Basic definitions

A sequence $a_1, a_2, \ldots, a_k$ of distinct integers is **alternating** if

$$a_1 > a_2 < a_3 > a_4 < \cdots,$$

and **reverse alternating** if

$$a_1 < a_2 > a_3 < a_4 > \cdots.$$
\( \mathfrak{S}_n \) : symmetric group of all permutations of \( 1, 2, \ldots, n \)
\( \mathfrak{S}_n \): symmetric group of all permutations of 1, 2, \ldots, n

**Euler number:**

\[
E_n = \# \{ w \in \mathfrak{S}_n : w \text{ is alternating} \} \\
= \# \{ w \in \mathfrak{S}_n : w \text{ is reverse alternating} \}
\]

(via \( a_1 \cdots a_n \mapsto n + 1 - a_1, \ldots, n + 1 - a_n \))
Euler numbers

\( \mathcal{S}_n \) : symmetric group of all permutations of 1, 2, \ldots, \( n \)

**Euler number:**

\[
E_n = \# \{ w \in \mathcal{S}_n : w \text{ is alternating} \}
= \# \{ w \in \mathcal{S}_n : w \text{ is reverse alternating} \}
\]

(via \( a_1 \cdots a_n \mapsto n + 1 - a_1, \ldots, n + 1 - a_n \))

E.g., \( E_4 = 5 \) : 2143, 3142, 3241, 4132, 4231
André’s theorem

**Theorem** (Désiré André, 1879)

\[ y := \sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x \]

\[ = 1 + 1x + 1 \frac{x^2}{2!} + 2 \frac{x^3}{3!} + 5 \frac{x^4}{4!} + 16 \frac{x^5}{5!} \]

\[ + 61 \frac{x^6}{6!} + 272 \frac{x^7}{7!} + \cdots \]
**André’s theorem**

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*\(E_{2n}\) is a secant number.*

*\(E_{2n+1}\) is a tangent number.*
Proof of André’s theorem

\[ y := \sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x \]
Proof of André’s theorem

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Choose \( S \subseteq \{1, 2, \ldots, n\} \), say \( \#S = k \).
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Choose a reverse alternating permutation \( u = a_1 a_2 \cdots a_k \) of \( S \).
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Choose a reverse alternating permutation \( \mathbf{u} = a_1 a_2 \cdots a_k \) of \( S \).

Choose a reverse alternating permutation \( \mathbf{v} = b_1 b_2 \cdots b_{n-k} \) of \( [n] - S \).
Proof of André’s theorem

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Choose \( S \subseteq \{1, 2, \ldots, n\} \), say \( \#S = k \).

Choose a reverse alternating permutation \( u = a_1 a_2 \cdots a_k \) of \( S \).

Choose a reverse alternating permutation \( v = b_1 b_2 \cdots b_{n-k} \) of \( [n] - S \).

Let \( w = a_k \cdots a_2 a_1, n + 1, b_1 b_2 \cdots b_{n-k} \).
Proof (continued)

\[ w = a_k \cdots a_2a_1, n + 1, b_1b_2 \cdots b_{n-k} \]

Given \( k \), there are:

- \( \binom{n}{k} \) choices for \( \{a_1, a_2, \ldots, a_k\} \)
- \( E_k \) choices for \( a_1a_2 \cdots a_k \)
- \( E_{n-k} \) choices for \( b_1b_2 \cdots b_{n-k} \).
Proof (continued)

\[ w = a_k \cdots a_2 a_1, n + 1, b_1 b_2 \cdots b_{n-k} \]

Given \( k \), there are:

- \( \binom{n}{k} \) choices for \( \{a_1, a_2, \ldots, a_k\} \)
- \( E_k \) choices for \( a_1 a_2 \cdots a_k \)
- \( E_{n-k} \) choices for \( b_1 b_2 \cdots b_{n-k} \).

We obtain each alternating and reverse alternating \( w \in S_{n+1} \) once each.
\[ 2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}, \quad n \geq 1 \]
\[ \Rightarrow 2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}, \ n \geq 1 \]

Multiply by \( x^{n+1}/(n + 1)! \) and sum on \( n \geq 0 \):

\[ 2y' = 1 + y^2, \ y(0) = 1. \]
\[
2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}, \quad n \geq 1
\]

Multiply by \( \frac{x^{n+1}}{(n+1)!} \) and sum on \( n \geq 0 \):

\[
2y' = 1 + y^2, \quad y(0) = 1.
\]

\[
\Rightarrow y = \sec x + \tan x.
\]
A new subject?

\[
\sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x
\]
Define

\[
\sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x
\]

\[
\tan x = \sum_{n \geq 0} E_{2n+1} \frac{x^{2n+1}}{(2n + 1)!}
\]

\[
\sec x = \sum_{n \geq 0} E_{2n} \frac{x^{2n}}{(2n)!}
\]
A new subject?

Define

\[ \tan x = \sum_{n \geq 0} E_{2n+1} \frac{x^{2n+1}}{(2n + 1)!} \]

\[ \sec x = \sum_{n \geq 0} E_{2n} \frac{x^{2n}}{(2n)!} \]

⇒ combinatorial trigonometry
sec^2 x = 1 + \tan^2 x

Exercises on combinatorial trig.

Alternating Permutations – p.
Exercises on combinatorial trig.

\[ \sec^2 x = 1 + \tan^2 x \]

\[ \tan(x + y) = \frac{\tan x + \tan y}{1 - (\tan x)(\tan y)} \]
Exercises on combinatorial trig.

\[
\sec^2 x = 1 + \tan^2 x
\]

\[
\tan(x + y) = \frac{\tan x + \tan y}{1 - (\tan x)(\tan y)}
\]

EC2, Exercise 5.7
boustrophedon:
boustrophedon: an ancient method of writing in which the lines are inscribed alternately from right to left and from left to right.
boustrophedon: an ancient method of writing in which the lines are inscribed alternately from right to left and from left to right.

From Greek *boustrophēdon* (βουστροφηδόν), turning like an ox while plowing: *bous*, ox + *strophē*, a turning (from *strephein*, to turn)
The boustrophedon array

\[
\begin{align*}
1 \\
0 & \rightarrow 1 \\
1 & \leftarrow 1 \leftarrow 0 \\
0 & \rightarrow 1 \rightarrow 2 \rightarrow 2 \\
5 & \leftarrow 5 \leftarrow 4 \leftarrow 2 \leftarrow 0 \\
0 & \rightarrow 5 \rightarrow 10 \rightarrow 14 \rightarrow 16 \rightarrow 16 \\
61 & \leftarrow 61 \leftarrow 56 \leftarrow 46 \leftarrow 32 \leftarrow 16 \leftarrow 0. \\
\ldots
\end{align*}
\]
The boustrophedon array

\[
\begin{array}{cccc}
1 \\
0 & \rightarrow & 1 \\
1 & \leftarrow & 1 & \leftarrow & 0 \\
0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 2 \\
5 & \leftarrow & 5 & \leftarrow & 4 & \leftarrow & 2 & \leftarrow & 0 \\
0 & \rightarrow & 5 & \rightarrow & 10 & \rightarrow & 14 & \rightarrow & 16 & \rightarrow & 16 \\
61 & \leftarrow & 61 & \leftarrow & 56 & \leftarrow & 46 \leftarrow & 32 \leftarrow & 16 & \leftarrow & 0. \\
\text{\ldots}
\end{array}
\]
Boustrophedon entries

- last term in row $n$: $E_{n-1}$
- sum of terms in row $n$: $E_n$
- $k$th term in row $n$: number of alternating permutations in $\mathcal{S}_n$ with first term $k$, the **Entringer number** $E_{n-1,k-1}$.
Boustrophedon entries

- last term in row \( n \): \( E_{n-1} \)
- sum of terms in row \( n \): \( E_n \)
- \( k \)th term in row \( n \): number of alternating permutations in \( \mathfrak{S}_n \) with first term \( k \), the Entringer number \( E_{n-1,k-1} \).

\[
\sum_{m \geq 0} \sum_{n \geq 0} E_{m+n,[m,n]} \frac{x^m y^n}{m! n!} = \frac{\cos x + \sin x}{\cos(x+y)}, \]

\[
[m, n] = \begin{cases} 
  m, & m + n \text{ odd} \\
  n, & m + n \text{ even.}
\end{cases}
\]
Some occurrences of Euler numbers

(1) \( E_{2n-1} \) is the number of complete increasing binary trees on the vertex set \([2n + 1] = \{1, 2, \ldots, 2n + 1\}\).
Five vertices
Five vertices

Slightly more complicated for $E_{2n}$
Proof for $2n + 1$

\[ b_1 b_2 \cdots b_m : \text{sequence of distinct integers} \]

\[ b_i = \min\{b_1, \ldots, b_m\} \]
Proof for $2n + 1$

$b_1 b_2 \cdots b_m$: sequence of distinct integers

$$b_i = \min\{b_1, \ldots, b_m\}$$

Define recursively a binary tree $T(b_1, \ldots, b_m)$ by

$$T(b_1, \ldots, b_{i-1}) \quad \text{and} \quad T(b_{i+1}, \ldots, b_m)$$
Example. 439172856

Let $w \in S_n^{\geq 1}$. Then $T(w)$ is complete if and only if $w$ is alternating, and the map $w \mapsto T(w)$ gives the desired bijection.
Example. 439172856

Let \( w \in \mathcal{S}_{2n+1} \). Then \( T(w) \) is complete if and only if \( w \) is alternating, and the map \( w \mapsto T(w) \) gives the desired bijection.
(2) Start with $n$ one-element sets $\{1\}, \ldots, \{n\}$.
Orbits of mergings

\(1\) Start with \(n\) one-element sets \(\{1\}, \ldots, \{n\}\).

Merge together two at a time until reaching \(\{1, 2, \ldots, n\}\).
(2) Start with $n$ one-element sets $\{1\}, \ldots, \{n\}$. 
Merge together two at a time until reaching $\{1, 2, \ldots, n\}$.

125 – 34 – 6, 125 – 346, 123456
Orbits of mergings

(2) Start with \( n \) one-element sets \( \{1\}, \ldots, \{n\} \). Merge together two at a time until reaching \( \{1, 2, \ldots, n\} \).

\[
1 - 2 - 3 - 4 - 5 - 6, \quad 12 - 3 - 4 - 5 - 6, \quad 12 - 34 - 5 - 6 \\
125 - 34 - 6, \quad 125 - 346, \quad 123456
\]

\( \mathfrak{S}_n \) acts on these sequences.
(2) Start with \( n \) one-element sets \( \{1\}, \ldots, \{n\} \).
Merge together two at a time until reaching \( \{1, 2, \ldots, n\} \).

\[
1-2-3-4-5-6, \quad 12-3-4-5-6, \quad 12-34-5-6 \\
125-34-6, \quad 125-346, \quad 123456
\]

\( \mathfrak{S}_n \) acts on these sequences.

**Theorem.** *The number of \( \mathfrak{S}_n \)-orbits is \( E_{n-1} \).*
(2) Start with \( n \) one-element sets \( \{1\}, \ldots, \{n\} \). Merge together two at a time until reaching \( \{1, 2, \ldots, n\} \).

\[
1-2-3-4-5-6, \quad 12-3-4-5-6, \quad 12-34-5-6 \\
125-34-6, \quad 125-346, \quad 123456
\]

\( S_n \) acts on these sequences.

**Theorem.** *The number of \( S_n \)-orbits is \( E_{n-1} \).*

**Proof.**
Orbits of mergings

(2) Start with $n$ one-element sets $\{1\}, \ldots, \{n\}$. Merge together two at a time until reaching $\{1, 2, \ldots, n\}$.

\begin{align*}
&1-2-3-4-5-6, \quad 12-3-4-5-6, \quad 12-34-5-6 \\
&\quad 125-34-6, \quad 125-346, \quad 123456
\end{align*}

$\mathfrak{S}_n$ acts on these sequences.

**Theorem.** The number of $\mathfrak{S}_n$-orbits is $E_{n-1}$.

**Proof.** Exercise.
Orbit representatives for $n = 5$

<table>
<thead>
<tr>
<th>12 – 3 – 4 – 5</th>
<th>123 – 4 – 5</th>
<th>1234 – 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>12 – 3 – 4 – 5</td>
<td>123 – 4 – 5</td>
<td>123 – 45</td>
</tr>
<tr>
<td>12 – 3 – 4 – 5</td>
<td>12 – 34 – 5</td>
<td>125 – 34</td>
</tr>
<tr>
<td>12 – 3 – 4 – 5</td>
<td>12 – 34 – 5</td>
<td>12 – 345</td>
</tr>
<tr>
<td>12 – 3 – 4 – 5</td>
<td>12 – 34 – 5</td>
<td>1234 – 5</td>
</tr>
</tbody>
</table>
Let $\mathcal{E}_n$ be the convex polytope in $\mathbb{R}^n$ defined by

\[ x_i \geq 0, \quad 1 \leq i \leq n \]

\[ x_i + x_{i+1} \leq 1, \quad 1 \leq i \leq n - 1. \]
Let $\mathcal{E}_n$ be the convex polytope in $\mathbb{R}^n$ defined by

$$x_i \geq 0, \quad 1 \leq i \leq n$$

$$x_i + x_{i+1} \leq 1, \quad 1 \leq i \leq n - 1.$$ 

**Theorem.** The volume of $\mathcal{E}_n$ is $E_n/n!$. 
Naive proof

\[
\text{vol}(\mathcal{E}_n) = \int_{x_1=0}^{1} \int_{x_2=0}^{1-x_1} \int_{x_3=0}^{1-x_2} \cdots \int_{x_n=0}^{1-x_{n-1}} \, dx_1 \, dx_2 \cdots \, dx_n
\]
Naive proof

\[ \text{vol}(E_n) = \int_{x_1=0}^{1} \int_{x_2=0}^{1-x_1} \int_{x_3=0}^{1-x_2} \cdots \int_{x_n=0}^{1-x_{n-1}} dx_1 \, dx_2 \cdots dx_n \]

\[ f_n(t) := \int_{x_1=0}^{t} \int_{x_2=0}^{1-x_1} \int_{x_3=0}^{1-x_2} \cdots \int_{x_n=0}^{1-x_{n-1}} dx_1 \, dx_2 \cdots dx_n \]
Naive proof

\[ \text{vol}(E_n) = \int_{x_1=0}^{1} \int_{x_2=0}^{1-x_1} \int_{x_3=0}^{1-x_2} \cdots \int_{x_n=0}^{1-x_{n-1}} dx_1 \, dx_2 \cdots dx_n \]

\[ f_n(t) := \int_{x_1=0}^{t} \int_{x_2=0}^{1-x_1} \int_{x_3=0}^{1-x_2} \cdots \int_{x_n=0}^{1-x_{n-1}} dx_1 \, dx_2 \cdots dx_n \]

\[ f'_n(t) = \int_{x_2=0}^{1-t} \int_{x_3=0}^{1-x_2} \cdots \int_{x_n=0}^{1-x_{n-1}} dx_2 \, dx_3 \cdots dx_n \]

\[ = f_{n-1}(1 - t). \]
\begin{align*}
  f'_n(t) &= f_{n-1}(1 - t), \quad f_0(t) = 1, \quad f_n(0) = 0 \quad (n > 0)
\end{align*}
\[ f'_n(t) = f_{n-1}(1 - t), \quad f_0(t) = 1, \quad f_n(0) = 0 \quad (n > 0) \]

\[
F(y) = \sum_{n \geq 0} f_n(t)y^n
\]

\[
\Rightarrow \frac{\partial^2}{\partial t^2} F(y) = -y^2 F(y),
\]

etc.
Conclusion of proof

\[ F(y) = (\sec y)(\cos(t - 1)y + \sin ty) \]

\[ \Rightarrow F(y)|_{t=1} = \sec y + \tan y. \]
Tridiagonal matrices

An $n \times n$ matrix $M = (m_{ij})$ is tridiagonal if $m_{ij} = 0$ whenever $|i - j| \geq 2$.

doubly-stochastic: $m_{ij} \geq 0$, row and column sums equal 1

$\mathcal{T}_n$: set of $n \times n$ tridiagonal doubly stochastic matrices
Easy fact: the map

\[ \mathcal{T}_n \rightarrow \mathbb{R}^{n-1} \]

\[ M \mapsto (m_{12}, m_{23}, \ldots, m_{n-1,n}) \]

is a (linear) bijection from \( \mathcal{T} \) to \( \mathcal{E}_n \).
Polytope structure of $\mathcal{T}_n$

**Easy fact:** the map

$$\mathcal{T}_n \rightarrow \mathbb{R}^{n-1}$$

$$M \mapsto (m_{12}, m_{23}, \ldots, m_{n-1,n})$$

is a (linear) bijection from $\mathcal{T}$ to $\mathcal{E}_n$.

**Application** (*Diaconis* et al.): random doubly stochastic tridiagonal matrices and random walks on $\mathcal{T}_n$
A modification

Let $\mathcal{F}_n$ be the convex polytope in $\mathbb{R}^n$ defined by

\[
x_i \geq 0, \quad 1 \leq i \leq n
\]

\[
x_i + x_{i+1} + x_{i+2} \leq 1, \quad 1 \leq i \leq n - 2.
\]

\[V_n = \text{vol}(\mathcal{F}_n)\]
A modification

Let $\mathcal{F}_n$ be the convex polytope in $\mathbb{R}^n$ defined by

\[
\begin{align*}
  x_i & \geq 0, \quad 1 \leq i \leq n \\
  x_i + x_{i+1} + x_{i+2} & \leq 1, \quad 1 \leq i \leq n - 2.
\end{align*}
\]

\[
V_n = \text{vol}(\mathcal{F}_n)
\]

\[
\begin{array}{c||c|c|c|c|c|c|c|c}
 n & 1–3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
n!V_n & 1 & 2 & 5 & 14 & 47 & 182 & 786 & 3774
\end{array}
\]
A “naive” recurrence

\[ V_n = f_n(1, 1), \]

where

\[ f_0(a, b) = 1, \quad f_n(0, b) = 0 \text{ for } n > 0 \]

\[ \frac{\partial}{\partial a} f_n(a, b) = f_{n-1}(b - a, 1 - a). \]
$f_n(a, b)$ for $n \leq 3$

\begin{align*}
    f_1(a, b) &= a \\
    f_2(a, b) &= \frac{1}{2}(2ab - a^2) \\
    f_3(a, b) &= \frac{1}{6}(a^3 - 3a^2 - 3ab^2 + 6ab)
\end{align*}
$f_n(a, b)$ for $n \leq 3$

\[
\begin{align*}
  f_1(a, b) &= a \\
  f_2(a, b) &= \frac{1}{2}(2ab - a^2) \\
  f_3(a, b) &= \frac{1}{6}(a^3 - 3a^2 - 3ab^2 + 6ab)
\end{align*}
\]

Is there a “nice” generating function for $f_n(a, b)$ or $V_n = f_n(1, 1)$?
Distribution of $\text{is}(w)$

$\text{is}(w) = \text{length of longest increasing subsequence of } w \in \mathfrak{S}_n$
Distribution of $\text{is}(w)$

$\text{is}(w) = \text{length of longest increasing subsequence of } w \in S_n$

$\text{is}(48361572) = 3$
Distribution of $\text{is}(w)$

$\text{is}(w) = \text{length of longest increasing subsequence of } w \in \mathcal{S}_n$

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Distribution of $\text{is}(w)$

$\text{is}(w) =$ length of longest increasing subsequence of $w \in \mathfrak{S}_n$

$\text{is}(48361572) = 3$

**Vershik-Kerov, Logan-Shepp:**

$$E(n) := \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \text{is}(w)$$

$\sim 2\sqrt{n}$
Baik-Deift-Johansson:

For fixed \( t \in \mathbb{R} \),

\[
\lim_{n \to \infty} \text{Prob} \left( \frac{\text{is}_n(w) - 2\sqrt{n}}{n^{1/6}} \leq t \right) = F(t),
\]

the **Tracy-Widom distribution**.
Alternating analogues

Length of longest alternating subsequence of $w \in \mathcal{S}_n$. The first is much easier!
Alternating analogues

- Length of longest alternating subsequence of $w \in \mathfrak{S}_n$

- Length of longest increasing subsequence of an alternating permutation $w \in \mathfrak{S}_n$. 
Alternating analogues

- Length of longest alternating subsequence of $w \in \mathfrak{S}_n$

- Length of longest increasing subsequence of an alternating permutation $w \in \mathfrak{S}_n$.

The first is much easier!
as(\(w\)) = \text{length of longest alt. subseq. of } w

\[ w = 56218347 \Rightarrow as(w) = 5 \]
Longest alternating subsequences

\[ \text{as}(w) = \text{length of longest alt. subseq. of } w \]

\[ w = 56218347 \Rightarrow \text{as}(w) = 5 \]

\[ D(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \text{as}(w) \sim ? \]
Definitions of $a_k(n)$ and $b_k(n)$

\[
a_k(n) = \# \{ w \in \mathcal{S}_n : \text{as}(w) = k \}
\]

\[
b_k(n) = a_1(n) + a_2(n) + \cdots + a_k(n) = \# \{ w \in \mathcal{S}_n : \text{as}(w) \leq k \}
\]
### The case $n = 3$

<table>
<thead>
<tr>
<th>$w$</th>
<th>$as(w)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>1</td>
</tr>
<tr>
<td>132</td>
<td>2</td>
</tr>
<tr>
<td>213</td>
<td>3</td>
</tr>
<tr>
<td>231</td>
<td>2</td>
</tr>
<tr>
<td>312</td>
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<tr>
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The case $n = 3$

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<td>3</td>
</tr>
<tr>
<td>321</td>
<td>2</td>
</tr>
</tbody>
</table>

$a_1(3) = 1$, $a_2(3) = 3$, $a_3(3) = 2$

$b_1(3) = 1$, $b_2(3) = 4$, $b_3(3) = 6$
Lemma. $\forall \omega \in \mathfrak{S}_n \exists$ alternating subsequence of maximal length that contains $n$. 
Lemma. \( \forall w \in \mathfrak{S}_n \exists \) alternating subsequence of maximal length that contains \( n \).

Corollary.

\[
\Rightarrow a_k(n) = \sum_{j=1}^{n} \binom{n-1}{j-1}
\]

\[
\sum_{2r+s=k-1} (a_{2r}(j-1) + a_{2r+1}(j-1)) a_s(n-j)
\]
The main generating function

\[ B(x, t) = \sum_{k,n \geq 0} b_k(n) t^k \frac{x^n}{n!} \]

Theorem.

\[ B(x, t) = \frac{2/\rho}{1 - \frac{1-\rho}{t} e^{\rho x}} - \frac{1}{\rho}, \]

where \( \rho = \sqrt{1 - t^2}. \)
Corollary.

\[ b_1(n) = 1 \]
\[ b_2(n) = n \]
\[ b_3(n) = \frac{1}{4}(3^n - 2n + 3) \]
\[ b_4(n) = \frac{1}{8}(4^n - (2n - 4)2^n) \]
\[ \vdots \]
Corollary.

\[ \Rightarrow b_1(n) = 1 \]
\[ b_2(n) = n \]
\[ b_3(n) = \frac{1}{4}(3^n - 2n + 3) \]
\[ b_4(n) = \frac{1}{8}(4^n - (2n - 4)2^n) \]

\[ \vdots \]

no such formulas for longest increasing subsequences
Mean (expectation) of $as(w)$

\[ D(n) = \frac{1}{n!} \sum_{w \in \mathcal{S}_n} as(w) = \frac{1}{n!} \sum_{k=1}^{n} k \cdot a_k(n), \]

the expectation of $as(w)$ for $w \in \mathcal{S}_n$
Mean (expectation) of $\text{as}(w)$

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the expectation of $\text{as}(w)$ for $w \in \mathfrak{S}_n$

Let

$$A(x, t) = \sum_{k,n \geq 0} a_k(n) t^k \frac{x^n}{n!} = (1 - t) B(x, t)$$

$$= (1 - t) \left( \frac{2/\rho}{1 - \frac{1-\rho}{t} e^{\rho x}} - \frac{1}{\rho} \right).$$
Formula for $D(n)$

$$\sum_{n \geq 0} D(n) x^n = \frac{\partial}{\partial t} A(x, 1)$$

$$= 6x - 3x^2 + x^3$$

$$= \frac{6(1 - x)^2}{6(1 - x)^2}$$

$$= x + \sum_{n \geq 2} \frac{4n + 1}{6} x^n.$$
Formula for $D(n)$

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\[
\Rightarrow D(n) = \frac{4n + 1}{6}, \quad n \geq 2
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\[ \Rightarrow D(n) = \frac{4n + 1}{6}, \quad n \geq 2 \]

Compare $E(n) \sim 2 \sqrt{n}$. 
Variance of $\text{as}(w)$

$$V(n) = \frac{1}{n!} \sum_{w \in \mathcal{S}_n} \left( \text{as}(w) - \frac{4n + 1}{6} \right)^2, \quad n \geq 2$$

the variance of $\text{as}(w)$ for $w \in \mathcal{S}_n$
Variance of $\text{as}(w)$

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Corollary.

$$V(n) = \frac{8}{45}n - \frac{13}{180}, \ n \geq 4$$
Variance of $\text{as}(w)$

$$V(n) = \frac{1}{n!} \sum_{w \in \mathcal{S}_n} \left( \text{as}(w) - \frac{4n + 1}{6} \right)^2, \ n \geq 2$$

the variance of $\text{as}(w)$ for $w \in \mathcal{S}_n$

Corollary.

$$V(n) = \frac{8}{45}n - \frac{13}{180}, \ n \geq 4$$

similar results for higher moments
A new distribution?

\[ P(t) = \lim_{n \to \infty} \text{Prob}_{w \in S_n} \left( \frac{\text{as}(w) - 2n/3}{\sqrt{n}} \leq t \right) \]
A new distribution?

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Stanley distribution?
Theorem (Pemantle, Widom, (Wilf)).

\[
\lim_{n \to \infty} \Prob_{w \in \mathfrak{S}_n} \left( \frac{\text{as}(w) - 2n/3}{\sqrt{n}} \leq t \right)
\]

\[
= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{t\sqrt{45/4}} e^{-s^2} \, ds
\]

(Gaussian distribution)
Theorem (Pemantle, Widom, (Wilf)).

\[ \lim_{n \to \infty} \operatorname{Prob}_{w \in \mathfrak{S}_n} \left( \frac{\text{as}(w) - 2n/3}{\sqrt{n}} \leq t \right) \]

\[ = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{t\sqrt{45/4}} e^{-s^2} \, ds \]

(Gaussian distribution)
Umbral enumeration

**Umbral formula:** involves $E^k$, where $E$ is an indeterminate (the **umbra**). Replace $E^k$ with the Euler number $E_k$. (Technique from 19th century, modernized by Rota et al.)
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Example.

$$(1 + E^2)^3 = 1 + 3E^2 + 3E^4 + E^6$$
$$= 1 + 3E_2 + 3E_4 + E_6$$
$$= 1 + 3 \cdot 1 + 3 \cdot 5 + 61$$
$$= 80$$
Another example

\[
(1 + t)^E = 1 + Et + \frac{E}{2} t^2 + \frac{E}{3} t^3 + \cdots \\
= 1 + Et + \frac{1}{2} E(E - 1)t^2 + \cdots \\
= 1 + E_1 t + \frac{1}{2} (E_2 - E_1) t^2 + \cdots \\
= 1 + t + \frac{1}{2} (1 - 1)t^2 + \cdots \\
= 1 + t + O(t^3).
\]
An umbral quiz

Let $B$ be the Bell number umbra. Then
An umbral quiz

Let $B$ be the Bell number umbra. Then

$$(1 + t)^B = ??$$
An umbral quiz

Let $B$ be the Bell number umbra. Then

$$(1 + t)^B = e^t$$
fixed point free involution $w \in S_{2n}$: all cycles of length two (number $= 1 \cdot 3 \cdot 5 \cdots (2n - 1)$)
**fixed point free involution** $w \in \mathfrak{S}_{2n}$: all cycles of length two (number $= 1 \cdot 3 \cdot 5 \cdots (2n - 1)$)

Let $f(n)$ be the number of **alternating** fixed-point free involutions in $\mathfrak{S}_{2n}$. 
fixed point free involution $w \in S_{2n}$: all cycles of length two (number $= 1 \cdot 3 \cdot 5 \cdots (2n - 1)$)

Let $f(n)$ be the number of alternating fixed-point free involutions in $S_{2n}$.

$n = 3$:

$214365 = (1, 2)(3, 4)(5, 6)$

$645231 = (1, 6)(2, 4)(3, 5)$

$f(3) = 2$
Theorem.

\[ F(x) = \sum_{n \geq 0} f(n) x^n \]
An umbral theorem

Theorem.

\[ F(x) = \sum_{n \geq 0} f(n) x^n \]

\[ = \left( \frac{1 + x}{1 - x} \right)^{\left( E^2 + 1 \right)/4} \]
Proof idea

Proof. Uses representation theory of the symmetric group $\mathfrak{S}_n$. 
Proof idea

Proof. Uses representation theory of the symmetric group $\mathfrak{S}_n$.

There is a character $\chi$ of $\mathfrak{S}_n$ (due to H. O. Foulkes) such that for all $w \in \mathfrak{S}_n$,

$$\chi(w) = 0 \text{ or } \pm E_k.$$
Proof idea

**Proof.** Uses representation theory of the symmetric group $\mathfrak{S}_n$.

There is a character $\chi$ of $\mathfrak{S}_n$ (due to H. O. Foulkes) such that for all $w \in \mathfrak{S}_n$,

$$\chi(w) = 0 \text{ or } \pm E_k.$$ 

Now use known results on combinatorial properties of characters of $\mathfrak{S}_n$. 

Alternating Permutations – p. 47
Theorem (Ramanujan, Berndt, implicitly) As $x \to 0+$,

$$2 \sum_{n \geq 0} \left( \frac{1 - x}{1 + x} \right)^{n(n+1)} \sim \sum_{k \geq 0} f(k) x^k = F(x),$$

an **analytic** (non-formal) identity.
A formal identity

Corollary (via Ramanujan, Andrews).

\[ F(x) = 2 \sum_{n \geq 0} q^n \frac{\prod_{j=1}^{n} (1 - q^{2j-1})}{\prod_{j=1}^{2n+1} (1 + q^j)} \]

where \( q = \left( \frac{1-x}{1+x} \right)^{2/3} \), a formal identity.
Recall: number of $n$-cycles in $\mathfrak{S}_n$ is $(n - 1)!$. 
Recall: number of $n$-cycles in $\mathfrak{S}_n$ is $(n - 1)!$.

**Theorem.** Let $b(n)$ be the number of **alternating** $n$-cycles in $\mathfrak{S}_n$. Then if $n$ is odd,

$$b(n) = \frac{1}{n} \sum_{d | n} \mu(d) (-1)^{(d-1)/2} E_{n/d}.$$
Corollary. Let $p$ be an odd prime. Then

$$b(p) = \frac{1}{p} \left( E_p - (-1)^{(p-1)/2} \right).$$
Corollary. Let \( p \) be an odd prime. Then

\[
    b(p) = \frac{1}{p} \left( E_p - (-1)^{(p-1)/2} \right). 
\]

Combinatorial proof?
Recall: $\text{is}(w) =$ length of longest increasing subsequence of $w \in S_n$. Define

$$C(n) = \frac{1}{E_n} \sum w \text{is}(w),$$

where $w$ ranges over all $E_n$ alternating permutations in $S_n$. 

Inc. subsequences of alt. perms.
Crude estimate: what is

\[ \beta = \lim_{n \to \infty} \frac{\log C(n)}{\log n} \, ? \]

I.e., \( C(n) = n^{\beta + o(1)} \).
Crude estimate: what is

$$\beta = \lim_{n \to \infty} \frac{\log C(n)}{\log n}?$$

I.e., $C(n) = n^{\beta + o(1)}$.

J. Shearer, 2010: $\beta = \frac{1}{2}$. 
What is the (suitably scaled) limiting distribution of \( \text{is}(w) \), where \( w \) ranges over all alternating permutations in \( \mathfrak{S}_n \)?
Limiting distribution?

What is the (suitably scaled) limiting distribution of $\text{is}(w)$, where $w$ ranges over all alternating permutations in $\mathfrak{S}_n$?

Is it the Tracy-Widom distribution?
What is the (suitably scaled) limiting distribution of \( \text{is}(w) \), where \( w \) ranges over all alternating permutations in \( \mathfrak{S}_n \)?

Is it the Tracy-Widom distribution?

**Possible tool:** ∃ “umbral analogue” of Gessel’s determinantal formula.
What is the (suitably scaled) limiting distribution of $\text{is}(w)$, where $w$ ranges over all alternating permutations in $\mathfrak{S}_n$?

Is it the Tracy-Widom distribution?

**Possible tool:** ∃ “umbral analogue” of Gessel’s determinantal formula.
Darn!

That’s the end...