**Department of Mathematical Sciences**
**The Johns Hopkins University**

## SEMINAR

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October 10, 2002
304 Whitehead Hall
Refreshments: 3:30 p.m.
Seminar: 4:00 p.m.

# THE LATTICE THEORY OF THE SYMMETRIC GROUP

## Episode II: f- and h-vectors of distributive lattices

### Abstract

Let $L$ be a finite distributive lattice. Let $c_i$ be the number of $i$-element chains (totally ordered subsets) of $L$. One of the main implications of the Stanley–Neggers Conjecture is that the sequence $c_0, c_1, c_2, \ldots$ is unimodal. We prove that the unimodality conjecture is three-quarters true (joint work with Anders Bjorner). Coxeter groups, polytopes, simplicial complexes, and matchings arise.

Let "Fred" be a finite partially ordered set, labeled by the numbers $1, 2, 3, \ldots, n$ so that, whenever an element $p$ is below an element $q$ in the poset, the label of $p$ (a natural number) is less than the label of $q$. (The permutation $123 \cdots n$ is a "linear extension" of the poset Fred.) For example, consider the zig-zag-shaped poset with four elements $1, 2, 3, 4$ whose partial ordering is given by $1 < 3 > 2 < 4$. Look at the linear extensions, that is, the permutations in $S_n$ that respect the partial ordering of Fred, by which we mean the following: If the element labelled $i$ in Fred is below the element labelled $j$ in Fred, then the number $i$ must come before the number $j$ in the permutation. In our example, there are 5 linear extensions: 1234, 2134, 1243, 2143, and 2413. (If Fred were a totally unordered poset, every permutation in $S_n$ would be a linear extension.)

Now take your favourite linear extension and count the number of "descents," that is, the places where a bigger number immediately precedes a smaller number. In our example, the numbers of descents in the linear extensions are 0, 1, 1, 2, and 1, respectively. Let $h_i$ be the number of linear extensions with $i$ descents. The zig-zag has $h_0 = 1$, $h_1 = 3$, and $h_2 = 1$. We usually call this sequence $(1, 3, 1, 0, 0, \ldots)$ the $h$-vector of the distributive lattice corresponding to the poset.

We prove that the number of linear extensions with at most $i$ descents is greater than or equal to the number of linear extensions with at least $n - 1 - i$ descents. We do this by proving that there is a bijection between the set of permutations in $S_n$ with at least $i$ lower covers (descents) and the set of permutations with at most $i$ upper covers (ascents) which sends every permutation to something greater than or equal to it in the weak Bruhat order. In fact, our proof works for any finite bounded-homormorphic image of a free lattice, in particular any finite Coxeter group with the weak Bruhat order.

Finally, we resolve the conjecture of Mark Skandera (Electronic Journal of Combinatorics, vol. 8), who asks, "Is the h-vector of a distributive lattice the f-vector of a poset?" To find out the answer, you must come to the talk ...