Problem 1. Find the general solution to $xy'(x) - y(x) = 1$.

$$y' - \frac{1}{x} y = \frac{1}{x}$$

integrating factor is $e^{-\int \frac{1}{x} dx} = e^{-\ln x} = e^{\ln(\frac{1}{x})} = \frac{1}{x}$.

So $\left(\frac{1}{x} y\right)' = \frac{1}{x^2} \Rightarrow \frac{1}{x} y = -\frac{1}{x} + c$

$\boxed{y(x) = -1 + Cx}$ for any constant C.

check: $y'(x) = C$ and $xy' - y = cx - (-1 + cx) = 1$ ✓.

Problem 2. Solve $\frac{dy}{dx} = -(x-y)^2$, $y(0) = 0$.

Set $u = x - y$ $\quad \frac{du}{dx} = 1 - \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = 1 - \frac{du}{dx}$.

$1 - \frac{du}{dx} = -u^2 \Rightarrow \frac{du}{dx} = 1 + u^2 \Rightarrow \frac{du}{1+u^2} = dx$

$\arctan(u) = x + C$

$u = \tan(x + c)$.

$x - y = \tan(x + c)$

$y(0) = 0 \Rightarrow 0 = \tan(0 + c) \Rightarrow c = 0$ (or any odd multiple of $\frac{\pi}{2}$).

$\boxed{y = x - \tan(x)}$.

Problem 3. Consider the differential equation $(4x - y)\,dx + (6y - x)\,dy = 0$.
(a) Show this equation is exact.
(b) Solve it.

(a) $\frac{\partial}{\partial y}(4x-y) = -1$ and $\frac{\partial}{\partial x}(6y-x) = -1$ agree

Therefore the above equation is exact.

(b) We have $G_x(x,y) = 4x - y$ implies $G(x,y) = 2x^2 - xy + h(y)$.

Taking a deriv. in y : $G_y(x,y) = -x + h'(y) = 6y - x \Rightarrow h'(y) = 6y$

$$\Rightarrow h(y) = 3y^2 + c \quad (\text{take } C = 0)$$

and

$$G(x,y) = 2x^2 - xy + 3y^2.$$

Our exact equation is therefore,

$$dG(x,y) = 0 \Rightarrow G(x,y) = C$$

i.e, the solution is given implicitly as

$$\boxed{2x^2 - xy + 3y^2 = c}$$

Problem 4. Label each of the following statements as either True or False.
(a) For any two row-equivalent $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, $\det(\mathbf{A}) = \det(\mathbf{B})$.
(b) If $\det(\mathbf{A}) \neq 0$, then the linear system $\mathbf{Ax} = \mathbf{b}$ is consistent for every choice of vector $\mathbf{b}$.
(c) For every symmetric $n \times n$ matrix $\mathbf{A}$ the reduced echelon form of $\mathbf{A}$ is also symmetric.
(d) If $\mathbf{E}_1$ and $\mathbf{E}_2$ are elementary matrices, then their product $\mathbf{E}_1\mathbf{E}_2$ is an elementary matrix.
(e) If $\mathbf{E}_1$ and $\mathbf{E}_2$ are elementary matrices, then their product $\mathbf{E}_1\mathbf{E}_2$ is row-equivalent to the Identity.

(a) False.
(b) True
(c) False
(d) False
(e) True

Problem 5. Consider the 4×4 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

(a) Compute $\det(\mathbf{A})$.
(b) Find $\mathbf{A}_{4,3}$. (Hint: This is the cofactor of some entry.)
(c) Compute $\mathbf{A}^{-1}$.

(a) $\det(A) = 1 \cdot 2 \cdot 3 \cdot 4 = 24$ (since A is triangular)

(b) $A_{43} = (-1)^{4+3} M_{43}$

$$= -M_{43} = -\begin{vmatrix} 1 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{vmatrix} = -8$$

(c) Augmenting with the corresp. identity matrix and row reducing

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{-R_4+R_2 \\ -R_4+R_3}]{-R_4+R_1} \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 0 & 1 & 0 & 0 & -1 \\ 0 & 2 & 3 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-R_3+R_2]{-R_3+R_1} \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{\frac{1}{3}R_3 \\ \frac{1}{4}R_4}]{\frac{1}{2}R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1/3 & -1/3 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$$

Problem 6. Consider the following linear system:

$$\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & + & x_3 & + & 2x_4 & = & b_1 \\ 2x_1 & - & 4x_2 & + & 2x_3 & + & x_4 & = & b_2 \\ x_1 & - & 2x_2 & + & x_3 & & & = & b_3 \end{array}$$

The corresponding augmented coefficient matrix of this linear system in reduced echelon form is given by

$$\begin{bmatrix} 1 & -2 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 0 & d_3 \end{bmatrix}.$$

(a) For which values of d_1, d_2 and d_3 is the corresponding linear system consistent?
(b) For which values of d_1, d_2 and d_3 is the corresponding linear system inconsistent?
(c) Identify the leading variables and the free variables. Clearly label each.
(d) Find the solution set of the corresponding homogeneous linear system.

(a) d_1, d_2 can be any constants, d_3 must be 0. in order for the LS to be consistent.

(b) Regardless of d_1, d_2, if $d_3 \neq 0$, the LS is inconsistent.

(c) Leading variables are x_1, x_4; and the free variables are x_2, x_3

(d) $x_2 = s, \quad x_3 = t. \quad (d_1 = d_2 = d_3 = 0)$.

$x_4 = 0$

$x_1 - 2x_2 + x_3 = 0 \Rightarrow x_1 = 2s - t$

The Solution set is the set of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$ for some $s, t \in \mathbb{R}$.