8.7.2 The Rao-Blackwell Theorem

In the preceding section, we argued for the importance of sufficient statistics on essentially qualitative grounds. The Rao-Blackwell theorem gives a quantitative rationale for basing an estimator of a parameter \( \theta \) on a sufficient statistic if one exists.

**THEOREM A  Rao-Blackwell Theorem**

Let \( \hat{\theta} \) be an estimator of \( \theta \) with \( E(\hat{\theta}^2) < \infty \) for all \( \theta \). Suppose that \( T \) is sufficient for \( \theta \), and let \( \bar{\theta} = E(\hat{\theta}|T) \). Then, for all \( \theta \),

\[
E(\bar{\theta} - \theta)^2 \leq E(\hat{\theta} - \theta)^2
\]

The inequality is strict unless \( \hat{\theta} = \bar{\theta} \).

**Proof** We first note that, from the property of iterated conditional expectation (Theorem A of Section 4.4.1),

\[
E(\bar{\theta}) = E[E(\hat{\theta}|T)] = E(\hat{\theta})
\]

Therefore, to compare the mean squared error of the two estimators, we need only compare their variances. From Theorem B of Section 4.4.1, we have

\[
\text{Var}(\hat{\theta}) = \text{Var}[E(\hat{\theta}|T)] + E[\text{Var}(\hat{\theta}|T)]
\]

or

\[
\text{Var}(\bar{\theta}) = \text{Var}(\hat{\theta}) + E[\text{Var}(\hat{\theta}|T)]
\]

Thus, \( \text{Var}(\hat{\theta}) > \text{Var}(\bar{\theta}) \) unless \( \text{Var}(\hat{\theta}|T) = 0 \), which is the case only if \( \hat{\theta} \) is a function of \( T \), which would imply \( \hat{\theta} = \bar{\theta} \). \( \blacksquare \)

Since \( E(\hat{\theta}|T) \) is a function of the sufficient statistic \( T \), the Rao-Blackwell theorem gives a strong rationale for basing estimators on sufficient statistics if they exist. If an estimator is not a function of a sufficient statistic, it can be improved.

*Why must \( T \) be sufficient, for the above result?*