KIRCHOFF'S CURRENT LAW AND KIRCHOFF'S VOLTAGE LAW

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Recall the Maxwell's equations :

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon} && : \text{Gauss's law} \\
\nabla \cdot \mathbf{B} &= 0 && : \text{Gauss's law for magnetism} \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && : \text{Faraday's law of induction} \\
\nabla \times \mathbf{B} &= \mu \Big(\epsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} \Big) && : \text{Ampère's law,}
\end{aligned}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, $\mathbf{E}$ is the electric field, $\mathbf{J}$ denotes the total electric current density, and ρ is the charge density. The constants ϵ and μ denote the permittivity and permeability respectively. Sometimes, it is convenient to use $\mathbf{D} = \epsilon \mathbf{E}$, and $\mathbf{H} = \frac{\mathbf{B}}{\mu}$. The field $\mathbf{D}$ is called as the electric displacement field, and $\mathbf{H}$ is known as the magnetizing field.
Using this notation the Gauss's law is written as

$$\nabla \cdot \mathbf{D} = \rho$$

and the Ampère's law takes the form:

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}.$$

1. Kirchoff's current law

Consider a point in a network of conductors, with N number of wires emanating from it. Enclose the node with some open bounded nonempty region $\Omega \subseteq \mathbb{R}^3$, with boundary S. We denote by S_k the intersection of the k^{th} wire and

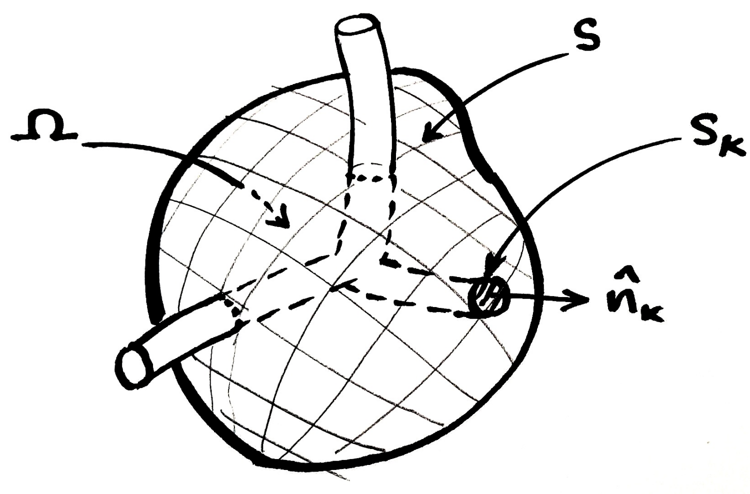

Figure 1. A node with three wires.

the surface S. Using the divergence theorem we obtain:

$$\begin{aligned}
\iiint\limits_{\Omega} \nabla \cdot \mathbf{J}\, dV &= \iint\limits_{S} \mathbf{J} \cdot \hat{n}\, dS \\
&= \sum_{k=1}^{N} \iint\limits_{S_k} \mathbf{J} \cdot \hat{n}\, dS + \iint\limits_{S \setminus (\bigcup_{k=1}^{N} S_k)} \mathbf{J} \cdot \hat{n}\, dS \\
&= \sum_{k=1}^{N} \iint\limits_{S_k} \mathbf{J} \cdot \hat{n}\, dS + 0 \text{ ...(as the current density } \mathbf{J} \text{ on } S \setminus \bigcup_{k=1}^{N} S_k \text{ is zero.)} \\
&= \sum_{k=1}^{N} I_k. \qquad (1)
\end{aligned}$$

Note that the current I_k is signed, with negative current indicating that the current is going inside the point.

Applying the gradient operator on both sides of the Ampère's law: $\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} = \nabla \times \mathbf{H}$, we get

$$\nabla \cdot \frac{\partial \mathbf{D}}{\partial t} + \nabla \cdot \mathbf{J} = \nabla \cdot (\nabla \times \mathbf{H}).$$

We can show that $\nabla \cdot (\nabla \times \mathbf{H}) = 0$ for any vector field $\mathbf{H}$. Thus,

$$\begin{aligned}
\nabla \cdot \frac{\partial \mathbf{D}}{\partial t} + \nabla \cdot \mathbf{J} &= 0 \\
\text{i.e. } \nabla \cdot \mathbf{J} &= -\nabla \cdot \frac{\partial \mathbf{D}}{\partial t}. \qquad (2)
\end{aligned}$$

Let us denote the components of $\mathbf{D}$ by D_1, and D_2, i.e. $\mathbf{D} = \langle D_1, D_2 \rangle$. Assuming the continuity of mixed second partial derivatives $\frac{\partial^2 D_1}{\partial t \partial x}$, and $\frac{\partial^2 D_2}{\partial t \partial y}$, we can use the Clairaut's theorem to switch the order of derivatives to get $\nabla \cdot \frac{\partial \mathbf{D}}{\partial t} \equiv \frac{\partial}{\partial t} \nabla \cdot \mathbf{D}$. Now, using the Gauss's law $\nabla \cdot \mathbf{D} = \rho$, we obtain $\nabla \cdot \frac{\partial \mathbf{D}}{\partial t} \equiv \frac{\partial}{\partial t} \nabla \cdot \mathbf{D} = \frac{\partial}{\partial t} \rho$. Using this in (2), we obtain

$$\nabla \cdot \mathbf{J} = -\frac{\partial}{\partial t} \rho.$$

Integrating over the region Ω, we get

$$\iiint\limits_{\Omega} \nabla \cdot \mathbf{J}\, dV = -\iiint\limits_{\Omega} \frac{\partial}{\partial t} \rho\, dV.$$

Furthermore using if the charge density ρ, and $\frac{\partial \rho}{\partial t}$ are continuous functions, we can take $\frac{\partial}{\partial t}$ outside the integral to obtain

$$\iiint\limits_{\Omega} \nabla \cdot \mathbf{J}\, dV = -\frac{d}{dt} \iiint\limits_{\Omega} \rho\, dV.$$

But the integral $\iiint\limits_{\Omega} \rho\, dV$ is nothing but the total charge $Q(t)$ enclosed in the region Ω at time t.

$$\iiint\limits_{\Omega} \nabla \cdot \mathbf{J}\, dV = -\frac{d}{dt} Q(t).$$

If the charge $Q(t)$ is conserved, i.e. $Q(t) = Q$, a constant with respect to t, then $\frac{d}{dt} Q(t) = 0$. Furthermore, using (1), we obtain the Kirchoff's current law:

$$\sum_{k=1}^{N} I_k = 0.$$

In other words, **the algebraic sum of currents in a network of conductors meeting at a point is zero**.

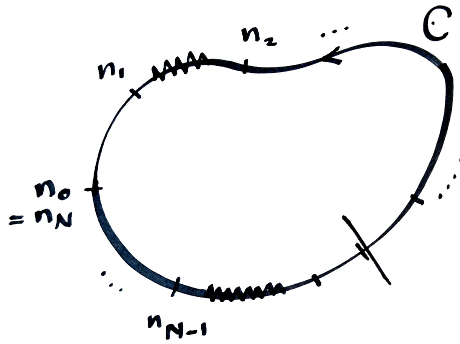

FIGURE 2. Close path in a circuit.

2. KIRCHOFF'S VOLTAGE LAW FOR DC CIRCUITS

The Kirchoff's voltage law is stated as: **the algebraic sum of all the voltages around any closed path in a circuit equals zero**. This applies to DC stationary circuits, with magnetic field density constant with respect to time. It is an application of the Faraday's law and the Stokes's theorem. Consider a closed electrical circuit with along a path C. Let us S be any surface with the boundary C. Integrating the equation of the Faraday's law, i.e. $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, on the surface S we obtain

$$\iint_S (\nabla \times \mathbf{E}) \cdot \hat{n}\, dS = -\iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{n}\, dS. \tag{3}$$

By the Stokes's theorem we get

$$\iint_S (\nabla \times \mathbf{E}) \cdot \hat{n}\, dS = \oint_C \mathbf{E} \cdot d\mathbf{r}. \tag{4}$$

Equating the right hand sides of (4) and (3) we obtain

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = -\iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{n}\, dS. \tag{5}$$

Let $\{n_k\}_{k=0}^N$ with $n_0 = n_N$ be any ordered points along the closed curve C, in the sense that for any smooth parametrization $\mathbf{r} : [a, b] \to C$, with $r(t_k) = n_k$, we have $t_k < t_{k+1}$ for $k = 0$ to $N-1$; moreover, $\mathbf{r}(t_0 = a) = \mathbf{r}(t_N = b) = n_0 = n_N$. Then

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = \sum_{k=1}^N \oint_{C_k} \mathbf{E} \cdot d\mathbf{r}, \tag{6}$$

where C_k denotes the segment of the curve C between the points n_{k-1} and n_k. By definition, $\oint_{C_k} \mathbf{E} \cdot d\mathbf{r} = V_k$ is the voltage between the points n_{k-1} and n_k. Thus, with this notation we get from (6) and (5)

$$\sum_{k=1}^N V_k = -\iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{n}\, dS. \tag{7}$$

If the magnetic field density constant with respect to time, i.e. $\frac{\partial \mathbf{B}}{\partial t} = 0$, we get the familiar version of the Kirchoff's voltage law:

$$\sum_{k=1}^N V_k = 0.$$

□