Recall the Maxwell’s equations:

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon} \quad : \text{Gauss’s law} \]
\[ \nabla \cdot \mathbf{B} = 0 \quad : \text{Gauss’s law for magnetism} \]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad : \text{Faraday’s law of induction} \]
\[ \nabla \times \mathbf{B} = \mu \left( \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} \right) \quad : \text{Ampère’s law,} \]

where \( \mathbf{E} \) is the electric field, \( \mathbf{B} \) is the magnetic field, \( \mathbf{E} \) is the electric field, \( \mathbf{J} \) denotes the total electric current density, and \( \rho \) is the charge density. The constants \( \varepsilon \) and \( \mu \) denote the permittivity and permeability respectively. Sometimes, it is convenient to use \( \mathbf{D} = \varepsilon \mathbf{E} \), and \( \mathbf{H} = \frac{\mathbf{B}}{\mu} \). The field \( \mathbf{D} \) is called as the electric displacement field, and \( \mathbf{H} \) is known as the magnetizing field.

Using this notation the Gauss’s law is written as

\[ \nabla \cdot \mathbf{D} = \rho \]

and the Ampère’s law takes the form:

\[ \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}. \]

1. **Kirchoff’s current law**

Consider a point in a network of conductors, with \( N \) number of wires emanating from it. Enclose the node with some open bounded nonempty region \( \Omega \subseteq \mathbb{R}^3 \), with boundary \( S \). We denote by \( S_k \) the intersection of the \( k^{th} \) wire and

![Figure 1. A node with three wires.](image-url)
the surface $S$. Using the divergence theorem we obtain:

$$
\iiint_{\Omega} \nabla \cdot J \, dV = \iint_{S} J \cdot \hat{n} \, dS
$$

$$
= \sum_{k=1}^{N} \int_{S_k} J \cdot \hat{n} \, dS + \int_{S \setminus \bigcup_{k=1}^{N} S_k} J \cdot \hat{n} \, dS
$$

$$
= \sum_{k=1}^{N} \int_{S_k} J \cdot \hat{n} \, dS + 0 \quad \text{(as the current density $J$ on $S \setminus \bigcup_{k=1}^{N} S_k$ is zero.)}
$$

$$
= \sum_{k=1}^{N} I_k.
$$

(1)

Note that the current $I_k$ is signed, with negative current indicating that the current is going inside the point.

Applying the gradient operator on both sides of the Ampère’s law: $\frac{\partial D}{\partial t} + J = \nabla \times H$, we get

$$
\nabla \cdot \frac{\partial D}{\partial t} + \nabla \cdot J = \nabla \cdot (\nabla \times H).
$$

We can show that $\nabla \cdot (\nabla \times H) = 0$ for any vector field $H$. Thus,

$$
\nabla \cdot \frac{\partial D}{\partial t} + \nabla \cdot J = 0
$$

i.e. $\nabla \cdot J = -\nabla \cdot \frac{\partial D}{\partial t}$.

(2)

Let us denote the components of $D$ by $D_1$, and $D_2$, i.e. $D = (D_1, D_2)$. Assuming the continuity of mixed second partial derivatives $\frac{\partial^2 D_1}{\partial x \partial t}$, and $\frac{\partial^2 D_2}{\partial y \partial t}$, we can use the Clairaut’s theorem to switch the order of derivatives to get $\nabla \cdot \frac{\partial D}{\partial t} = \frac{\partial}{\partial t} \nabla \cdot D$.

Now, using Gauss’s law $\nabla \cdot D = \rho$, we obtain $\nabla \cdot \frac{\partial D}{\partial t} = \frac{\partial}{\partial t} \nabla \cdot D = \frac{\partial}{\partial t} \rho$. Using this in (2), we obtain

$$
\nabla \cdot J = -\frac{\partial}{\partial t} \rho.
$$

Integrating over the region $\Omega$, we get

$$
\iiint_{\Omega} \nabla \cdot J \, dV = -\iiint_{\Omega} \frac{\partial}{\partial t} \rho \, dV.
$$

Furthermore using if the charge density $\rho$, and $\frac{\partial}{\partial t}$ are continuous functions, we can take $\frac{\partial}{\partial t}$ outside the integral to obtain

$$
\iiint_{\Omega} \nabla \cdot J \, dV = -\frac{d}{dt} \iiint_{\Omega} \rho \, dV.
$$

But the integral $\iiint_{\Omega} \rho \, dV$ is nothing but the total charge $Q(t)$ enclosed in the region $\Omega$ at time $t$.

$$
\iiint_{\Omega} \nabla \cdot J \, dV = -\frac{d}{dt} Q(t).
$$

If the charge $Q(t)$ is conserved, i.e. $Q(t) = Q$, a constant with respect to $t$, then $\frac{d}{dt} Q(t) = 0$. Furthermore, using (1), we obtain the Kirchoff’s current law:

$$
\sum_{k=1}^{N} I_k = 0.
$$

In other words, the algebraic sum of currents in a network of conductors meeting at a point is zero.
2. Kirchoff’s voltage law for DC circuits

The Kirchoff’s voltage law is stated as: the algebraic sum of all the voltages around any closed path in a circuit equals zero. This applies to DC stationary circuits, with magnetic field density constant with respect to time. It is an application of the Faraday’s law and the Stokes’s theorem. Consider a closed electrical circuit with along a path $C$. Let us $S$ be any surface with the boundary $C$. Integrating the equation of the Faraday’s law, i.e. $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, on the surface $S$ we obtain

$$\int_S (\nabla \times \mathbf{E}) \cdot \hat{n} dS = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{n} dS. \tag{3}$$

By the Stokes’s theorem we get

$$\int_S (\nabla \times \mathbf{E}) \cdot \hat{n} dS = \oint_C \mathbf{E} \cdot d\mathbf{r}. \tag{4}$$

Equating the right hand sides of (4) and (3) we obtain

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{n} dS. \tag{5}$$

Let $\{n_k\}_{k=0}^N$ with $n_0 = n_N$ be any ordered points along the closed curve $C$, in the sense that for any smooth parametrization $\mathbf{r} : [a, b] \to C$, with $r(t_k) = n_k$, we have $t_k < t_{k+1}$ for $k = 0$ to $N - 1$; moreover, $\mathbf{r}(t_0 = a) = \mathbf{r}(t_N = b) = n_0 = n_N$. Then

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = \sum_{k=1}^{N} \oint_{C_k} \mathbf{E} \cdot d\mathbf{r}, \tag{6}$$

where $C_k$ denotes the segment of the curve $C$ between the points $n_{k-1}$ and $n_k$. By definition, $\oint_{C_k} \mathbf{E} \cdot d\mathbf{r} = V_k$ is the voltage between the points $n_{k-1}$ and $n_k$. Thus, with this notation we get from (6) and (5)

$$\sum_{k=1}^{N} V_k = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{n} dS. \tag{7}$$

If the magnetic field density constant with respect to time, i.e. $\frac{\partial \mathbf{B}}{\partial t} = 0$, we get the familiar version of the Kirchoff’s voltage law:

$$\sum_{k=1}^{N} V_k = 0.$$