1. [10 pt] Solve $y''(x) = 0$, $y(0) = \pi$ and $y'(0) = 1$.

$$y'(x) = c_1$$
$$y(x) = c_1 x + c_2$$

$y(0) = \pi \Rightarrow c_2 = \pi$

$y'(0) = 1 \Rightarrow c_1 = 1$

Thus, $\boxed{y(x) = x + \pi.}$

2. [15 pt] Find the general solution to $\dfrac{dy}{dx} = e^{x-2y}$.

(Separable)

$$e^{2y} \cdot dy = e^{x} dx$$

$$\int e^{2y} dy = \int e^{x} dx$$

$$\frac{e^{2y}}{2} = e^{x} + C$$

$$2y = \ln(2e^{x} + A)$$

$$\boxed{y = \frac{1}{2} \ln(2e^{x} + A),}$$

where A is a constant

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3. [15 pt] Solve $\frac{dy}{dx} = 1 + (y-x)^2$, $y(1) = 1/2$.

Substitute $u = y - x$, $\frac{du}{dx} = \frac{dy}{dx} - 1 \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + 1$

Therefore,

$$\frac{du}{dx} + 1 = 1 + u^2 \Rightarrow \frac{du}{dx} = u^2 \Rightarrow \frac{du}{u^2} = dx \Rightarrow \frac{-1}{u(x)} = x + C$$

and

$$u(x) = \frac{-1}{x+C} \Rightarrow y(x) = x - \frac{1}{x+C} \text{ for some constant } C$$

$y(1) = \frac{1}{2} \Rightarrow C = 1$. Thus, $\boxed{y(x) = x - \frac{1}{x+1}}$ is the solution.

4. [15 pt] Find the general solution to $y' + 2xy = 3$.

An integrating factor is $e^{\int 2x\,dx} = e^{x^2}$.

and

$$(e^{x^2} y)' = 3e^{x^2} \Rightarrow e^{x^2} y(x) = 3\int_0^x e^{u^2}\,du + C$$

and

$$\boxed{y(x) = 3e^{-x^2}\int_0^x e^{u^2}\,du + Ce^{-x^2}.}$$

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5. Let $\mathbf{A}=\begin{bmatrix} 2 & -1 & 3 \\ -2 & 0 & 2 \end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(a) [5 pt] Is it possible to compute **A+B**? If so, compute it. If not, briefly explain why not.

No, it is not possible, since A and B are different sizes.

(b) [10 pt] Of the products **AA**, **AB**, **BA**, and **BB** which one makes sense? Compute the ones that make sense.

Only BA and BB make sense.

$$BA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \\ -2 & 0 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} -2 & -1 & 7 \\ -2 & -3 & 17 \end{bmatrix}}$$

$$BB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \boxed{\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}}.$$

6. [10 pt] Find $\mathbf{A}^{-1}$ when

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -3 & 3 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[-R_3]{-R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 3 & -1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \xrightarrow[-R_3+R_2]{-3R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

and $\boxed{A^{-1} = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}}.$

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7. [20 pt] Find the solution set for the *homogeneous* linear system of equations:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 0 \\ 6x_1 - x_2 + 2x_3 &= 0 \\ 12x_1 + 6x_2 + 4x_3 &= 0 \end{aligned}$$

Hint: the linear system has the following coefficient matrix:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ 6 & -1 & 2 \\ 12 & 6 & 4 \end{bmatrix}$$

and the problem is asking you to compute the Nullspace(**A**).

$$\begin{bmatrix} 3 & 2 & 1 \\ 6 & -1 & 2 \\ 12 & 6 & 4 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 3 & 2 & 1 \\ 0 & -5 & 0 \\ 12 & 6 & 4 \end{bmatrix} \xrightarrow{-4R_1+R_3} \begin{bmatrix} 3 & 2 & 1 \\ 0 & -5 & 0 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{5}R_2} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{2R_2+R_3} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

leading variables are x_1, x_2

free variable is x_3

Set $x_3 = s$

$x_2 = 0$

$3x_1 + 2x_2 + x_3 = 0 \Rightarrow 3x_1 + 0 + s = 0 \Rightarrow x_1 = -\frac{1}{3}s$

and the solution set is all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} s \quad \text{for any scalar } s.$$