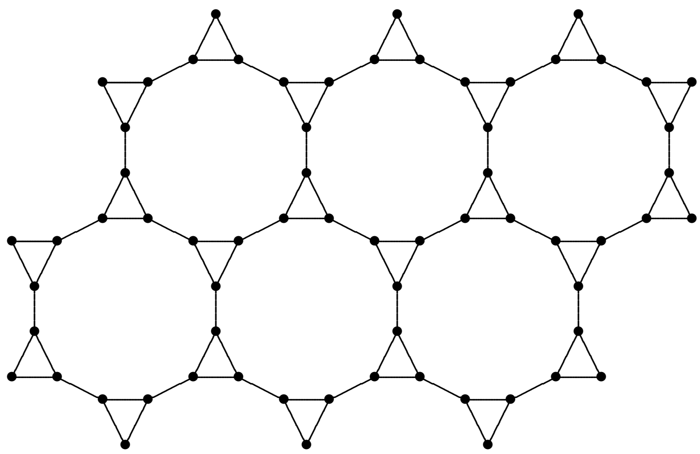

FIGURE 1: A portion of the $(3, 12^2)$ lattice, also known as the "extended Kagomé lattice.

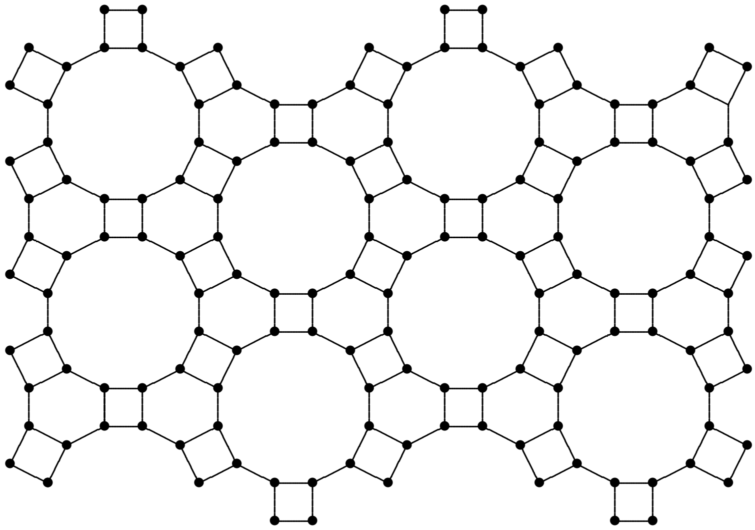

FIGURE 2: A portion of the $(4, 6, 12)$ lattice, also known as the “cross” lattice.

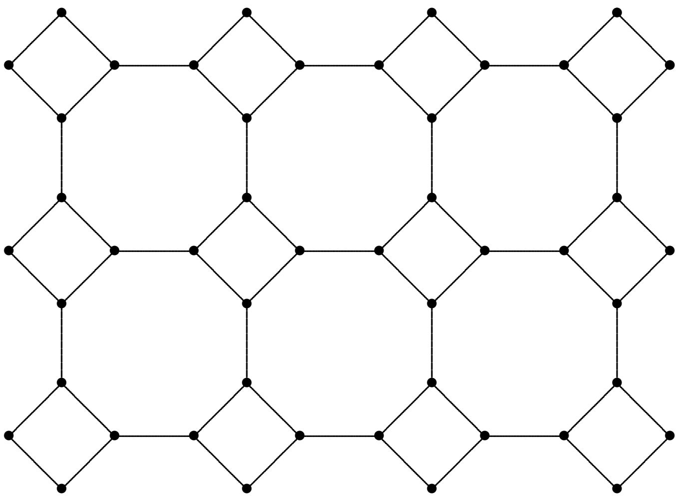

FIGURE 3: A portion of the $(4, 8^2)$ lattice, also known as the “square-octagon,” “bathroom tile,” or “Briarwood” lattice.

The site percolation critical probability of the $(3, 12^2)$ lattice is exactly known, with value $\sqrt{1 - 2\sin(\pi/18)} \approx .807900764$, because it is the line graph of the subdivided hexagonal lattice [19, 22]. Since the bond percolation critical probability is smaller that the site percolation critical probability for any graph [5], this provides an upper bound:

$$p_c((3,12^2) \text{ bond}) \leq .80791.$$

Based on a derivation of the exact phase diagram of the nearest-neighbor q-state Potts ferromagnet in the fully anisotropic $(3,12^2)$ lattice, Tsallis [23] conjectured that the bond percolation critical probability of the $(3,12^2)$ lattice is exactly .739830.... We prove accurate bounds for the bond percolation critical probability, which are consistent with the Tsallis conjecture, in section 2:

Theorem 1: $.7385 \leq p_c((3,12^2) \text{ bond}) \leq .7449.$

We are not aware of any estimates or non-trivial bounds for the $(4,6,12)$ lattice bond percolation critical probability. Suding and Ziff [22] estimated its site percolation critical probability as .747806. Since the site percolation critical probability is an upper bound for the bond percolation critical probability, their estimate is consistent with our bounds, which will be proved in section 3:

Theorem 2: $.6430 < p_c((4,6,12) \text{ bond}) < .7376.$

Wierman [26] proved that

$$.6281 < p_c((4,8^2) \text{ bond}) < .7288.$$

Suding and Ziff [22] report an estimate of the site percolation critical probability of the line graph (also called "covering graph") of the $(4,8^2)$ lattice, which is equal to the bond percolation critical probability of the $(4,8^2)$ lattice, of .6768. This estimate is consistent with the improved bounds:

Theorem 3: $.6281 < p_c((4,8^2) \text{ bond}) < .7201.$

The bounds are derived by the substitution method, which has produced the most accurate bounds for several other critical probabilities [27, 28]. The reader is referred to these papers for justification and examples of use of the method. With future improvements in computational efficiency, the bounds in Theorems 1 – 3 may be improved.

The bounds are relevant to a conjecture of Häggström [9] regarding bond percolation critical probabilities of vertex-transitive graphs. While some vertex-transitive graphs have bond percolation critical probability equal to one, it is conjectured that there exists $B < 1$ such that every vertex-transitive graph has critical probability equal to one or less than or equal to B. Häggström asked which vertex-transitive graph has the optimal value B. Natural candidates are the hexagonal lattice, the $(4,8^2)$ lattice, the $(4,6,12)$, and the $(3,12^2)$ lattice. The hexagonal lattice is exactly solved [24], with

$$p_c(\text{Hexagonal bond}) = 1 - 2\sin(\pi/18) \approx .6527.$$

Combining Theorems 1 – 3, we have

$$p_c((3,12^2) \text{ bond}) > p_c((4,6,12) \text{ bond}),$$

$$p_c((3,12^2) \text{ bond}) > p_c(\text{Hexagonal bond}),$$

and

$$p_c((3,12^2) \text{ bond}) > p_c((4,8^2) \text{ bond}).$$

Combined with elementary bounds for the bond percolation critical probabilities of the other Archimedean lattices, in section 4 we conclude that the $(3,12^2)$ lattice has the largest bond percolation critical probability of all the Archimedean lattices. Unfortunately, however, the bounds do not determine any inequalities among the $(4,8^2)$ lattice, $(4,6,12)$ lattice, and hexagonal lattice critical probabilities.

In the calculations which follow, the computed decimal values given as solutions were rounded up or down, as appropriate, to provide upper or lower bounds in the final results.

2 Bounds for the $(3,12^2)$ Lattice

To apply the substitution method, we must decompose the $(3,12^2)$ lattice into isomorphic edge-disjoint subgraphs, and substitute alternative subgraphs in order to obtain another lattice which is exactly solved. To facilitate this, we first subdivide each edge between two triangles in the $(3,12^2)$ lattice, i.e. replace it by two edges in series. To maintain equivalence of the percolation models, each of the new edges is open with probability $\sqrt{p}$. The lattice may then be decomposed into subgraphs consisting of a triangle with three incident edges. Substituting three-stars for these subgraphs produces a subdivided hexagonal lattice. See Figure 4.

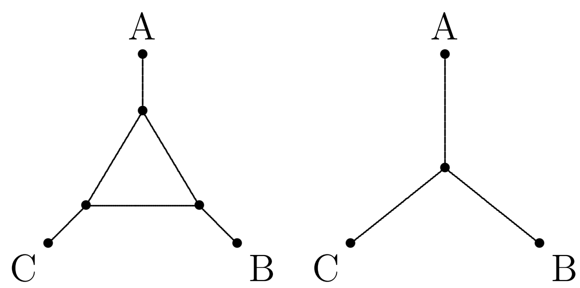

FIGURE 4: The substitutions used in the proof of Theorem 1. The $(3,12^2)$ is decomposed into copies of the subgraph shown on the left, while the subdivided hexagonal lattice is decomposed into copies of the subgraph shown on the right.

To compare probabilities of open connections on the two lattices, we compute probabilities of partitions of the vertices on the boundary of the subgraph. A partition is denoted by a sequence of vertices and vertical bars, where vertices not separated by a vertical bar are in the same cluster. For the $(3, 12^2)$ lattice model with parameter p, we have

$$P_p(ABC) = p^{3/2}(3p^2 - 2p^3),$$

$$P_p(AB|C) = P_p(A|BC) = P_p(AC|B) = p^2 + p^3 - 3p^{7/2} - p^4 + 2p^{9/2},$$

and

$$P_p(A|B|C) = 1 - 3p^2 - 3p^3 + 6p^{7/2} + 3p^4 - 4p^{9/2},$$

while for the hexagonal lattice with parameter q, using Q_q to denote the probability measure, we have

$$Q_q(ABC) = q^3,$$

$$Q_q(AB|C) = Q_q(A|BC) = Q_q(AC|B) = q^2(1 - q),$$

and

$$Q_q(A|B|C) = q^3 + 3q^2(1 - q).$$

The set of partitions, ordered by refinement, form a partially ordered set: α is a refinement of β if every cluster of α is contained in a cluster of β. ABC is the maximum element and $A|B|C$ is the minimum element, and, for example, $AB|C$ is a refinement of ABC. An upset U is a set of partitions such that if α is a refinement of β and $\alpha \in U$ then $\beta \in U$. The probability of an upset is the sum of the probabilities of the partitions in the upset.

By the substitution method, if q is set equal to the critical probability of the subdivided hexagonal lattice, i.e.

$$q = \sqrt{1 - 2\sin(\pi/18)} \approx .8079,$$

then upper and lower bounds for $p_c((3, 12^2)$ bond) are the largest and smallest (respectively) solutions for p of the equations

$$P_p(U) = Q_q(U)$$

for any nontrivial upset U.

Thus, we need only solve four equations:

$$p^{3/2}(3p^2 - 2p^3) + i(p^2 + p^3 - 3p^{7/2} - p^4 + 2p^{9/2}) = q^3 + iq^2(1 - q)$$

for $i = 0, 1, 2, 3$. The largest solution, for $i = 3$, is .7448997, and the smallest solution, for $i = 0$, is .738598, establishing Theorem 1.

3 Bounds for the $(4, 6, 12)$ and $(4, 8^2)$ Lattices

Fortunately, a common substitution region works for the $(4, 6, 12)$ and the $(4, 8^2)$ lattices.

To decompose the $(4, 6, 12)$ lattice into isomorphic edge-disjoint subgraphs, we replace each edge that is on the boundary of two twelve-sided faces by two edges in series. To maintain the equivalence of the bond percolation models, each new edge is open with probability $b = \sqrt{p}$, while original edges are open with probability b^2. The $(4, 6, 12)$ lattice can then be decomposed in subgraphs consisting of a square with four incident edges. Substituting four-stars for these subgraphs produces a subdivided Kagomé lattice. See Figure 5.

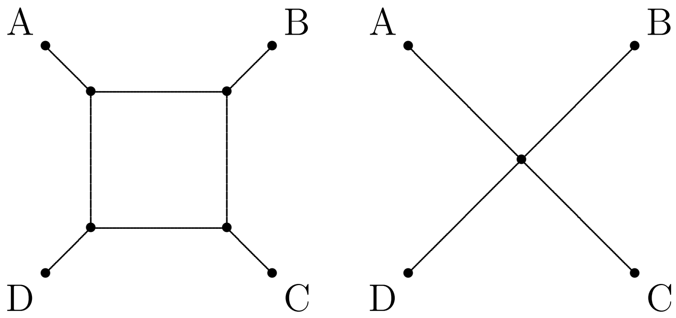

FIGURE 5: Substitutions used in the proofs of Theorems 2 and 3. The $(4, 8^2)$ and $(4, 6, 12)$ lattices are decomposed into subgraphs isomorphic to that shown on the left, while the subdivided graphs are decomposed into subgraphs isomorphic to that shown on the right.

Similarly, to decompose the $(4, 8^2)$ lattice into isomorphic edge-disjoint subgraphs, replace each edge that is on the border of two octagonal faces by two edges in series. Again, each new edge is open with probability $b = \sqrt{p}$, while original edges are with probability b^2. The $(4, 8^2)$ lattice then decomposes into subgraphs consisting of a square with four incident edges. Substituting four-stars for these subgraphs produces a subdivided square lattice.

Let P_b denote the probability measure on partitions of the boundary vertices in the $(4, 6, 12)$ and $(4, 8^2)$ lattice subgraphs, and calculate the partition probabilities:

$$P_b(ABCD) = 4b^{10} - 3b^{12}.$$

$$P_b(ABC|D) = P_b(ABD|C) = P_b(ACD|B) = P_b(BCD|A) =$$
$$= b^7 + 2b^9 - 4b^{10} - 2b^{11} + 3b^{12}.$$

$$P_b(AB|CD) = P_b(AD|BC) = b^8 - 2b^{10} + b^{12}.$$

$$P_b(AB|C|D) = P_b(BC|A|D) = P_b(CD|A|B) = P_b(AD|B|C) =$$
$$= b^4 - 2b^7 - 4b^9 + 5b^{10} + 4b^{11} - 4b^{12}.$$

$$P_b(AC|B|D) = P_b(BD|A|C) = 2b^6 - 2b^7 - 4b^9 + 3b^{10} + 4b^{11} - 3b^{12}.$$

$$P_b(A|B|C|D) = 1 - 4b^4 - 4b^6 + 8b^7 - 2b^8 + 16b^9 - 10b^{10} - 16b^{11} + 11b^{12}.$$

Let Q_q denote the probability measure on partitions of the boundary vertices of the four-star, and calculate the partition probability functions:

$$Q_q(ABCD) = q^4.$$

$$Q_q(ABC|D) = Q_q(ABD|C) = Q_q(ACD|B) = Q_q(BCD|A) = q^3 - q^4.$$

$$Q_q(AB|CD) = Q_q(AD|BC) = 0.$$

$$Q_q(AB|C|D) = Q_q(BC|A|D) = Q_q(CD|A|B) = Q_q(AD|B|C) =$$
$$= q^2 - 2q^3 + q^4.$$

$$Q_q(AC|B|D) = Q_q(BD|A|C) = q^2 - 2q^3 + q^4.$$

$$Q_q(A|B|C|D) = 1 - 6q^2 + 8q^3 - 3q^4.$$

As in the proof of Theorem 1, we set q equal to the bond percolation critical probability of the corresponding lattice. The critical probability of the subdivided Kagomé lattice is $\sqrt{p_c(\text{Kagomé bond})}$. To derive bounds for the $(4, 6, 12)$ lattice, we would like to set $q = \sqrt{p_c(\text{Kagomé bond})}$, and find the maximal and minimal solutions of the upset probability equations

$$P_b(U) = Q_q(U).$$

However, the Kagomé lattice bond model is not exactly solved, although very accurate bounds,

$$.5182 < p_c(\text{Kagomé bond}) < .5335,$$

have been proved [27], and recently [29] improved to

$$.5209 < p_c(\text{Kagomé bond}) < .5291.$$

Thus, for calculating the lower bound we set $q = \sqrt{.5209}$, and for calculating the upper bound we set $q = \sqrt{.5291}$.

Note that the maximum partition $ABCD$ must be in every proper upset, and the minimum partition $A|B|C|D$ cannot be in any proper subset. The remaining partitions that have positive probability in either probability measure fall into four groups – two of size two, and two of size four. Within each group, all partitions have the same probability function in each of the two models. By considering the number of partitions in each group that may be included in an upset, we see that all upset equations have the form

$$4b^{10} - 3b^{12} + i(b^7 + 2b^9 - 4b^{10} - 2b^{11} + 3b^{12}) + j(b^8 - 2b^{10} + b^{12})$$

$$+m(2b^6-2b^7-4b^9+3b^{10}+4b^{11}-3b^{12})+k(b^4-2b^7-4b^9+5b^{10}+4b^{11}-4b^{12})$$
$$= q^4 + i(q^3 - q^4) + (m+k)(q^2 - 2q^3 + q^4),$$

for all $i, k = 0, 1, 2, 3, 4$ and $j, m = 0, 1, 2$. These 225 equations cover all possible subsets of the partitions, including some which are not upsets. However, if the maximal and minimal solutions do not correspond to an upset, they can be eliminated until the maximal and minimal solutions corresponding to upsets are found.

For the $(4, 6, 12)$ lattice, the maximal solution for an upset is .858807, which provides an upper bound of .737550 for the bond model. The corresponding upset arises when $i = 4$, $j = 0$, $k = 0$, and $m = 2$. The minimal solution for an upset is .801928, which provides a lower bound of .643088. It corresponds $i = 0$, $j = 2$, $k = 0$, and $m = 0$.

Since the square lattice bond percolation critical probability is $1/2$, the bond percolation critical probability of the subdivided square lattice is $1/\sqrt{2}$. As above, we set $q = 1/\sqrt{2} \approx .7071067$ and find the maximal and minimal solutions of the upset probability equations. The upsets which give the maximal and minimal solutions are the same as for the $(4, 6, 12)$ lattice. The maximal solution is .8485341, which produces an upper bound of .720011, while the minimal solution is .792584, corresponding to a lower bound of .6281893. This lower bound of .6281 does not improve the previous lower bound.

Remark: The computational approach above is sufficient to prove Theorems 2 – 3. However, there is a more efficient way to verify that the upsets given are optimal, i.e. they correspond to the largest and smallest solutions among all upset probability equations. We will briefly illustrate the approach for verifying the correctness of the upper bounds given above.

Let $U^* = \{ ABCD, ABC|D, ABD|C, ABD|C, ACD|B, BCD|A, AC|B|D, BD|A|C \}$ denote the candidate for the optimal upset. Let s^* be the solution of its upset probability equation, i.e.

$$P_{s^*}(U^*) = Q_{q_0}(U^*),$$

where $q_0 = 1/\sqrt{2}$ or $\sqrt{.5291}$ for the $(4, 8^2)$ and $(4, 6, 12)$ lattices respectively.

Suppose U is any other nontrivial upset. Denote set difference by $A \setminus B = A \cap B^c$, where B^c is the complement of B. Then

$$U = \{U^* \cup (U \setminus U^*)\} \setminus (U^* \setminus U),$$

so

$$P_{s^*}(U) = P_{s^*}(U^*) + P_{s^*}(U \setminus U^*) - P_{s^*}(U^* \setminus U),$$

and

$$Q_{q_0}(U) = Q_{q_0}(U^*) + Q_{q_0}(U \setminus U^*) - Q_{q_0}(U^* \setminus U).$$

Since $ABCD$ is the maximum element, it must be in every upset, so $ABCD$ is not in $U^* \setminus U$. For each $\pi \in U^* \setminus U$, $\pi \neq ABCD$, check that

$$P_{s^*}(\pi) \leq Q_{q_0}(\pi),$$

and by summing over such π,

$$P_{s^*}(U^* \setminus U) \le Q_{q_0}(U^* \setminus U).$$

On the other hand, $U \setminus U^*$ consists of partitions from only two classes, $\{\ AB|C|D,\ BC|A|D,\ CD|A|B,\ AD|B|C\ \}$ and $\{\ AB|CD,\ AD|BC\ \}$. Since U is an upset, inclusion of partitions from the first group implies inclusion of the covering partitions in the second group. As above, use k and j, respectively, for the numbers of partitions from those two groups that are in $U - U^*$. Check all possible combinations for (k, j): (0,1), (0,2), (1,1), (1,2), (2,1), (2,2), (3,2), and (4,2), to find that in each case the P_{s^*} probability is greater than or equal to the Q_{q_0} probability. Thus,

$$P_{s^*}(U \setminus U^*) \ge Q_{q_0}(U \setminus U^*).$$

Combining these equations and inequalities, we have that

$$\begin{aligned} P_{s^*}(U) &= P_{s^*}(U^*) + P_{s^*}(U \setminus U^*) - P_{s^*}(U^* \setminus U) \\ &\ge Q_{q_0}(U^*) + Q_{q_0}(U \setminus U^*) - Q_{q_0}(U^* \setminus U) \\ &= Q_{q_0}(U). \end{aligned}$$

Since $P_s(U)$ is nondecreasing in s for any fixed upset U, the solution to the upset probability equation for U must be less than or equal to s^*. Since U is an arbitrary upset, s^* is the largest solution, and thus, the upper bound.

4 Upper Bounds for Other Archimedean Lattices

The bounds that are proved in the previous sections establish that

$$p_c((3, 12^2) \text{ bond}) > p_c((4, 6, 12) \text{ bond}) > p_c(\text{Hexagonal bond})$$

and

$$p_c((3, 12^2) \text{ bond}) > p_c((4, 8^2) \text{ bond}).$$

The exact solutions for the square and triangular lattices show that

$$p_c((3, 12^2) \text{ bond}) > .5000 = p_c(\text{Square bond})$$

and

$$p_c((3, 12^2) \text{ bond} > .3473 = p_c(\text{Triangular bond}),$$

and bounds of Wierman [29] give

$$p_c((3, 12^2) \text{ bond} > .5291 > p_c(\text{Kagomé bond}).$$

To establish that the $(3, 12^2)$ bond model has the largest critical probability of all Archimedean lattice bond models, we must consider the remaining four Archimedean lattices. The $(3^3, 4^2)$ and $(3^2, 4, 3, 4)$ lattices both contain the square lattice, so by Fisher's containment principle [4],

$$p_c(\text{Square bond}) \ge p_c((3^3, 4^2) \text{ bond})$$

and

$$p_c(\text{Square bond}) \geq p_c((3^2, 4, 3, 4) \text{ bond}).$$

The containment principle also proves that

$$p_c((3,4,6,4) \text{ bond}) \geq p_c((3^4, 6) \text{ bond}),$$

so it remains to prove an appropriate upper bound for the $(3,4,6,4)$ lattice.

We will use the fact that, by Kesten's duality result ([11], Chapter 3),

$$p_c((3,4,6,4) \text{ bond}) = 1 - p_c((3,4,6,4) \text{ dual bond})$$

and find a lower bound on the critical probability of the dual lattice using self-avoiding walk counts. (The $(3,4,6,4)$ dual lattice is shown in Figure 6.) To find an appropriate bound, we first count three step self-avoiding walks. Starting from vertices of degree 3, 4, and 6, respectively, there are 26, 45, and 45 three-step self-avoiding walks. By considering an initial step, then blocks of three steps, we compute an upper bound on the number of self-avoiding walks of length $3n+1$ starting from any fixed vertex to be $6(45)^n \leq 6(3.56)^{3n}$. The expected number of open self-avoiding paths of length $3n+1$ starting from a vertex v is then at most $6(3.56)^{3n}p^{3n+1}$, which converges to zero as $n \to \infty$ if $p < \frac{1}{3.56}$. Thus, the probability that v is in an infinite open component is zero if $p < .2809$. Since there are countably many vertices in the lattice, the probability that there is an infinite open component in the lattice is also zero for such p. Therefore, $p_c((3,4,6,4) \text{ dual bond}) > .2809$, so

$$p_c((3,4,6,4) \text{ bond}) < .7191 < p_c((3, 12^2) \text{ bond}).$$

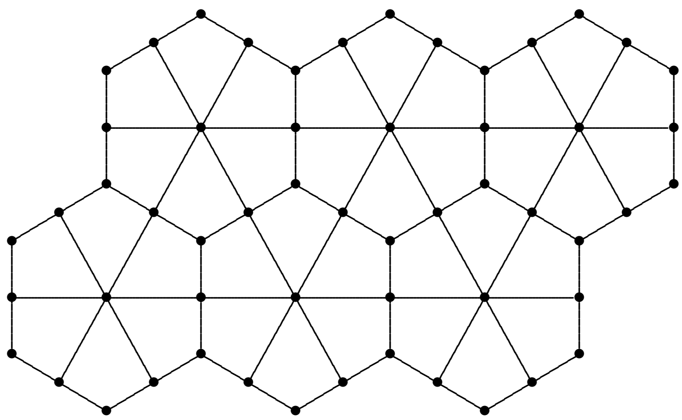

FIGURE 6: A portion of the dual graph of the $(3,4,6,4)$ lattice.

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